USA-Chile Workshop on Nonlinear Analysis,
Electron. J. Diff. Eqns., Conf. 06, 2001, pp. 303–311.
http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu or ejde.math.unt.edu (login: ftp)

Recent progress in the anisotropic electrical impedance problem *

Gunther Uhlmann

Abstract

We survey some recent progress on the problem of determining an anisotropic conductivity of a medium by making voltage and current measurements at the boundary of the medium.

1 Introduction

We give more details on open problem 5 stated in [13] which was only briefly discussed there for lack of space. We also survey some recent developments on the same problem.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $\gamma = (\gamma^{ij}(x))$ be the electrical conductivity of Ω which is assumed to be a positive definite, smooth, symmetric matrix on $\overline{\Omega}$. Muscle tissue in the human body is a prime example of an anisotropic conductivity since the conductivity in the transverse direction (for cardiac muscle this is 2.3 mho) is quite different than in the longitudinal direction (for cardiac muscle this is 6.3 mho).

Under the assumption of no sources or sinks of current in Ω, the equation for the potential, given a voltage potential f on $\partial\Omega$, is given by the solution of the Dirichlet problem

$$\begin{gathered} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\gamma^{ij}\frac{\partial u}{\partial x_j}\right) = 0 \text{ on } \Omega \\ u\big|_{\partial\Omega} = f. \end{gathered} \tag{1}$$

The Dirichlet-to-Neumann map (DN) is defined by

$$\Lambda_\gamma(f) = \sum_{i,j=1}^{n} \nu^i \gamma^{ij} \frac{\partial u}{\partial x_j}\big|_{\partial\Omega} \tag{2}$$

Mathematics Subject Classifications: 35R30.
Key words: Dirichlet to Neumann map, Electrical Impedance Tomography, Anisotropic conductivities.

Published January 8, 2001.
Partly supported by NSF grant DMS–0070488

where $\nu = (\nu^1, \dots, \nu^n)$ denotes the unit outer normal to $\partial\Omega$ and u is the solution of (1). Λ_γ is also called the *voltage* to *current* map since $\Lambda_\gamma(f)$ measures the induced current flux at the boundary.

The inverse problem is whether one can determine γ by knowing Λ_γ. Calderón proposed this problem in [4]. He worked as an engineer for YPF (Yacimientos Petroleros Fiscales) in Argentina and he thought of this problem and his contribution during that time. This problem arises naturally in geophysical prospection. In fact the Schlumberger-Doll company was founded in the early part of the century to find oil using electrical prospection (see [15] for an account). We are grateful to Alberto Grünbaum who convinced Calderón to publish his result in 1980 (personal communication). Paul Malliavin in his lecture at the conference held at the University of Chicago to honor the 75$^{\text{th}}$ birthday of Calderón mentioned that Calderón told him of his inverse result in 1954 (see footnote in page 228 of [5]). More recently this inverse problem has been proposed as a valuable diagnostic tool in medicine (see for instance [2]) and it has been called *electrical impedance tomography* (EIT). Unfortunately, Λ_γ doesn't determine γ uniquely. This observation is due to L. Tartar (see [6] for an account). To see this we define first the Dirichlet integral associated to a solution of (1). Let

$$Q_\gamma(f) = \sum_{i,j=1}^{n} \int_\Omega \gamma^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx \tag{3}$$

with u a solution of (1).

A standard application of the divergence theorem gives that

$$Q_\gamma(f) = \int_{\partial\Omega} \Lambda_\gamma(f) f dS, \tag{4}$$

where dS denotes surface measure in $\partial\Omega$. In other words, Λ_γ is the linear operator associated to the quadratic form Q_γ so that Λ_γ and Q_γ carry the same information.

Let $\psi : \overline{\Omega} \to \overline{\Omega}$ be a C^∞ diffeomorphism with $\psi\big|_{\partial\Omega} = \text{Identity}$. Let $v = u \circ \psi^{-1}$. Then a straightforward calculation shows that v satisfies

$$\begin{gathered} \sum_{i,j=1}^{n} \tfrac{\partial}{\partial x_i} \left(\widetilde{\gamma}_{ij} \tfrac{\partial v}{\partial x_j} \right) = 0 \\ v\big|_{\partial\Omega} = f \end{gathered} \tag{5}$$

where

$$\widetilde{\gamma} = \left(\frac{(D\psi)^T \circ \gamma \circ (D\psi)}{|\det D\psi|} \right) \circ \psi^{-1} =: \psi_* \gamma. \tag{6}$$

Here $D\psi$ denotes the (matrix) differential of ψ, $(D\psi)^T$ its transpose and the composition in (6) is to be interpreted as composition of matrices.

By making the change of variables $v = u \circ \psi^{-1}$ in the quadratic form (3) we see that

$$Q_{\widetilde{\gamma}}(f) = Q_\gamma(f) \tag{7}$$

and therefore $\Lambda_{\tilde{\gamma}} = \Lambda_\gamma$.

We have found a large number of conductivities with the same DN map: any change of variables of Ω that leaves the boundary fixed gives rise to a new conductivity with the same electrical boundary measurements. The question is then whether this is the only obstruction to unique identifiability of the conductivity. As we outline below this is a problem of geometrical nature and we proceed to state it in invariant form.

2 Geometric Formulation

Let (M, g) be a compact Riemannian manifold with boundary. The Laplace-Beltrami operator associated to the metric g is given in local coordinates by

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}g^{ij}\frac{\partial u}{\partial x_j}\right) \tag{1}$$

where (g^{ij}) is the inverse of the metric g. Let us consider the Dirichlet problem associated to (1)

$$\begin{aligned} \Delta_g u &= 0 \text{ on } \Omega \\ u\big|_{\partial\Omega} &= f \end{aligned} \tag{2}$$

We define the DN map in this case by

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^i g^{ij} \frac{\partial u}{\partial x_j}\sqrt{\det g}\big|_{\partial\Omega} \tag{3}$$

where $(\nu^i) = \nu$ is the outer unit normal to $\partial\Omega$. The inverse problem is to recover g from Λ_g.

By using a similar argument to the one outlined above we have that

$$\Lambda_{\psi^* g} = \Lambda g \tag{4}$$

where ψ is a C^∞ diffeomorphism of $\overline{M}$ which is the identity on the boundary. As usual $\psi^* g$ denotes the pull back of the metric g by the diffeomorphism ψ.

In the case that M is an open, bounded subset of $\mathbb{R}^n$ with smooth boundary, it is easy to see that ([7]) for $n \geq 3$

$$\Lambda_g = \Lambda_\gamma \tag{5}$$

where

$$g_{ij} = (\det \gamma^{kl})^{\frac{1}{n-2}}(\gamma^{ij})^{-1}, \quad \gamma^{ij} = (\det g_{kl})^{1/2}(g_{ij})^{-1}. \tag{6}$$

In the two dimensional case (1.12) is not valid. In fact in $n = 2$ the Laplace-Beltrami operator is conformally invariant. More precisely

$$\Delta_{\alpha g} = \frac{1}{\alpha}\Delta_g$$

for any function α, $\alpha \neq 0$. Therefore we have that for $n = 2$

$$\Lambda_{\alpha(\psi^* g)} = \Lambda_g \tag{7}$$

for any smooth function $\alpha \neq 0$ so that $\alpha\big|_{\partial M} = 1$.

Now we give an invariant formulation of the EIT problem in the two dimensional case. In the Euclidean case a current is a one form given by

$$i(x) = \gamma(x) du(x)$$

where u is the voltage potential. Then, in two dimensions, the conductivity γ can be viewed as a linear map from 1-forms to 1-forms. Now let (M, g) be a two dimensional Riemannian manifold. Let γ be a positive definite symmetric mapping (with respect to the inner product defined by the metric g) from one forms to one forms. In this case (1) takes the form

$$\begin{aligned} \delta(\gamma du) &= 0 \text{ in } M \\ u\big|_{\partial M} &= f \end{aligned} \tag{8}$$

where d denotes differentiation and δ codifferentiation with respect to the metric g.

The DN map is given by the 1-form

$$\Lambda_{g,\gamma} f = \gamma du\big|_{\partial M}. \tag{9}$$

An argument similar to the one outlined above shows that

$$\Lambda_{g,\psi_*\gamma} = \Lambda_\gamma \tag{10}$$

for every diffeomorphism $\psi : \overline{M} \to \overline{M}$ which is the identity at the boundary. Here $\psi_*\gamma$ denotes the push-forward by the diffeomorphism ψ of the one form γ. We remark that Riemannian metrics pullback naturally under smooth maps and conductivities push-forward naturally under smooth maps.

Now we are in position to state the main conjectures.

Conjecture A ($n \geq 3$).

Let (M, g) be a compact Riemannian manifold with boundary. The pair $(\partial M, \Lambda_g)$ determines (M, g) uniquely. Of course uniquely means up to an isometric copy.

Conjecture B ($n = 2$).

Let (M, g) be a compact Riemannian surface. Then the pair $(\partial M, \Lambda_g)$ determines uniquely the conformal class of (M, g). Uniquely means again up to an isometric copy.

Conjecture C ($n = 2$).

Let (M, g) be a compact Riemannian surface with boundary and γ a positive definite symmetric map from one forms to one forms on M. Suppose we know $(M, g, \partial M, \Lambda_{g,\gamma})$ with $\Lambda_{g,\gamma}$ defined as in (9), then we can recover uniquely γ. Uniquely means here up to an isometry which is the identity on the boundary as in (1.6)

3 The results

A basic result which is used in all the anisotropic results stated below is the following Lemma proved in [7]:

Lemma 3.1 *(a) $n \geq 3$. Let (M, g) be a compact Riemannian manifold with boundary. Then Λ_g determines the C^∞-jet of the metric at the boundary in the following sense. If g' is another Riemannian metric on M such that $\Lambda_g = \Lambda_{g'}$, then there exists a diffeomorphism $\varphi : M \to M$, $\varphi\big|_{\partial M} = \text{Identity}$ such that $g' = \varphi^* g$ to infinite order at ∂M.*

(b) $n = 2$. Let (M, g) be a compact Riemannian manifold with boundary. Then Λ_g determines the conformal class of the C^∞-jet of the metric at the boundary.

(c) $n = 2$. Let (M, g) be a compact Riemannian surface with boundary. Let γ be a positive definite symmetric map from one forms to one forms. Then the mapping $\Lambda_{g,\gamma}$, as defined in (6), determines the C^∞-jet of the map γ at the boundary in the following sense: If γ' is another such one form such that $\Lambda_{g,\gamma} = \Lambda_{g,\gamma'}$. Then there exists a diffeomorphism $\varphi : M \to M$, $\varphi\big|_{\partial M} = \text{Identity}$ such that $\gamma' = \varphi_* \gamma$ to infinite order at ∂M.

In other words Lemma 3.1 shows that Conjectures A, B, C above are valid at the boundary. The proof of this result is done in case a) by showing that Λ_g is a pseudodifferential operator of order 1. Its full symbol, calculated in appropriate coordinates, determines the C^∞-jet of the metric g at the boundary. The proofs of b) and c) are similar.

The only case of Conjecture A that has been settled in general is the isotropic case in Euclidean space. Namely we have in the case that $M = \Omega$ an open, bounded subset of $\mathbb{R}^n$ with a smooth boundary and the metric g is given by

$$g_{ij} = \alpha(x)\delta_{ij}, \qquad \alpha > 0 \tag{1}$$

where δ_{ij} is the Krönecker delta.

Suppose $g^{(1)}$, $g^{(2)}$ are two isotropic Riemannian metrics

$$g^{(i)} = \alpha^i(x)(\delta_{kl}) \qquad i = 1, 2, \qquad \alpha^i > 0. \tag{2}$$

Then it is straightforward to show that if $\psi^* g_1 = g_2$, $\psi\big|_{\partial\Omega} = \text{Identity}$, then $\psi = \text{Identity}$. So the Conjecture A in this case is that $g_1 = g_2$. This was proven in [12]:

Theorem 3.2 *Let $\Omega \subseteq \mathbb{R}^n$ $n \geq 3$ be a bounded domain with smooth boundary. Let $g^{(i)}$, $i = 1, 2$ be two isotropic Riemannian manifolds satisfying (2). Then $\Lambda_{g_1} = \Lambda_{g_2}$ implies $g_1 = g_2$.*

We won't outline the proof here. We mention that a crucial ingredient in the proof is the construction of complex geometrical optics solutions of the Laplace-Beltrami operator when the Riemannian metric is isotropic. More precisely

Lemma 3.3 *Let g be an isotropic Riemannian metric as in (1), with $\alpha = 1$ outside a large ball. Let $\rho \in \mathbb{C}^n$, $\rho \cdot \rho = 0$. Then for $|\rho|$ sufficiently large, there exist solutions of $\Delta_g u = 0$ of the form*

$$u = e^{x\cdot\rho}\alpha^{-\frac{1}{2}}(1 + \psi_g(x, \rho)) \tag{3}$$

with $\psi_g \underset{|\rho|\to\infty}{\longrightarrow} 0$ uniformly in compact sets.

For more precise statements and a recent survey of other results using complex geometrical optics solutions, see [14].

One of the main difficulties in extending Theorem 3.2 to the general anisotropic case even in the case when M is an open subset of Euclidean space is to construct an analog of (3) for the Laplace-Beltrami operator.

Lassas and the author ([M-U]) proved Conjecture A in the real-analytic case and Conjecture B in general. Moreover these results assume that Λ_g is measured only on an open subset of the boundary.

Let Γ be an open subset of ∂M. we define for f, $\operatorname{supp} f \subseteq \Gamma$

$$\Lambda_{g,\Gamma}(f) = \Lambda_g(f)\big|_{\Gamma}.$$

The first result of [7] is:

Theorem 3.4 **($n \geq 3$)** *Let (M, g) be a real-analytic compact, connected Riemannian manifold with boundary. Let $\Gamma \subseteq \partial M$ be real-analytic and assume that g is real-analytic up to Γ. Then $(\Lambda_{g,\Gamma}, \partial M)$ determines uniquely (M, g).*

Notice that Theorem 3.4 doesn't assume any condition on the topology of the manifold except for connectedness. An earlier result of [7] assumed that (M, g) was strongly convex and simply connected and $\Gamma = \partial M$.

The second result of [8] is the proof of Conjecture B assuming we only measure the DN map on an open subset of the boundary.

Theorem 3.5 **($n = 2$)** *Let (M, g) be a compact Riemannian surface with boundary. let $\Gamma \subseteq \partial M$ be an open subset. Then $(\Lambda_{g,\Gamma}, \partial M)$ determines uniquely the conformal class of (M, g).*

Sketch of proof of Theorems 3.4 and 3.5. We'll sketch the proof of Theorem 3.5. Theorem 3.4 follows along similar lines. Using Lemma 3.1 we know that Λ_g determines $g\big|_{\partial M}$.

We add to M a collar neighborhood to construct $\widetilde{M} = M \cup (\partial M \times [0, 1])$ with the metric given on $\partial M \times [0, 1]$ by

$$g\big|_{\partial M\times[0,1]} = g\big|_{\partial M} + ds^2.$$

With this definition $g \in C^{0,1}(\widetilde{M})$. The Green's kernel is defined by

$$\Delta_g h_y = \delta_y \text{ in } \widetilde{M}$$
$$h_y\big|_{\partial\widetilde{M}} = 0$$

It is proven in [M-U] that the DN map determines the Green's functions in the collared neighborhood. More precisely we have:

Lemma 3.6 *Λ_g determines $h_y(x)$, $x, y \in \widetilde{M} - M$.*

In two dimensions there are special coordinates that change any Riemannian metric to a conformal multiple of the Euclidean one. These are called isothermal coordinates ([A]). Given any point $x \in M$, there exists a coordinate system (U, ϕ), $\phi : U \subseteq M \to \mathbb{R}^2$ so that

$$g \circ \phi^{-1} = \alpha(x)(\delta_{ij}) \tag{4}$$

that is, the metric g is isotropic in these coordinates.

Let V be an open neighborhood of $\partial\widetilde{M}$ so that $\partial\widetilde{M} \subseteq V \subseteq \widetilde{M}$. A fundamental step in the proof is to show the following result that states, roughly speaking that we can use the Green's functions based on points of V as coordinates.

Lemma 3.7 *Given any point $x \in \overline{M}$, there exists a neighborhood U of x and points $y_1, y_2 \in V$ so that $H_{y_1,y_2} = (h_{y_1}, h_{y_2})$ form a coordinate system on U.*

The next observation is that H_{y_1,y_2} are real-analytic in isothermal coordinates on M. These follow since the Laplacian in two dimensions is conformally invariant and therefore

$$\Delta h_y \circ \phi^{-1} = 0 \text{ on } \phi(U) \text{ if } y \in \widetilde{M} - M$$

and harmonic functions are real-analytic. Let us take a point $x \in \overline{M}$. Then we find a coordinate system (U, ϕ) near x and $y_1, y_2 \in V$ so that H_{y_1,y_2} is real-analytic in these coordinates.

Now we continue analytically $h_y, y \in V$ in these coordinates as much as possible. When this is no longer possible we use Lemma 3.7 to find new points $\widetilde{y}_1, \widetilde{y}_2 \in V$ so that $H_{\widetilde{y}_1,\widetilde{y}_2}$ is a system of coordinates and we continue this analytic continuation process again. This is done in [8] using the theory of sheaves. Let $\mathcal{A}$ be the sheaf of sequences of real-analytic maps. We define an equivalent class $\mathcal{B}$ in this sheaf by identifying elements that are obtained from each other by using real-analytic diffeomorphisms. Let $p \in \mathcal{B}$ be the element corresponding to the germs of the Green's kernel at a point $x \in \widetilde{M} - M$. The isometric copy of the manifold (M, g) is constructed by taking the path connected component of $\mathcal{B}$ containing the point p.

As for Conjecture C the only known result is the case when $M = \Omega$ is an open subset of $\mathbb{R}^n$ with smooth boundary and $g = (\delta_{ij}) =: e$ is the Euclidean metric. More precisely we have

Theorem 3.8 $(n = 2)$ *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with smooth boundary. Let γ_1, γ_2 be two anisotropic conductivities so that*

$$\Lambda_{e,\gamma_1} = \Lambda_{e,\gamma_2}.$$

Then there exists $\psi : \overline{\Omega} \to \overline{\Omega}$ diffeomorphism $\psi\big|_{\partial\Omega} = \text{Identity}$ so that

$$\psi_* \gamma_1 = \gamma_2.$$

The proof of Theorem 3.8 is a combination of the results of [9] and [10]. In [9] it was proven Theorem 3.8 for isotropic conductivities. Then one uses the results of [10] to reduce the anisotropic case to the isotropic one by using the analog of isothermal coordinates in this case. The result is that given an anisotropic conductivity, we can find a diffeomorphism ϕ so that $\phi_*\gamma$ is isotropic. We end by mentioning that the result of [9] uses the complex geometrical solutions, (3) for all complex frequencies $\rho \in \mathbb{C}^n - 0$, $\rho \cdot \rho = 0$ (not just large frequencies). For another construction of these solutions which allow Lipschitz conductivities see [3] (the result of [9] works for C^2 conductivities). Theorem 3.8 has been extended to anisotropic non-linear conductivities in [11].

References

[1] L. Ahlfors, *Quasiconformal maps*, Van Nostrand (1966).

[2] D. C. Barber and B. H. Brown, *Applied potential tomography*, J. Physics E **17** (1984), 723-733.

[3] R. Brown R. and G. Uhlmann, *Uniqueness in the inverse conductivity problem with less regular conductivities in two dimensions*, Comm. PDE **22**(1997), 1009-10027.

[4] A. P. Calderón, *On an inverse boundary-value problem*, Seminar on numerical analysis and its applications to continuum physics, Soc. Brasileira de Matemática, Río de Janeiro (1980), 65-73.

[5] *Harmonic Analysis and Partial Differential Equations*, Essays in Honor of Alberto P. Calderón, University of Chicago Press (1999), M. Christ, C. Kenig and C. Sadosky editors.

[6] R. Kohn and M. Vogelius, *Identification of an unknown conductivity by means of measurements at the boundary*, in Inverse Problems, edited by D. McLaughlin, SIAM-AMS Proceedings **14**, (1984), 113-123.

[7] J. Lee and G. Uhlmann, *Determining anisotropic real-analytic conductivities by boundary measurements*, Comm. Pure Appl. Math. **42** (1989), 1097-1112.

[8] M. Lassas and G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, to appear Annales Scientifiques de L'École Normale Supérieure.

[9] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Annals of Math. **143**(1996), 71-96.

[10] J. Sylvester, *An anisotropic inverse boundary-value problem*, Comm. Pure Appl. Math. **43**(1990), 201-232.

[11] Z. Sun and G. Uhlmann, *Inverse problems in quasilinear anisotropic media*, Amer. J. Math. **119**(1997), 771-797.

[12] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary-value problem*, Ann. of Math. **125**(1987), 153-169.

[13] G. Uhlmann, *Inverse boundary value problems for partial differential equations*, Proceedings of the International Congress of Mathematicians, Berlin (1998), Documenta Mathematica Vol. III, 77-86.

[14] G. Uhlmann, *Developments in inverse problems since Calderón's foundational paper*, Chapter 19 in "Harmonic Analysis and Partial Differential Equations", University of Chicago Press (1999), 295-345, edited by M. Christ, C. Kenig and C. Sadosky.

[15] M. S. Zhdanov and G. V. Keller, *The Geoelectrical Methods in Geophysical exploration*, Methods in Geochemistry and Geophysics **31**, Elsevier (1994).

Gunther Uhlmann
Department of Mathematics
University of Washington
Box 354350
Seattle, WA 98195, USA
e-mail: gunther@math.washington.edu