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Properties of the solution map for a first order linear problem *

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Abstract

We are interested in establishing properties of the general mathematical model

$$\frac{d\vec{u}}{dt} = T(t,\vec{u}) + \vec{b} + \vec{g}(t), \quad \vec{u}(t_0) = \vec{u}_0$$

for the dynamical system defined by the (possibly nonlinear) operator $T(t,\cdot) : V \to V$ with state space V. For one state variable where $V = \mathbb{R}$ this may be written as $dy/dx = f(x,y)$, $y(x_0) = y_0$. This paper establishes some mapping properties for the operator $L[y] = dy/dx + p(x)y$ with $y(x_0) = y_0$ where $f(x,y) = -p(x)y + g(x)$ and $T(x,y) = -p(x)y$ is linear. The conditions for the one-to-one property of the solution map as a function of $p(x)$ appear to be new or at least undocumented. This property is needed in the development of a solution technique for a nonlinear model for the agglomeration of point particles in a confined space (reactor).

1 Introduction

We begin with a family of initial value problems (IVP's) each consisting of a (possibly nonlinear) first order ordinary differential equation (ODE)

$$\frac{dy}{dx} = f(x,y), \tag{1}$$

and an arbitrary initial condition (IC) at an arbitrary point:

$$y(x_0) = y_0\,. \tag{2}$$

To specify a problem in this family, we must give the point (x_0, y_0) in the plane $\mathbb{R}^2 = \{(x,y) : x, y \in \mathbb{R}\}$ where $\mathbb{R}$ is the set of real numbers and the function $f : \Omega \to \mathbb{R}$ where Ω is an open connected region in $\mathbb{R}^2$ containing (x_0, y_0). Constraints on f are needed to make the problem reasonable and more likely to have a physical application. We assume immediately that $\Omega \supseteq R$. where R

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is a closed rectangle containing (x_0, y_0) in its interior and that f is continuous on R. Thus to specify a problem, we choose, in order, $(x_0, y_0) \in \mathbb{R}^2$, $R \subseteq \Omega$ (the exact definition of Ω really does not matter) as a closed rectangle in $\mathbb{R}^2$ containing (x_0, y_0) in its interior, and $f \in C(R)$, the set of functions $f : R \to \mathbb{R}$ that are continuous. We let $\Omega_{CR}(x_0, y_0)$ be the set of all closed rectangles in $\mathbb{R}^2$ that contain (x_0, y_0) in their interior, and $C(\Omega_{CR}(x_0, y_0)) = \{f \in C(R) : R \in \Omega_{CR}(x_0, y_0)\}$. Then $\mathbb{R}^2 \times \Omega_{CR}(x_0, y_0) \times C(\Omega_{CR}(x_0, y_0))$ is in a one-to-one correspondence with the set of problems of interest.

If (1) is nonlinear, the **interval of validity** (i.e, the open interval I containing x_0 where (1) and (2) are satisfied) is part of the problem which is therefore impredicative. However, we state the problem predicatively by assuming that I is given and look for solutions to (1) on I. That is, we look for solutions to (1) in a set $\Sigma(I)$ of functions whose common domain is I (i.e., a subset of $F(I) = \{f : I \to \mathbb{R}\}$). Let $I_{IV}(x_0, y_0,\ R)$ be the set of intervals I containing x_0 where $I \times \{y_0\} \subseteq R \subseteq \mathbb{R}^2$ and $\mathrm{Prob}((x_0, y_0), R, f, I)$ denote the initial value problem (1) and (2) associated with $(x_0, y_0) \in \mathbb{R}^2$, $R \in \Omega_{CR}(x_0, y_0)$, $f \in C(R)$, and $I \in I_{IV}(x_0, y_0, R)$. A minimum requirement for this IVP to be **well-posed** is that there is exactly one solution in $\Sigma(I)$ that satisfies both (1) and (2). Since we elect to specify the interval I, we denote by $\mathrm{Prob}(\mathbb{R}^2 \times \Omega_{CR}(x_0, y_0) \times C(\Omega_{CR}(x_0, y_0)) \times I_{IV}(x_0, y_0, R))$ the set of all initial value problems of interest.

There are at least four problem solving contexts for (1)–(2).

Traditional: If $f(x, y)$ is given as an elementary function and has one of several specific forms, the solution process starts with the ODE and uses calculus to obtain the "general" solution (i.e., a formula for all or at least almost all solutions) to the ODE as a parameterized family of functions (or curves). The IC is then applied to obtain the parameter and hence the (name of the) unique solution function. The interval of validity is then obtained as the largest (open) interval where the solution is valid. The solution algorithm (usually) establishes uniqueness and, if all steps are reversible, existence. If all steps are not reversible, existence can be established by substituting the proposed solution back into the ODE and the IC. (Or this can be used simply as a check.) Often a formula can be found for the (name of the) solution function for a whole class of problems by allowing parameters such as x_0 and y_0 to be arbitrary.

Classical: For a class of problems, existence and uniqueness of a solution in $\Sigma(I)$ is established using properties of $f(x, y)$, without necessarily obtaining a solution algorithm to obtain the (name of the) solution function.
Classical I: $\Sigma(I) = A(I) = \{y : I \to \mathbb{R} : y \text{ is analytic on } I\}$ (e.g., if f is analytic).
Classical II: $\Sigma(I) = C^1(I) = \{y : I \to \mathbb{R} : y' \text{ exists and is continuous on } I\}$.

Modern: A weak form of the problem is developed which allows weak solutions; that is, things that need not be functions (e.g., equivalence classes of functions and distributions).

In the Classical II context, the standard condition that f, $\partial f/\partial y \in C(R)$ assures local uniqueness (i.e., that for any $I \in I_{IV}(x_0, y_0, R), \Sigma(I) = C^1(I)$ contains at most one solution), but only local (and not global) existence (i.e., there exists an $I \in I_{IV}(x_0, y_0, R)$ such that $\Sigma(I) = C^1(I)$ contains a solution). Thus, in this context, the problem then focuses on finding the extent of the interval of validity for a class of problems rather than on finding the (name of the) solution function for a specific problem.

2 The linear solution map

Even though different contexts may define the problem differently, Traditional, Classical, and Modern all come together with the assumption of linearity, that is, when $f(x, y) = -p(x)y + g(x)$. In this case we have the ordinary differential equation (ODE)

$$\frac{dy}{dx} + p(x)y = g(x) \tag{3}$$

with the initial condition (IC)

$$y(x_0) = y_0\,. \tag{4}$$

We switch from viewing (1) as an equation to viewing (3) and (4) as a mapping problem. Thus we keep x_0, y_0, I, and p fixed and only vary g. To keep solutions as functions, we continue with the Classical II context, let $\Sigma(I) = C^1(I)$, and assume $p, g \in C(I) = \{f : I \to \mathbb{R}$ that are continuous$\}$ where $x_0 \in I$ so that $f, \partial f/y \in C(R)$ where R is the strip $R = I\times\mathbb{R} = \{(x, y) : x \in I\}$. Now let $L_p[y] : C^1(I) \to C(I)$ be defined by $L_p[y] = dy/dx + p(x)y$ and $N_{p,y_0}; D_{y_0}(I) \to C(I)$ be the restriction of L_p to the hyper-plane $D_{y_0}(I) = \{y \in C^1(I) : y(x_0) = y_0\}$. Not only are we assured that for any $g \in C(I)$, a unique (global) solution to the IVP (3) and (4) exists in $\Sigma(I) = C^1(I)$ so that inverse mapping exists, but, using the integrating factor $\mu_p(x) = \exp\{\int_{t=x_0}^{t=x} p(t)dt\}$, we have a calculus formula for $y(x) = N_{p,y_0}^{-1}[g](x)$:

$$\begin{aligned} y(x) &= \Big(y_0 + \int_{t=x_0}^{t=x} g(t)\exp\{\int_{s=x_0}^{s=t} p(s)ds\}dt\Big)\exp\{-\int_{t=x_0}^{t=x} p(t)dt\} \\ &= y_0 \exp\{-\int_{t=x_0}^{t=x} p(t)dt\} + \int_{t=x_0}^{t=x} g(t)\exp\{\int_{s=x}^{s=t} p(s)ds\}dt \end{aligned} \tag{5}$$

If $y_0 \neq 0$, $D_{y_0}(I)$ will not pass through the origin and not be a subspace of $C^1(I)$ so that N_{p,y_0} and N_{p,y_0}^{-1} will not be linear operators. However,

$$y(x) = N_{p,y_0}^{-1}[g](x) = y_0\mu_{-p}(x) + L_{p,0}^{-1}[g](x) \tag{6}$$

where the (linear) Voltera operator $L_{p,0}^{-1}[g](x) = \int_{t=x_0}^{t=x} G(x,t)g(t)dt$ with kernel (Green's function) $G(x, t) = \exp\{\int_{s=x}^{s=t} p(s)ds\}$ is the inverse of $N_{p,0}$ which we might also call $L_{p,0}$ since when $y_0 = 0$, $N_{p,y_0}$is linear.

In a traditional context, p and g are given elementary functions in $C(I)$. If possible, the Riemann integrals in (5) are computed explicitly. In a classical context, the interval of validity $I = (a, b)$ can be extended to closed (and half-open) intervals by requiring p and g to be analytic (in a neighborhood of) or continuous at the end points. In one modern context, the Riemann integrals become Lebesgue integrals and act on equivalence classes of functions, for example, piecewise continuous functions which need not be defined at the points of discontinuity (as these points form a set of measure zero). We continue to keep x_0 and I fixed, but now allow y_0 and p as well as g to vary. To simplify our notation, we write $y(x) = y(x; y_0, p, g)$ instead of $y(x) = N^{-1}_{p,y_0}[g](x) = y_0\mu_{-p}(x) + L^{-1}_{p,0}[g](x)$.

3 One-to-one properties

To understand how $y(x; y_0, p, g)$ given by (5) depends on each of the parameters y_0, p, and g, we need several relations. For i=1,2, let y_i be the solution to the IVP (3) and (4) (on I where I may be open or closed) when $p = p_i, g = g_i$, and $y_0 = y_i^0$. If $p_1 = p_2 = p, g_1 = g_2 = g$ and $p, g \in C(I)$, then for all x in I using (5) we obtain

$$\begin{aligned} |y_1(x) - y_2(x)| &= \left|y_1^0 - y_2^0\right| \exp\{-\int_{t=x_0}^{t=x} p(t)dt\} \\ &\leq \left|y_1^0 - y_2^0\right| \exp\{\int_{t=x_0}^{t=x} |p(t)|\, dt\} \end{aligned} \tag{7}$$

If $y_1^0 = y_2^0 = y_0, p_1 = p_2 = p$ and $p, g_1, g_2 \in C(I)$, then for all x in I using (3) we obtain

$$d(y_1 - y_2)/dx + p(x)(y_1(x) - y_2(x)) = g_1(x) - g_2(x) \tag{8}$$

and using (5)

$$|y_1(x) - y_2(x)| = \left|\int_{t=x_0}^{t=x} [g_1(t) - g_2(t)] \exp\{\int_{s=x}^{s=t} p(s)ds\}dt\right| \tag{9}$$

$$\leq \int_{t=x_0}^{t-x} |g_1(t) - g_2(t)| \exp\{\int_{s=x}^{s=t} |p(s)|\, ds\}dt \tag{10}$$

If $y_1^0 = y_2^0 = y_0$, $g_2 = g_1 = g$ and $p_1, p_2, g \in C(I)$, then for all x in I from (3) we obtain

$$d(y_1 - y_2)/dx + p_1(x)y_1(x) - p_2(x)y_2(x) = 0 \tag{11}$$

and using (5)

$$
\begin{aligned}
&|y_1(x) - y_2(x)| \\
=&\Big| y_0 \exp\{-\int_{t=x_0}^{t=x} p_1(t)dt\} - y_0 \exp\{-\int_{t=x_0}^{t=x} p_2(t)dt\} \\
&+ \int_{t=x_0}^{t=x} g(t) \exp\{\int_{s=x}^{s=t} p_1(s)ds\}dt - \int_{t=x_0}^{t=x} g(t) \exp\{\int_{s=x}^{s=t} p_2(s)ds\}dt \Big| \quad (12) \\
=&\Big| y_0 \exp\{-\int_{t=x_0}^{t=x} p_1(t)dt\} \big[1 - \exp\{\int_{t=x_0}^{t=x} [-p_2(t) + p_1(t)]dt\}\big] \\
&+ \int_{t=x_0}^{t=x} g(t) \big[\exp\{\int_{s=x}^{s=t} p_1(s)ds\}\big] \big[1 - \exp\{\int_{s=x}^{s=t} [p_2(s) - p_1(s)]ds\}\big] dt \Big| \\
\leq& |y_0| \exp\{\int_{t=x_0}^{t=x} |p_1(t)|\, dt\} \big[\exp\{\int_{t=x_0}^{t=x} |p_2(t) - p_1(t)| dt\} - 1\big] \\
&+ \int_{t=x_0}^{t=x} |g(t)| \, \big[\exp\{\int_{s=x}^{s=t} |p_1(s)|\, ds\}\big] \big[\exp\{\int_{s=x}^{s=t} |p_2(s) - p_1(s)| ds\} - 1\big] dt
\end{aligned}
\tag{13}
$$

where we have used the inequality $\left|1 - e^b\right| = \left|e^b - 1\right| \leq \left|e^{|b|} - 1\right| = e^{|b|} - 1$.

Standard theory [1] implies that for each $y_0 \in \mathbb{R}$ and $p \in C(I)$, $N_{p,y_0}^{-1}[g](x)$ provides a one-to-one correspondence between $g \in C(I)$ and $y \in D_{y_0}(I)$ as well as establishing that for fixed p and g, the solutions to (3) (parameterized by y_0) do not cross each other. Interestingly, with some restrictions, the solution map from $(y_0, p, g) \in \mathbb{R} \times C(I) \times C(I)$ to $y \in C^1(I)$ is one-to-one if any two of these three variables are held constant.

Theorem 1 *If $p, g \in C(I)$, then the solution map defined by (5) from $y_0 \in \mathbb{R}$ to $y \in C^1(I)$ is one-to-one. Also, the solutions to (3) in $C^1(I)$ do not cross each other.*

Proof. If y_1^0 and y_2^0 are different initial conditions and $p, g \in C(I)$, then for all x in I we have from (7) that $|y_1(x) - y_2(x)| = |y_1^0 - y_2^0| \{\exp - \int_{t=x_0}^{t=x} p(t)dt\}$. Hence if $y_1(x) = y_2(x)$ for any $x \in I$, then we must have $y_1^0 = y_2^0$. That is, if we change the initial condition, the solution changes everywhere. If p and g are fixed, not only is the the mapping from $y_0 \in \mathbb{R}$ to $y \in C^1(I)$ one-to-one, but the family of solutions to (3) parameterized by y_0 do not cross each other.

Theorem 2 *If $y_0 \in I, p \in C(I)$, then the solution map $y(x) = N_{p,y_0}^{-1}[g](x)$ defined by (5) from $g \in C(I)$ to $y \in C^1(I)$ is one-to-one. Also $N_{p,y_0}^{-1}[g](x)$ provides a one-to-one correspondence between $C(I)$ and $D_{y_0}(I)$.*

Clearly L_p maps $C^1(I)$ into $C(I)$. N_{p,y_0}^{-1} given by (6) shows N_{p,y_0} is one-to-one and that the domain of N_{p,y_0}^{-1} is all of $C(I)$. Hence N_{p,y_0}^{-1} is one-to-one. Alternately, this follows directly from (8) and is just the statement that

$N_{p,y_0} = \left(N_{p,y_0}^{-1}\right)^{-1}$ is a well defined operator. Hence N_{p,y_0}^{-1} provides a one-to-one correspondence between $C(I)$ and $D_{y_0}(I)$. (And N_{p,y_0} provides a one-to-one correspondence between $D_{y_0}(I)$ and $C(I)$.)

Restrictions on y_0 and g are needed for the mapping from $p \in C(I)$ to $y \in C^1(I)$ to be one-to-one. To see why, note from (5) that if $y_0 = 0$ and g is the zero function, then for any p in $C(I), y$ is identically zero.

Definition $f \in C(I)$ is said to have only dispersed zeros on an interval I (either open, closed, or half-opened), if there exists no open interval $J \subseteq I$ where f is identically zero; that is, if for any open interval $J \subseteq I$ there exists $x \in J$ such that $f(x) \neq 0$ (i.e., $Z_f = \{x \in I : f(x) = 0\}$ has no interior points).

We show that for any y_0 and p, if g has only dispersed zeros, then the solution y also has only dispersed zeros. On the other hand, if g is identically zero, then y is either never zero or always zero, depending on y_0.

Theorem 3 *If $g \in C(I)$ and has only dispersed zeros on an interval I, then the solution y has only dispersed zeros on I.*

Proof. We prove the contrapositive. Assume y does not have only dispersed zeros on I. By definition there exists an open interval $J \subseteq I$ such that $y(x) = 0$ for all x in J. Then $dy/dx = 0$ on J and by (3), $g(x) = 0$ on J. Hence g does not have only dispersed zeros on I.

Theorem 4 *Let I be an interval. If $g(x) = 0$ on an interval $J \subseteq I$, then either y is identically zero on J or is never zero on J. In particular, if $g(x) = 0$ on I, then either $y_0 = 0$ and y is identically zero or $y_0 \neq 0$ and y is never zero on I.*

Proof. Suppose $g(x) = 0$ on an interval $J \subseteq I$. Choose $x_1 \in J$. Applying (5) at x_1 with $g(x) = 0$ on J, we have that $y(x) = y(x_1)exp\{-\int_{t=x_0}^{t=x} p(t)dt\}$ on J. If $y(x_1) = 0$, then y is identically zero on J. If $y(x_1) \neq 0$, then y is never zero on J. Similarly for I.

Corollary 1 *If y has only dispersed zeros on an interval I, then on any interval $J \subseteq I$ where $g(x) = 0$, y is never zero.*

Proof. Assume y has only dispersed zeros and that $g(x) = 0$ on an interval $J \subseteq I$. But if y has only dispersed zeros, there exists $x_1 \in J$ such that $y(x_1) \neq 0$. Then by Theorem 4, we have that $y(x) \neq 0$ for all x in J.

Theorem 5 *If g is identically zero on an interval I and $y_0 \neq 0$, then the solution map defined by (5) from $p \in C(I)$ to $y \in C^1(I)$ is one-to-one.*

Proof. Suppose $y_0 \neq 0$, $g = 0$ and $p_1, p_2 \in C(I)$. If $y_1 = y_2 = y$, then for all $x \in I$ we have from (11) that $(p_1(x) - p_2(x))y(x) = 0$. From Theorem 4, $y(x)$ is never zero so that for all $x \in I, p_1(x) = p_2(x)$. Hence $p_1 = p_2$ so that the solution map is one-to-one.

Theorem 6 *If g has only dispersed zeros on I, then the solution map defined by (5) from $p \in C(I)$ to $y \in C^1(I)$ is one-to-one.*

Proof. Suppose g has only dispersed zeros on I, and $p_1, p_2, g \in C(I)$. If $y_1 = y_2 = y$, then for all x in I we have from (11) that $(p_1(x)-p_2(x))y(x) = 0$. Hence if $y(x) \neq 0$, then $p_1(x) = p_2(x)$. Given $x \in I$, if there exists a sequence $\{x_n\}_{n=1}^{\infty}$ such that $y(x_n) \neq 0$ and $\lim_{n\to\infty} x_n = x$; then by continuity, $p_1(x) = p_2(x)$. Thus $p_1 = p_2$ everywhere if the zeros of y are dispersed. It remains to show that if g has only dispersed zeros on I, then y has only dispersed zeros on I. But this is just Theorem 3.

4 Continuity properties

Finally, for the IVP (3) and (4) to be a well-posed problem, the solution map should be continuous with respect to $y_0, p,$and$g$. From (7), we see that $y(x; y_1^0, p(x), g(x))$ will be pointwise close to $y(x; y_2^0, p(x), g(x))$ if y_0^1 is close to y_0^2. From (10), we see that $y(x; y_0, p(x), g_1(x))$ will be pointwise close to $y(x; y_0, p(x), g_2(x))$ if $g_1(x)$ is everywhere pointwise close to $g_2(x)$. Finally, from (13), we see that $y(x; y_0, p_1(x), g(x))$ will be pointwise close to $y(x; y_0, p_2(x), g(x))$ if $p_1(x)$ is everywhere pointwise close to $p_2(x)$.

To obtain a global notion of closeness, we require the domain of y to be the closed interval $\overline{I} = [a, b]$ where $I = (a, b)$ and redefine $C(I)$ and $C^1(I)$ as $C(I) = \{f : \overline{I} \to \mathbb{R}$ such that f is continuous on $I = (a, b)\}$ and $C^1(I) = \{f : \overline{I} \to \mathbb{R}$ such that f' exists and is continuous on $I = (a, b)\} \subseteq C(I)$. Then $C(\overline{I}) = \{f : I \to \mathbb{R}$ such that f is continuous on $I = [a, b]\} \subseteq C(I)$, and $C^1(\overline{I}) = \{f : I \to \mathbb{R}$ such that, using one-sided limits, f' exists and is continuous on $\overline{I} = [a, b]\} \subseteq C^1(I)$. As we have said, if $p, g \in C(\overline{I})$, then, using one-sided limits, y given by (5) can be considered to solve (3) on $\overline{I}$ so that $\Sigma(\overline{I}) = C^1(\overline{I})$. Now let $D_{y_0}(\overline{I}) = \{y \in C^1(\overline{I}) : y(x_0) = y_0\}$. Then N_{p,y_0} maps $D_{y_0}(\overline{I})$ to $C(\overline{I})$, the solution maps $(y_0, p, g) \in \mathbb{R} \times C(\overline{I}) \times C(\overline{I})$ to $y \in C^1(\overline{I})$ and Theorems 1, 2, 6, and 7 remain valid with $C(I)$ replaced by $C(\overline{I})$, $C^1(I)$ replaced by $C^1(\overline{I})$ and $D_{y_0}(I)$ replaced by $D_{y_0}(\overline{I})$. We have $D_{y_0}(\overline{I}) \subseteq C^1(\overline{I}) \subseteq C(\overline{I}) \cap C^1(I) \subseteq C(I)$.

Now recall that $L^\infty(\overline{I})$ is the set of equivalent classes of functions that are equal except on a set of Lebesgue measure zero such that $\operatorname{ess\,sup}_{x\in\overline{I}} |f(x)| < \infty$. Since $C(\overline{I})$ can be considered as a subset of $L^\infty(\overline{I})$ we can use the $L^\infty(\overline{I})$ norm, $\|f\|_\infty = \operatorname{ess\,sup}_{x\in\overline{I}} |f(x)|$ for functions in $C(\overline{I})$ where $\|f\|_\infty = \max_{x\in\overline{I}} |f(x)|$ as well as for those in all of its subsets: $D_{y_0}(\overline{I}) \subseteq C^1(\overline{I}) \subseteq C(\overline{I}) \cap C^1(I) \subseteq C(\overline{I}) \subseteq L^\infty(\overline{I})$. In this restricted context, we can obtain the following global inequalities. If $p_1 = p_2 = p, g_1 = g_2 = g$ and $p, g \in C(\overline{I})$, then for all x in

$\overline{I} = [a, b]$ we have using (7) that

$$|y_1(x) - y_2(x)| \leq \left|y_1^0 - y_2^0\right| \exp\{\|p\|_\infty (b - a)\}$$

so that

$$\|y_1(x) - y_2(x)\|_\infty \leq \left|y_1^0 - y_2^0\right| \exp\{\|p\|_\infty (b - a)\} \tag{14}$$

If $y_1^0 = y_2^0 = y_0$, $p_1 = p_2 = p$, and $p, g_1, g_2 \in C(\overline{I})$, then for all $x \in \overline{I} = [a, b]$ we have from (9) that

$$|y_1(x) - y_2(x)| \leq \|g_1 - g_2\|_\infty (b - a) \exp\{\|p\|_\infty (b - a)\}$$

so that

$$\|y_1 - y_2\|_\infty \leq \|g_1 - g_2\|_\infty (b - a) \exp\{\|p\|_\infty (b - a)\} \tag{15}$$

If $y_1^0 = y_2^0 = y_0$, $g_1 = g_2 = g$, and $p_1, p_2, g, p \in C(\overline{I})$, then for all $x \in \overline{I} = [a, b]$ we have from (13) that

$$\begin{aligned}|y_1(x) - y_2(x)| \leq{}& |y_0| \exp\{\|p_1\|_\infty (b - a)\}[\exp\{\|p_1 - p_2\| (b - a)\} - 1] \\ &+ \|g\|_\infty \exp\{\|p_1\|_\infty (b - a)\}[\exp\{\|p\|_\infty (b - a)\} - 1](b - a)\end{aligned}$$

so that

$$\begin{aligned}\|y_1 - y_2\| \leq{}& |y_0| \exp\{\|p_1\|_\infty (b - a)\}[\exp\{\|p_1 - p_2\| (b - a)\} - 1] \\ &+ \|g\|_\infty \exp\{\|p_1\|_\infty (b - a)\}[\exp\{\|p_1 - p_2\|_\infty (b - a)\} - 1](b - a)\end{aligned} \tag{16}$$

To show that $y(x; y_0, p, g)$ depends continuously on $y_0, p,$and g when two are fixed, we use the norm (and hence metric) topologies for $\mathbb{R}$ and that induced on $C(\overline{I})$ from $L^\infty(\overline{I})$.

Theorem 7 *If $p, g \in C(\overline{I})$, then the solution map defined by (5) from $y_0 \in \mathbb{R}$ to $y \in C^1(\overline{I})$ is continuous.*

Proof. Let $p_1 = p_2 = p$, $g_1 = g_2 = g$, $p, g \in C(\overline{I})$, and $\epsilon > 0$. Then from (14), there exists $\delta > 0$ such that $|y_1^0 - y_2^0| < \delta$ implies $\|y_1 - y_2\| < \epsilon$. Hence the mapping from $y_0 \in \mathbb{R}$ to $y \in C^1(\overline{I})$ is continuous.

Theorem 8 *If $y_0 \in I, p \in C(\overline{I})$, then the solution map defined by (5) from $g \in C(\overline{I})$ to $y \in C^1(\overline{I})$ is continuous.*

Proof. Let $y_1^0 = y_2^0 = y_0$, $p_1 = p_2 = p$, $p, g_1, g_2 \in C(\overline{I})$, and $\epsilon > 0$. Then from (15), there exists $\delta > 0$ such that $\|g_1 - g_2\| < \delta$ implies $\|y_1 - y_2\| < \epsilon$. Hence the mapping is continuous.

Theorem 9 *If $y_0 \in I$, $p \in C(\overline{I})$, then the solution map defined by (5) from $p \in C(\overline{I})$ to $y \in C^1(\overline{I})$ is continuous.*

Proof. Let $y_1^0 = y_2^0 = y_0$, $g_1 = g_2 = g$, $p_1, p_2, g \in C(\overline{I})$, and $\epsilon > 0$. Then from (16) there exists $\delta > 0$ such that $\|p_1 - p_2\| < \delta$ implies $\|y_1 - y_2\| < \epsilon$. Hence the mapping is continuous.

Summary. Let $y_0 \in R$ and $p, g \in C(I)$ where I is an interval and consider the solution map for the IVP (3) and (4) given by (5) that maps $(y_0, p, g) \in \mathbb{R} \times C(I) \times C(I)$ to $y \in C^1(I)$.

1. If p and g are fixed, the mapping is one-to-one and solutions do not cross.
2. If y_0 and p are fixed, the mapping is one-to-one and has an inverse mapping that is linear if and only if $y_0 = 0$.
3. If y_0 and g are fixed and either $y_0 \neq 0$ and g is identically zero or g has only dispersed zeros, then the mapping is one-to-one.
4. If I is a closed interval, and two of y_0, p and g are fixed, then the solution map is continuous using the usual topology for $\mathbb{R}$ and the norm (metric) topologies for $C(\overline{I})$ and $C^1(\overline{I})$ as subspaces of $L^\infty(\overline{I})$.

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