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On the Opial-Olech-Beesack inequalities *

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Abstract

We investigate two integral inequalities. The first of these generalizes a result proved by Beesack [1] in 1962. We then use our inequality to generalize earlier results of Olech [3] and Opial [4] on a related problem.

1 Introduction.

In 1962 Beesack [1] proved the integral inequality

Theorem 1.1 *Let $b > 0$. If $y(x)$ is real, continuously differentiable on $[0, b]$, and $y(0) = 0$, then*

$$\int_0^b |y(x)y'(x)|dx \leq \frac{b}{2} \int_0^b |y'(x)|^2 \, dx \,. \tag{1}$$

Equality holds only for $y = mx$ where m is a constant.

Beesack used this result to obtain a simplification of proofs given earlier by Olech [3] and Opial[4] of the inequality

Theorem 1.2 *Let $c > 0$, and let $y(x)$ be real, continuously differentiable on $[0, c]$, with $y(0) = y(c) = 0$. Then*

$$\int_0^c |y(x)y'(x)|dx \leq \frac{c}{4} \int_0^c |y'(x)|^2 dx. \tag{2}$$

Equality holds for the function satisfying $y = x$ on $[0, \frac{c}{2}]$, and $y = c - x$ on $[\frac{c}{2}, c]$.

In 1964 Levinson [2] gave a simpler proof of Theorem 1.1 His proof generalizes to the class of functions which are complex valued.

In this paper we have two goals. The first of these is to extend Levinson's arguments and generalize Beesack's inequality. This is done below in Theorem1.3. Our second goal is to use the results of Theorem1.3 and obtain a generalization of Theorem1.2. This is done in Theorem1.4.

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Theorem 1.3 *Let $p > -1$. (i) Let a and b be real with $0 \le a < b$. If $y(x)$ is continuously differentiable on $[a, b]$, and $y(a) = 0$, then*

$$\int_a^b t^p |y(t)y'(t)|dt \le \frac{1}{2\sqrt{p+1}} \int_a^b (b^{p+1} - at^p)|y'(t)|^2 dt. \tag{3}$$

(ii) Let b and c be real with $0 \le b < c$. If $y(x)$ is continuously differentiable on $[b, c]$, and $y(c) = 0$, then

$$\int_b^c t^p |y(t)y'(t)|dt \le \frac{1}{2\sqrt{p+1}} \int_b^c (ct^p - b^{p+1})|y'(t)|^2 dt. \tag{4}$$

Remarks:

(R1) If $a = 0$ then (3) reduces to

$$\int_0^b t^p |y(t)y'(t)|dt \le \frac{b^{p+1}}{2\sqrt{p+1}} \int_0^b |y'(t)|^2 dt. \tag{5}$$

(R2) It remains an open problem to determine the sharpness of (3), (4) and (5).

We now state our second result which is a generalization of the inequality of Olech [3] and Opial [4] stated above in Theorem1.2.

Theorem 1.4 *Let $p > -1$ and $c > 0$. If $y(x)$ is continuously differentiable on $[0, c]$, and $y(0) = y(c) = 0$, then*

$$\int_0^c t^p |y(t)y'(t)|dt \le \frac{c^{p+1}}{4\sqrt{p+1}} \int_0^c |y'(t)|^2 dt + \frac{c}{2\sqrt{p+1}} \int_{c_p}^c (t^p - c^p)|y'(t)|^2 dt, \tag{6}$$

where $c_p = \frac{c}{2^{1/(p+1)}}$.

Remarks:

(R3) If $p = 0$ then (6) reduces to (2).

(R4) It remains an open problem to determine the sharpness of (6).

2 Proof of Theorem1.3

(i) We begin by defining the integral

$$I_1 = \int_a^b t^p |y(t)y'(t)|dt.$$

Then I_1 can be written in the form

$$I_1 = \int_a^b (t^{\frac{p}{2}}(t-a)^{\frac{1}{2}}|y'(t)|)(t^{\frac{p}{2}}(t-a)^{\frac{-1}{2}}|y(t)|)dt,$$

and an application of the Schwarz inequality leads to

$$I_1 \le I_2^{1/2} I_3^{1/2}, \tag{7}$$

where

$$I_2 = \int_a^b t^p (t-a)|y'(t)|^2 dt \tag{8}$$

and

$$I_3 = \int_a^b t^p (t-a)^{-1} |y(t)|^2 dt\,. \tag{9}$$

Because $y(a) = 0$ and $y(x)$ is continuously differentiable on $[a, b], y(x)$ satisfies

$$|y(t)|^2 = \left| \int_a^t y'(\eta) d\eta \right|^2, \qquad a \le t \le b. \tag{10}$$

A further application of Schwarz's inequality to (10) gives

$$|y(t)|^2 \le (t-a) \int_a^t |y'(\eta)|^2 d\eta, \qquad a \le t \le b. \tag{11}$$

Combining (9)and (11), we obtain

$$I_3 \le \int_a^b t^p \int_a^t |y'(\eta)|^2 d\eta dt. \tag{12}$$

Reversing the order of integration in (12) gives

$$I_3 \le \frac{b^{p+1}}{p+1} \int_a^b |y'(t)|^2 dt - \frac{1}{p+1} \int_a^b t^{p+1} |y'(t)|^2 dt. \tag{13}$$

Next, we recall a well known result: if $A \ge 0, B \ge 0$ and $\lambda > 0$, then

$$(AB)^{1/2} \le \frac{\lambda}{2} A + \frac{1}{2\lambda} B. \tag{14}$$

We now combine (7), (8), (13) and (14), to obtain

$$I_1 \le \frac{1}{2}\left(\lambda - \frac{1}{\lambda(p+1)}\right) \int_a^b t^{p+1} |y'(t)|^2 dt + \int_a^b \left(\frac{b^{p+1}}{2\lambda(p+1)} - \frac{a\lambda t^p}{2} \right) |y'(t)|^2 dt, \tag{15}$$

where λ is any positive number. Setting $\lambda = \frac{1}{\sqrt{p+1}}$ in (15), we obtain (3). This completes the proof of part (i).

(ii) The proof of part (ii) follows the method used above. We give the details for the sake of completeness. Thus, we define

$$I_4 = \int_b^c t^p |y(t) y'(t)| dt.$$

Then I_4 can be written in the form

$$I_4 = \int_b^c (t^{\frac{p}{2}}(c-t)^{\frac{1}{2}}|y'(t)|)(t^{\frac{p}{2}}(c-t)^{\frac{-1}{2}}|y(t)|)dt,$$

and once again an application of the Schwarz inequality leads to

$$I_4 \leq I_5^{1/2} I_6^{1/2}, \tag{16}$$

where

$$I_5 = \int_b^c t^p(c-t)|y'(t)|^2 dt \qquad \textit{and} \qquad I_6 = \int_b^c t^p(c-t)^{-1}|y(t)|^2 dt. \tag{17}$$

Because $y(c) = 0$ and $y(x)$ is continuously differentiable on $[b, c], y(x)$ satisfies

$$|y(t)|^2 = \left| \int_t^c y'(\eta) d\eta \right|^2, \qquad b \leq t \leq c. \tag{18}$$

An application of Schwarz's inequality to (18) gives

$$|y(t)|^2 \leq (c-t) \int_t^c |y'(\eta)|^2 d\eta, \qquad b \leq t \leq c. \tag{19}$$

Combining (17) and (19), we obtain

$$I_6 \leq \int_b^c t^p \int_t^c |y'(\eta)|^2 d\eta dt. \tag{20}$$

Reversing the order of integration in (20) leads to

$$I_6 \leq \frac{1}{p+1} \int_b^c (t^{p+1} - b^{p+1})|y'(t)|^2 dt. \tag{21}$$

Next, we combine (16), (17) and (21), and apply (14) to arrive at

$$I_4 \leq \frac{1}{2}\left(\frac{1}{\lambda(p+1)} - \lambda\right) \int_b^c t^{p+1}|y'(t)|^2 dt + \int_b^c \left(\frac{c\lambda t^p}{2} - \frac{b^{p+1}}{2\lambda(p+1)}\right)|y'(t)|^2 dt, \tag{22}$$

where λ is any positive number. Setting $\lambda = \frac{1}{\sqrt{p+1}}$ in (22), we obtain (4). This completes the proof of part (ii).

3 Proof of Theorem1.4

Let b be any positive number satisfying $0 < b < c$. Then, from (3) and (4) we obtain

$$\begin{aligned} \int_0^c t^p|y(t)y'(t)|dt &= \int_0^b t^p|y(t)y'(t)|dt + \int_b^c t^p|y(t)y'(t)|dt \\ &\leq \frac{b^{p+1}}{2\sqrt{p+1}} \int_0^b |y'(t)|^2 dt + \frac{1}{2\sqrt{p+1}} \int_b^c (ct^p - b^{p+1})|y'(t)|^2 dt \end{aligned}$$

This inequality can now be written in the form

$$\int_0^c t^p|y(t)y'(t)|dt = \frac{b^{p+1}}{2\sqrt{p+1}}\int_0^c |y'(t)|^2dt + \frac{1}{2\sqrt{p+1}}\int_b^c (ct^p - 2b^{p+1})|y'(t)|^2dt \tag{23}$$

Setting $b = c_p = c/2^{1/(p+1)}$ in (23), we obtain (6) and Theorem1.4 is proved.

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