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Nontrivial solutions for noncooperative elliptic systems at resonance *

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Abstract

In this article we establish the existence of a nonzero solution for variational noncooperative elliptic systems under Dirichlet boundary conditions and a resonant condition at infinity. Situations where the problem is nonresonant and resonant at the origin are considered. The results are based on a version of a critical point theorem for strongly indefinite functionals which are asymptotically quadratic at infinity and do not satisfy any Palais-Smale type condition.

1 Introduction

In this article we consider the existence of a nonzero solution for the variational noncooperative elliptic system

$$\begin{aligned} -\Delta u &= au - bv + f(x,u,v) =: \quad F_u(x,u,v), \quad \text{in } \Omega, \\ -\Delta v &= bu + cv - g(x,u,v) =: -F_v(x,u,v), \quad \text{in } \Omega, \\ & u = v = 0, \quad \text{on } \partial\Omega\,, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, the numbers a, b, c are real parameters and the nonlinearities $f, g : \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ satisfy $f(x, 0) \equiv 0$, $g(x, 0) \equiv 0$. To apply the infinite dimensional Morse theory to the functional associated with (1.1), we assume that $F : \bar{\Omega} \times \mathbb{R}^2 \to \mathbb{R}$ is of class C^2 and satisfies the growth condition

$$|D^2F(x,z)| \le c_1|z|^{\sigma-2} + c_2,\ \forall\ z \in \mathbb{R}^2,\ x \in \Omega,$$

for constants $c_1, c_2 > 0$ and $\sigma > 2$ ($\sigma < \frac{2N}{N-2}$, if $N \ge 3$). In this paper we represent by $\nabla F(x,z)$ and $D^2F(x,z)$, respectively, the gradient of F and the second derivative of F with respect to the variable $z \in \mathbb{R}^2$.

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We observe that there exists a vast literature on the use of nonlinear methods to the study of elliptic systems. We refer the readers to the works by Figueiredo-Mitidieri [13], Lazer-Mckenna [20], Silva [30], Costa-Magalhães [10, 11], Felmer-Figueiredo [12], Hulshof-van der Vorst [16], Kryszewski-Szulkin [18] and references therein.

One of our main goals in this article is to illustrate how the ideas introduced in [31] can be applied to handle the problem of existence of a nonzero solution for the system (1.1) under a resonant condition at infinity [19, 26, 11]. More specifically, here we assume

(F_1) $\lim_{|z|\to\infty} \frac{|(f,g)(x,z)|}{|z|} = 0$, for a. e. $x \in \Omega$, $|(f,g)(x,z)| \leq A(x)|z| + B(x)$, for all $z \in \mathbb{R}^2$, for a. e. x in Ω, where $B \in L^1(\Omega)$ and $A \in L^{p_1}(\Omega)$, $1 < p_1 < \infty$ ($p_1 = N/2$, if $N \geq 3$)),

We also assume that the associated linear system

$$\begin{aligned} -\Delta u &= au - bv, \quad \text{in } \Omega, \\ -\Delta v &= bu + cv, \quad \text{in } \Omega, \\ u &= v = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

possesses a nontrivial solution.

In this work the existence of a nonzero solution for (1.1) is based on the relation between the index of the second derivative of the associated functional at zero and at points away from zero and on an appropriate region containing the space of solutions of (1.2). In order to obtain such estimates, we impose conditions on the behavior of the nonlinearity at the origin and at infinity:

(F_2) $F(x,0) \equiv 0$, $\nabla F(x,0) \equiv 0$, and $D^2F(x,0) = RA_0$,

(F_3) $\liminf_{|z|\to\infty} D^2F(x,z) > RA_1$, for a.e. $x \in \Omega$, $D^2F(x,z) \geq -C(x)I_2$, for all z in $\mathbb{R}^2$, for a.e. x in Ω, where $C \in L^{p_2}(\Omega)$, $1 < p_2 < \infty$ ($p_2 = N/2$, if $N \geq 3$), I_2 is the identity 2×2 matrix, and

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_i = \begin{pmatrix} a_i & -b_i \\ b_i & c_i \end{pmatrix}, \ i = 0, 1.$$

Before stating our basic theorems, we recall some facts about the spectrum of the linear system (1.2) and introduce some notation.

Let $0 < \lambda_1 < \ldots \leq \lambda_j \leq \ldots$ be the sequence of eigenvalues of the operator $-\Delta$ on $H_0^1(\Omega)$. Denoting by A the anti-symmetric 2×2 matrix associated to (1.2), it can be proved (see [10]) that μ is an eigenvalue of the corresponding linear operator on $H_0^1(\Omega) \times H^1(\Omega)$ if, and only if, $\det(A + \mu R - \lambda_j I_2) = 0$. Therefore, the corresponding sequence of eigenvalues of (1.2), $\{\mu_j^{\pm}\}$, is given by

$$\mu_j^{\pm} = \frac{c-a}{2} \pm \sqrt{\left(\frac{c-a}{2}\right)^2 + \det(\lambda_j I_2 - A)}.$$

For a given $k \in \mathbb{N}$, we denote by $i_k(A)$ the number of negative eigenvalues of the set $\{\mu_{-k}, \ldots, \mu_k\}$ minus k. Observing that $\mu_j^{(\pm)} \to (\pm)\infty$, as $j \to \infty$, we conclude that $i_k(A)$ is constant for k sufficiently large. That allows us to define the relative index of A, denoted $i(A)$, by

$$i(A) = \lim_{k\to\infty} i_k(A).$$

By the same reasoning, we may define the nullity of A, represented by $n(A)$, as the number of eigenvalues of (1.2) which are zero. Note that $i(A)$ is the relative Morse index of the quadratic form associated to (1.2) on $H_0^1(\Omega) \times H_0^1(\Omega)$ [18]. Furthermore, the linear system (1.2) has a nontrivial solution if, and only if, $n(A) \neq 0$. We now state our first result:

Theorem 1.1 *Suppose $F \in C^2(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ satisfies $(F_1) - (F_3)$, with $n(A) \neq 0$ and $n(A_0) = 0$. Then the system (1.1) possesses a nonzero solution provided $i(A_1) > i(A_0) + 1$.*

We remark that $n(A_0) \neq 0$ if, and only if, the origin is a nondegenerate critical point of the associated functional. On that case the problem (1.1) is said to be resonant at the origin. To deal with such possibility we suppose the local condition:

(F_4) there exists $r > 0$ such that $F(x, z) \leq \frac{1}{2}\langle A_0 z, z\rangle$, for all $|z| \leq r$, for a.e. $x \in \Omega$.

The following theorem give us a version of Theorem 1.1 under the resonant condition at the origin.

Theorem 1.2 *Suppose $F \in C^2(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ satisfies $(F_1) - (F_3)$, with $n(A) \neq 0$ and $n(A_0) \neq 0$. Then the system (1.1) possesses a nonzero solution provided F satisfies (F_4) and $i(A_1) > i(A_0) + 1$.*

Assuming the following version of (F_4)

$(\hat{F}_4)$ there exists $r > 0$ such that $F(x, z) \geq \frac{1}{2}\langle A_0 z, z\rangle$,for all $|z| \leq r$, for a.e. $x \in \Omega$,

we obtain

Theorem 1.3 *Suppose $F \in C^2(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ satisfies (F_1)-(F_3), with $n(A) \neq 0$ and $n(A_0) \neq 0$. Then the system (1.1) possesses a nonzero solution provided F satisfies $(\hat{F}_4)$ and $i(A_1) > i(A_0) + n(A_0) + 1$.*

Theorems 1.1, 1.2 and 1.3 are related to earlier results by Costa-Magalhães [10, 11] and Kryszewski-Szulkin [18]. We note that, under conditions $(F_1) - (F_3)$, the associated functional may not satisfy any version of the Palais-Smale compactness condition [27, 4] on levels belonging to a very general subset of the real line. This fact does not allow us to apply the abstract results considered in [10, 11, 18]. We should also note that versions of the above theorems can be proved when we assume that $\limsup_{|z|\to\infty} D^2F(x,z)$ is bounded from above by RA_1 (see Remark 5.8).

As observed earlier, the proofs of Theorems 1.1, 1.2 and 1.3 are based on a critical point theorem, due to the author [31], for strongly indefinite functionals which are asymptotically quadratic at infinity. In this article, we establish a slightly improved version of that result. We believe that the proof given here is more clarifying than the one in [31].

We should observe that the method applied here, and in [31], for establishing critical point theorems when the functional does not satisfy a compactness condition is based on perturbation arguments, a construction associated with the existence of a local linking structure [22, 28, 29], and the estimates established by Lazer-Solimini [21, 33] for the Morse index of a functional at a critical point associated to a given minimax level.

It is worthwhile mentioning that the above method is related to the methods used by Masiello-Pisane [25], Hirano-Li-Wang [15] and Li-Wang [23] to study the scalar problem under strongly resonant conditions at infinity. Note that the conditions $(F_1)-(F_3)$ include such class of problems and that we are assuming pointwise limits in conditions (F_1) and (F_3). Finally, we should remark that corresponding results for periodic solutions of asymptotically linear Hamiltonian systems have been derived in [5, 32, 9, 1].

This article has the following structure: in section 2, we state the main critical point theorem. In section 3, after presenting some preliminary results, we prove an abstract theorem that gives us an estimate for the Morse index of the the functional at a critical point belonging to a level $c \neq 0$. The main critical point theorem is proved in section 4 by applying the theorem of section 3 and a perturbation argument. Finally, we reserve the section 5 for the proofs of Theorems 1.1, 1.2 and 1.3.

2 A critical point theorem

Let H be a real separable Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and let $I : H \to \mathbb{R}$ be a functional of class C^2 having the origin as a critical point. Our goal in this section is to provide sufficient conditions for the existence of a nonzero critical point for the functional when I is of the form

$$I(u) = \frac{1}{2}\langle Lu, u\rangle + J(u), \tag{2.1}$$

where L is a bounded self-adjoint linear operator and J is a functional of class C^2 satisfying, respectively,

(I_1) the number zero is an isolated point of the spectrum of L with finite multiplicity,

(I_2) $J' : H \to H$ is compact and $\lim_{\|u\|\to\infty} \frac{\|J'(u)\|}{\|u\|} = 0$.

Note that the condition (I_2) implies that I is asymptotically quadratic at infinity, and the condition (I_1) says that the associated quadratic form is degenerate.

Invoking the spectral theory for self-adjoint operators, we may write $H = H^+ \oplus H^0 \oplus H^-$, where H^+, H^-, H^0 are the orthogonal closed subspaces of H on which L is strictly positive definite, strictly negative definite and null, respectively. Furthermore, H^0 is a nontrivial finite dimensional subspace of H. We also recall that the index of L, denoted ind(L), is the dimension of the subspace H^-.

Here, we are interested in the situation where I is strongly indefinite in the sense that both subspaces H^+ and H^- have infinite dimensions. This is indeed the case for the noncooperative elliptic system considered in this article.

In order to apply a Galerking type argument in our setting, we assume the existence of a family of closed subspaces $H_k = H_k^+ \oplus H^0 \oplus H_k^-$ of H such that $H_1^{\pm} \subset \ldots \subset H_k^{\pm} \subset \ldots \subset H^{\pm}$. If $\dim(H^{\pm}) < \infty$, we set $H_k^{\pm} = H^{\pm}$ for every $k \in \mathbb{N}$. We also suppose the existence of a basis $\{e_j \mid j \in J \subset \mathbb{N}\}$ of H such that, for every $j \in J$, there exists $k_j \in \mathbb{N}$ so that $\{e_1, \ldots e_j\} \subset H_{k_j}$. In the following we denote by I_k the restriction of the functional I to the subspace H_k.

We now recall the versions of the Palais-Smale condition associated to the family (H_k) [3, 22, 28, 29]:

Definition 2.1 *(i) Given $c \in \mathbb{R}$, we say that $(u_j) \subset H$ is a $(PS)_c^*$-sequence, for $c \in \mathbb{R}$, if $I(u_j) \to c$, as $j \to \infty$, and there exists $(k_j) \subset \mathbb{N}$, with $k_j \to \infty$, such that $u_j \subset H_{k_j}$, for every $j \in \mathbb{N}$, and $\|I'_{k_j}(u_j)\| \to 0$, as $j \to \infty$.*
(ii) we say that $I \in C^1(H, \mathbb{R})$ satisfies the [$(PSB)_c^*$] $(PS)_c^*$ condition if every [bounded] $(PS)_c^*$-sequence possesses a convergent subsequence.
(iii) If I satisfies [$(PSB)_c^*$] $(PS)_c^*$ on every level $c \in \mathbb{R}$, we simply say that I satisfies [$(PSB)^*$] $(PS)^*$.

It is worthwhile mentioning that we may have $\dim H_k = \infty$, for every $k \in \mathbb{N}$. Furthermore, when $H_k = H$ for k sufficiently large, the Definition 2.1 provides exactly the definitions of the (PS) and the (PSB) conditions.

In this work we also suppose the following local condition [22]:

(I_3) there exist $\rho > 0$ and subspaces X_k^i, $i = 1, 2$, of H_k, for every $k \in \mathbb{N}$, such that $H_k = X_k^1 \oplus X_k^2$, $\dim X_k^1 < \infty$, and
(i) $I(u) \geq 0, \ \forall\, u \in X_k^2 \cap B_\rho(0)$,
(ii) $I(u) \leq 0, \ \forall\, u \in X_k^1 \cap B_\rho(0)$.

We observe that the above generalization of the local linking condition has been introduced in [28] (see also [29, 7]). Note that under the condition (I_3) the origin may be a degenerate critical point of I and the Morse index of the functional at that point may be infinite.

Given $\alpha > 0$, we set $C_\alpha = \{u \in H : \|u\| \leq (1+\alpha)\|P_0 u\|\}$, where P_0 denotes the orthogonal projection of H onto $\ker(L) = H^0$. The following conditions allow us to compare the Morse index of the functional at the origin with the indexes of L and of $D^2 I(u)$, for $u \in C_\alpha$, with $\|u\|$ sufficiently large.

(I_4) $\dim X_k^1 + 1 \leq \dim(H_k^- \oplus H^0)$, for every $k \in \mathbb{N}$,

(I_5) there exist $\alpha, \mu > 0$ and $M > 0$ such that $\dim X_k^1 + 1 < \text{ind}(D^2 I_k(u) + \mu id)$, for every $u \in H_k \cap C_\alpha$, $\|u\| \geq M$.

Condition (I_5) is slightly weaker than the corresponding condition assumed in [31]. Now, we state our main abstract theorem.

Theorem 2.2 *Let H be a real separable Hilbert space. Suppose $I \in C^2(H, \mathbb{R})$ satisfies $(I_1)-(I_5)$ and $(PSB)^*$. Then the functional I possesses a critical point other than zero.*

The proof of Theorem 2.2 is based on another abstract result which is proved in section 3. In that section we suppose that L is an isomorphism and that $H_k = H$, for every $k \in \mathbb{N}$. We also assume that the condition (I_3) has some special structure (see Definition 3.2). Under those assumptions, we establish a critical point theorem that relates the Morse index at a nonzero critical point of the functional with the dimension of $X_k^1 \equiv X^1$. The Theorem 2.2 is proved in section 4 by applying the theorem of section 3 to an appropriate perturbation of the functional I_k, for k sufficiently large.

3 A version of Theorem 2.2

In this section we suppose that $H_k = H$ for every $k \in \mathbb{N}$. Under this hypothesis and supposing that L, given by (2.1), is an isomorphism, we are able to provide an estimate for the Morse index of the functional at a critical point associated to a level $c \neq 0$.

In this article we represent by $K(I)$ the set of critical points of I. Given $c \in \mathbb{R}$, we set $I^c = \{u \in H \,:\, I(u) \leq c\}$, $K_c(I) = \{u \in H \,:\, I(u) = c,\, I'(u) = 0\}$. We also set $K_a^b(I) = \cup_{c=a}^b K_c(I)$, for every $a < b \in \mathbb{R}$.

Denoting by $\mathcal{S}$ the family of continuous map $\Phi \in C([0,1] \times H, H)$ such that $\Phi(0, \cdot) = id$ and considering the subsets S and Q of H, we say that S and ∂Q link if $\Phi(t, Q) \cap S \neq \emptyset$, for every $0 \leq t \leq 1$, whenever $\Phi \in \mathcal{S}$ and $\Phi([0,1] \times \partial Q) \cap S = \emptyset$ [6, 27]. The following characterization of the link was introduced in [28] (see also [29, 7]):

Definition 3.1 *Given $I \in C^1(H, \mathbb{R})$, we say that the link between S and ∂Q is of deformation type (with respect to I) if there exist $\gamma' \leq \gamma$ and $\Phi \in \mathcal{S}$ such that*

(L_1) $\Phi(t, \partial Q) \cap S = \emptyset$, *for all* $t \in [0, 1]$,

(L_2) $\Phi(1, \partial Q) \subset I^{\gamma'}$,

(L_3) $I(u) > \gamma$, *for all* $u \in S$.

We now introduce the corresponding notion when the functional I satisfies the local condition (I_3).

Definition 3.2 *We say that the condition* (I_3) *is a local linking of deformation type (with respect to* I*) if there exist* $0 < c_1 < c$, $0 < d < r < \rho$, *a homeomorphism* $\psi : H \to H$ *and a continuous map* $\eta : [0,1] \times \partial B_\rho(0) \cap X^1 \to H$ *satisfying*

(d_1) $\psi(u) = u$, *for all* $u \in (H \setminus B_r(0)) \cup X^1$,

(d_2) $I(\psi(u)) \geq c > 0$, *for all* u $\partial B_d(0) \cap X^2$,

(d_3) $\eta(0,u) = u$, *for all* $u \in \partial B_\rho(0) \cap X^1$,

(d_4) $I(\eta(t,u)) \leq -ct$, *for all* $(t,u) \in [0,1] \times \partial B_\rho(0) \cap X^1$,

(d_5) $r < \|\eta(t,u)\| < 2\rho$, *for all* $(t,u) \in [0,1] \times \partial B_\rho(0) \cap X^1$,

(d_6) $I(u) > -c_1$, *for all* $u \in B_r(0)$.

Remark 3.3 *If the functional* I *has the origin as an isolated critical point and satisfies the* (PSB) *condition, we may apply the local deformation lemmas proved in [28, 29] to conclude that the condition* (I_3) *is a local linking of deformation type.*

Representing by $m(I,u)$ the Morse index of $I \in C^2(H,\mathbb{R})$ at the critical point $u \in H$, we may state the deformation lemma:

Lemma 3.4 *Let* $\phi : A \to I^b$ *be a continuous map, with* $A \subset \mathbb{R}^n$ *compact. Suppose* $I \in C^2(H,\mathbb{R})$ *satisfies* $(PS)_c$ *for every* $c \in [a,b]$. *Assume that* $K_a^b(I) = \{u_j\}_{j=1}^m$, *with* u_j *non-degenerate and* $m(I,u_j) > n$, *for every* $1 \leq j \leq m$. *Then, there exists a continuous map* $\tau : [0,1] \times A \to I^b$ *such that*

(τ_1) $\tau(0,u) = \phi(u)$, *for all* $u \in A$,

(τ_2) $\tau(t,u) = \phi(u)$, *for all* $t \in [0,1]$, *if* $\phi(u) \in I^a$,

(τ_3) $\tau(1,A) \subset I^a$.

Remark 3.5 *The Lemma 3.4 is proved by applying the second deformation lemma [8] combined with a local deformation based on the Morse Lemma for nondegenerate critical points (see Lemma 2.9 in [31]). As observed in [31] (see also Lemma 3.10), the above result and the perturbation argument due to Marino-Prodi [24, 21] provide some useful estimates for the Morse index at a nontrivial critical point for functionals satisfying the local linking condition* (I_3).

Assuming the version of the condition (I_1)

$(\hat{I}_1)$ L is an isomorphism,

we state the main result of the section:

Theorem 3.6 *Let H be a real Hilbert space. Suppose $I \in C^2(H,\mathbb{R})$ satisfies $(\hat{I}_1)$, (I_2), (I_3) and (I_4), with $H_k = H$, for every $k \in \mathbb{N}$, and (I_3) a local linking of deformation type. Then I possesses a critical point $u \in H$ such that either $I(u) \leq -c$ and $m(I,u) \leq \dim X^1$, or $I(u) \geq c$ and $m(I,u) \leq \dim X^1 + 1$, where $c > 0$ is given by Definition 3.2.*

Remark 3.7 *(i) The conditions $(\hat{I}_1)$ and (I_2) imply that $K(I)$ is compact. Consequently, under those hypotheses, the functional satisfies the Palais-Smale condition. (ii) If $X^1 = \{0\}$, we may apply the Mountain Pass Theorem [2] and the corresponding Morse index estimate [8] to obtain a critical point u of I such that $I(u) \geq c$ and $m(I,u) \geq 1$. (iii) If $\dim X^2 = 1$, we use (I_4) and $(\hat{I}_1)$ to conclude that $H = H^-$ and that I has a critical point u such that $I(u) = \max\{I(v) \,|\, v \in H\} \geq c$. Clearly, $m(I,u) \leq \dim X^1 + 1$.*

Proof of Theorem 3.6: Supposing that $m(I,u) > \dim X^1$ for every critical point $u \in H$ such that $I(u) \leq -c$, we prove the Theorem 3.6 by verifying that I has a critical point u_0 such that $I(u_0) \geq c$ and $m(I,u_0) \leq \dim X^1 + 1$. That verification is lengthy so we will sketch it first. We start by showing the existence of a functional which is quadratic away from zero and has the same set of critical points than I. Furthermore, that functional will be equal to I on a ball containing $K(I)$. Using a perturbation argument due to Marino-Prodi and Lemma 3.4, we verify the existence of a linking of deformation type and of a critical value $b \geq c$. We conclude the proof of Theorem 3.6 by applying the Morse index estimates for the minimax critical points provided by Lazer-Solimini [21, 33].

By Remark 3.7, we may assume without loss of generality that $X^1 \neq \{0\}$ and $\dim(X^2) \geq 2$. Considering

$(\hat{I}_2)$ I is bounded on bounded sets, and there exists $R_0 > 0$ such that $J(u) = 0$, for every $u \in H$ such that $\|u\| \geq R_0$,

The next lemma provides the first step for the proof of Theorem 3.6 (see [31]).

Lemma 3.8 *Suppose $I \in C^2(H,\mathbb{R})$ is given by (2.1) and satisfies $(\hat{I}_1)$ and $(I_2)-(I_4)$, with (I_3) a local linking of deformation type. Let $R_1 > 2\rho$, with $\rho > 0$ given by (I_3), be such that $K(I) \subset \mathrm{int}(B_{R_1}(0))$. Then there exists $I_1 \in C^2(H,\mathbb{R})$ satisfying (PS), $(\hat{I}_1)$, $(\hat{I}_2)$, (I_3), (I_4), and*

$$I_1(u) = I(u), \ \forall\, u \in H, \ \|u\| \leq R_1. \tag{3.1}$$

Furthermore, $K(I_1) = K(I)$.

Remark 3.9 *(i) Since $R_1 > 2\rho$, (I_3) is a local linking of deformation type with respect to I_1. Moreover, the constants $\{c_1, c, d, r, \rho\}$, given in Definition 3.2, are the same for I and I_1. (ii) By lemma 3.8, to prove the Theorem 3.6 it suffices to verify that I_1 has a critical point u_0 such that $I_1(u_0) \geq c$ and $m(I_1, u_0) \leq \dim X^1 + 1$.*

Let I_1 be the functional given by Lemma 3.8. Considering $r > 0$ given by Definition 3.2 and R_0 given in $(\hat{I}_2)$, we fix $R > R_0$ such that

$$\{u = v_1 + v_2,\, v_i \in X^i, i = 1,2, : \frac{\|v_i\|}{\rho} + \frac{\|v_2\|}{R} = 1\} \cap B_r(0) = \emptyset. \tag{3.2}$$

We now take $e \in H^- \cap X^2$, $\|e\| = 1$, and we define $Q = Q(R) \subset \mathbb{R}e \oplus X^1$ by

$$Q = \{u = se + v\ : v \in X^1 s \geq 0,\, \frac{s}{R} + \frac{\|v\|}{\rho} \leq 1\}.$$

Writing $u = u^+ + u^-$, $u^i \in H^i$, $i = +, -$, and considering that I_1 satisfies $(\hat{I}_1)$ and $(\hat{I}_2)$, we find $a < -c$ such that

$$I(u) > a,\ \forall\ u = u^+ + u^-,\ \|u^-\| \leq R. \tag{3.3}$$

Lemma 3.10 *Let $a \in \mathbb{R}$ be given by (3.3). Suppose $m(I,u) > \dim X^1$ for every $u \in K(I) \cap I^{-c}$. Then there exists a continuous map $\eta_1 : [0,1] \times \partial B_\rho(0) \to H$ satisfying*

(η_1) $\eta_1(0,u) = u$, *for all* $u \in \partial B_\rho(0) \cap X^1$,

(η_2) $I_1(\eta_1(t,u)) \leq 0$, *for all* $(t,u) \in [0,1] \times \partial B_\rho(0) \cap X^1$,

(η_3) $\|\eta_1(t,u)\| > r$, *for all* $(t,u) \in [0,1] \times \partial B_\rho(0) \cap X^1$,

(η_4) $\eta_1(1, \partial B_\rho(0) \cap X^1) \subset I_1^a$,

(η_5) $\|\eta_1^-(1,u)\| \geq R$, *for all* $u \in \partial B_\rho(0) \cap X^1$.

Proof: First, we claim that there exists $b \in (c_1, c)$ such that $m(I_1,u) > \dim X^1$, for every $u \in K(I_1) \cap I_1^{-b}$. Effectively, assuming otherwise, by Lemma 3.8, we find $(u_m) \subset K(I)$ such that $I(u_m) \to -c$, as $m \to \infty$, and $m(I,u_m) \leq \dim X^1$, for every $m \in \mathbb{N}$. Since $K(I)$ is compact, we may suppose that $u_m \to u \in K(I)$, as $m \to \infty$. Hence, $I(u) = -c$ and $m(I,u) \leq \dim X^1$. This prove the claim.

Now, considering $0 < \epsilon < (c-b)/2$, we use the condition (I_2), $K(I_1) = K(I) \subset \operatorname{int} B_{R_1}(0)$, (3.1) and the perturbation method introduced by Marino-Prodi [24] to find $I \in C^2(H,\mathbb{R})$ such that $K(I_2) \cap I_2^{-c+\epsilon}$ is a finite set possessing only non-degenerate critical points,

$$\|I_2(u) - I_1(u)\| < \epsilon,\ \forall\ u \in H. \tag{3.4}$$

and

$$m(I_2,u) > \dim X^1,\ \forall\ u \in K(I_2) \cap I_2^{-c+\epsilon}. \tag{3.5}$$

Taking η given by Definition 3.2, from (d_4) and (3.4), we have that $\eta(1,.) : \partial B_\rho(0) \cap X^1 \to I_2^{-c+\epsilon}$. Hence, invoking (3.5) and Lemma 3.4, we find a continuous map $\tau : [0,1] \times \partial B_\rho(0) \cap X^1 \to I_2^{-c+\epsilon}$ satisfying

$$\tau(0,u) = \eta(1,u),\ \forall\ u \in \partial B_\rho(0) \cap X^1, \tag{3.6}$$

$$\tau(1, \partial B_\rho(0) \cap X^1) \subset I_2^{a-\epsilon}. \tag{3.7}$$

Now, we define $\eta_1 : [0,1] \times \partial B_\rho(0) \cap X^1 \to H$ by

$$\eta_1(t,u) = \begin{cases} \eta(2t,u), & 0 \le t \le \frac{1}{2},\ u \in \partial B_\rho(0) \cap X^1, \\ \tau(2t-1, \eta(1,u)), & \frac{1}{2} < t \le 1,\ u \in \partial B_\rho(0) \cap X^1. \end{cases}$$

By (3.6), η_1 is a well defined continuous map. We shall verify that η_1 satisfies the conditions $(\eta_1) - (\eta_5)$. First, we note that (η_1) is a direct consequence of the definition of η_1 and (d_3). Considering that $-c + 2\epsilon < -c_1$, from (3.4), we get that

$$\tau([\frac{1}{2}, 1] \times \partial B_\rho(0) \cap X^1) \subset \text{int}(I_1^{-c_1}). \tag{3.8}$$

This fact and (d_4) show that η_1 satisfies (η_2). The relation (3.8), (d_5) and (d_6) imply that (η_3) holds. The condition (η_4) is a consequence of (3.4) and (3.7). Finally, we observe that (η_5) is implied by (η_4) and (3.3). The Lemma 3.10 is proved. ♢

Invoking (η_5), $R > R_0$ and $(\hat{I}_2)$, we obtain

$$I_1(s\eta_1^+(1,u) + \eta_1^-(1,u)) \le 0,\ \forall\ u \in \partial B_\rho(0) \cap X^1,\ 0 \le s \le 1.$$

Using the above relation, $(\eta_1) - (\eta_5)$ and the fact that $\dim(X^2) \ge 2$, by [28] (see also Lemma 1.25 in [29]), we have

Proposition 3.11 *There exists $\Phi \in \mathcal{S}$ satisfying*

$(\hat{L}_1)$ $\Phi([0,1] \times \partial Q) \subset \{u \in H \mid \|u\| \ge r\} \cup X^1$,

$(\hat{L}_2)$ $\Phi(1, \partial Q) \subset I_1^0$.

Conclusion of the proof of Theorem 3.6: From (3.2) and (d_1), we have that $S = \psi(\partial B_d(0) \cap X^2)$ and $\psi(\partial Q) = \partial Q$ since $0 < d < r$ and ψ is a homeomorphism. We claim that this link is of deformation type with respect to I_1. Effectively, taking $\Phi \in \mathcal{S}$ given by Proposition 3.11, we see easily that (L_2) and (L_3) hold with $\gamma \ge c > 0 = \gamma'$. Moreover, the condition (L_1) is a consequence of (d_1), $(\hat{L}_1)$ and $d < r$. The claim is proved.

As I_1 satisfies (PS), we may invoke [28, 29] to conclude that I_1 has a critical value $b \ge c > 0$ given by

$$b = \inf_{\Phi \in \Gamma} \max_{u \in Q} I_1(\Phi(1,u)), \tag{3.9}$$

where

$$\Gamma = \{\Phi \in \mathcal{S} \,:\, \Phi \text{ satisfies } (L_1) \text{ and } (L_2)\}. \tag{3.10}$$

Finally, considering that $Q \subset \mathbb{R}e \oplus X^1$ we may apply the Morse index estimates for minimax critical points given by [21, 33] to conclude that I_1 has a critical point $u_0 \in H$ such that $I_1(u_0) = b \ge c$ and $m(I_1, u_0) \le \dim X^1 + 1$. That concludes the proof of Theorem 3.6. ♢

4 Proof of Theorem 2.2

Arguing by contradiction, we suppose that $u = 0$ is the only critical point of I in H. First, we note that given $0 < r_1 < R_1$, from (I_1), (I_2) and $(PSB)^*$, there exist $\delta > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|I_k'(u)\| \geq \delta > 0, \ \forall \ r_1 \leq \|u\| \leq R_1, \ k \geq k_0. \tag{4.1}$$

The next lemma is a direct consequence of the local deformations lemmas proved in [28, 29] (see also Lemma 3.8 in [31]).

Lemma 4.1 *There exists $k_1 \in \mathbb{N}$ such that (I_3) is a local linking of deformation type with respect to I_k, for every $k \geq k_1$. Furthermore, the constants $\{c, c_1, r, d, \rho\}$ appearing in Definition 3.2 are independent of $k \geq k_1$.*

Consider $c, \rho > 0$ given by Lemma 4.1 and take $0 < \beta < c$. Since $I(0) = 0$, there exists $0 < r_1 < \rho$ so that

$$|I(u)| \leq \beta, \ \forall \ u \in B_{r_1}(0). \tag{4.2}$$

Fixing $R > 2\rho$ and using $(I_1) - (I_2)$ and $(PSB)^*$, we find $k_2 \geq k_1$ and $\hat{\delta} > 0$ such that, for every $k \geq k_2$,

$$\|I_k'(u)\| \geq \hat{\delta} > 0, \ \forall \ r_1 \leq \|u^0\| \leq R + 1. \tag{4.3}$$

Now, taking $\chi : \mathbb{R} \to [0, 1]$ of class C^∞ such that $\chi(s) = 0$, if $s \leq 0$, and $\chi(s) = 1$, if $s \geq 1$, we set $\chi_R(s) = \chi(s - R)$ and define $I_{k,\epsilon} \in C^2(H, \mathbb{R})$ by

$$I_{k,\epsilon} = I_k(u) - \frac{\epsilon}{2}\chi_R(\|u^0\|)\|u^0\|^2, \ \forall \ u \in H, \tag{4.4}$$

for $k \geq k_2$ and $\epsilon > 0$. We take $\hat{M} > \{M, (R+1)/(\alpha+1)\}$, $M, \alpha > 0$ given by (I_5), and use (4.4), $(I_1) - (I_2)$ and $(PSB)^*$ to obtain $k \geq k_2$ and $\epsilon > 0$ so that

$$\hat{M} < \|u\| < (1 + \alpha)\|u^0\|, \ \forall \ u \in K(I_{k,\epsilon}) \setminus B_{r_1}(0), \tag{4.5}$$

Setting $\hat{I} = I_{k,\epsilon}$, from (4.3), (4.5) and (I_5), we get

$$m(\hat{I}, u) > \dim X_k^1 + 1, \ \forall \ u \in K(\hat{I}) \setminus B_{r_1}(0). \tag{4.6}$$

On the other hand, by (4.4) and Lemma 4.1, we obtain that $\hat{I}$ satisfies $(\hat{I}_1)$, $(I_2) - (I_4)$, with (I_3) a linking of deformation type with respect to $\hat{I}$. Thus, by Theorem 3.6, $\hat{I}$ possesses a critical point $u_0 \in H$ such that $|\hat{I}(u_0)| \geq c > 0$, c given by Lemma 4.1, and $m(\hat{I}, u_0) \leq \dim X_k^1 + 1$. Noting that $\hat{I}(u) = I(u)$, for every $u \in B_{r_1}(0)$, we obtain a contradiction with $\beta < c$, (4.2) and (4.6). The proof of Theorem 2.2 is complete. ♢

We finish this section by presenting a version of Theorem 2.2 when (H_k) satisfies $H_k = H$, for every $k \in \mathbb{N}$.

Theorem 4.2 *Let H be a real Hilbert space. Suppose $I \in C^2(H,\mathbb{R})$ satisfies $(I_1)-(I_5)$, with $H_k = H$, for every $k \in \mathbb{N}$. Assume that the origin is an isolated critical point of I. Then there exists $c > 0$ such that I possesses a critical point $u \in H$ satisfying either $I(u) \le -c$ and $m(I,u) \le \dim X^1$, or $I(u) \ge c$ and $m(I,u) \le \dim X^1 + 1$.*

Remark 4.3 *The Theorem 4.2 generalizes the Theorem 2.18 in [31].*

5 Proofs of Theorems 1.1, 1.2 and 1.3

We start by recalling the variational structure associated to the problem (1.1). Consider the Hilbert space $H = H_0^1(\Omega)\times H_0^1(\Omega)$ endowed with the inner product

$$\langle z,\xi\rangle = \int_\Omega \langle \nabla z, \nabla \xi\rangle\, dx,\ \forall\ z,\xi \in H,$$

where $\nabla z = (\nabla u, \nabla v)$, for $z = (u,v) \in H$. Denoting by $I : H \to \mathbb{R}$ the functional associated to (1.1), we may write

$$I(z) = Q_A(z) + J(z),\ \forall\ z \in H, \tag{5.1}$$

with

$$Q_A(z) = \frac{1}{2}\langle L_A z, z\rangle = \frac{1}{2}\int_\Omega \left(\langle R\nabla z, \nabla z\rangle - \langle RAz, z\rangle\right) dx, \tag{5.2}$$

$$J(z) = \int_\Omega G(x,z)\, dx, \tag{5.3}$$

and $G(x,z) = \frac{1}{2}\langle RAz, z\rangle - F(x,z)$. We note that a standard argument [27] shows that $I \in C^2(H,\mathbb{R})$ and that the critical points of I are solutions of (1.1). Furthermore, we have that the self-adjoint linear operator $L_A : H \to H$, given by (5.2), satisfies the condition (I_1). In the following, we denote by $H^+(A)$, $H^0(A)$ and $H^-(A)$ the spectral decomposition of H associated to L_A. We also set $T_A = L_A - L_0$, where L_0 is the linear operator associated to the null matrix.

For every $j \in \mathbb{N}$, we set $e_j = (\varphi_j, 0)$ and $e_{-j} = (0, \varphi_j)$, where φ_j is the eigenvector associated to the eigenvalue λ_j of the operator $-\Delta$ on $H_0^1(\Omega)$. We note that $H^+(0) = \overline{\operatorname{span}\{e_j \mid j \in \mathbb{N}\}}$ and $H^-(0) = \overline{\operatorname{span}\{e_{-j} \mid j \in \mathbb{N}\}}$, where $H^+(0)$ and $H^-(0)$ are the subspaces given by the spectral decomposition of H associated to the linear operator L_0.

In order to apply the Theorem 2.2 to our problem, we consider the family of closed subspaces (H_k) of H defined by $H_k = H_k^-(0) \oplus H^+(0)$, where $H_k^-(0) = \operatorname{span}\{e_{-1}, \ldots, e_{-k}\}$, for every $k \in \mathbb{N}$. The following lemma is proved in [10, 11].

Lemma 5.1 *Let A be an anti-symmetric 2×2 matrix. Then the linear operator $T_A = L_A - L_0$ is compact. Furthermore, for every $k \in \mathbb{N}$, H_k is an invariant subspace of T_A.*

Remark 5.2 *(i) As a consequence of Lemma 5.1, we have that* $H_k = H^+(A) \oplus H^0(A) \oplus H_k^-(A)$*, for* k *sufficiently large, where* $H_k^-(A)$ *is a finite subspace of* $H^-(A)$*. (ii) Considering the definitions given in the introduction, we obtain that* $n(A) = \dim H^0(A)$ *and* $i(A) = i_k(L_A) - i_k(L_0) = \dim(H_k^-(A)) - k$*, for* k *sufficiently large, where* $i_k(L)$ *denotes the index of the operator* L *restricted to* H_k*.*

Taking $X^1 = H^-(A_0)$ and $X^2 = H^0(A_0) \oplus H^+(A_0)$, the proof of the next lemma can be based on the argument used in [28, 29] (see also [31]).

Lemma 5.3 *Suppose* $F \in C^2(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ *satisfies* (F_2)*, with* (F_4) *holding when* $n(A_0) \neq 0$*. Then there exists* $\rho > 0$ *such that*
(i) $I(u) \geq 0$*, for all* $u \in X^2 \cap B_\rho(0)$*,*
(ii) $I(u) \leq 0$*, for all* $u \in X^1 \cap B_\rho(0)$*.*

Setting $X_k^i = X^i \cap H_k$, $i = 1, 2$, for every $k \in \mathbb{N}$, as a direct consequence of Lemma 5.3, we obtain

Corollary 5.4 *Suppose* $F \in C^2(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ *satisfies* (F_2)*, with* (F_4) *holding when* $n(A_0) \neq 0$*. Then the functional* I *satisfies* (I_3)*.*

We omit the proof of the next result since it is similar to the proof of Lemma 5.6 (see also [14] and references therein),

Lemma 5.5 *Suppose* $F \in C^2(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ *satisfies* (F_1)*. Then the functional* J *given by (5.3) satisfies the condition* (I_2)*.*

The next lemma allows us to handle the pointwise limit assumed in condition (F_3).

Lemma 5.6 *Suppose* $F \in C^2(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ *satisfies* (F_3)*. Then, given* $\beta > 0$*, there exists* $M > 0$ *and* $\alpha > 0$ *such that*

$$\int_\Omega \big(\langle D^2F(x,z)w,w\rangle - \langle RA_1 w,w\rangle\big)\,dx \geq -\beta,\ \forall\ w \in \partial B_1(0), \tag{5.4}$$

for every $z \in C_\alpha$*,* $\|z\| > M$*.*

Proof: Arguing by contradiction, we suppose that there exist sequences $(z_n) \subset H$ and $(w_n) \subset \partial B_1(0)$ satisfying $\|z_n\| \to \infty$, $\frac{\|z_n^0\|}{\|z_n\|} \to 1$, as $n \to \infty$, and

$$\int_\Omega g_n(x)\,dx \leq -\beta,\ \forall\ n \in \mathbb{N}, \tag{5.5}$$

where $g_n(x) = \langle D^2F(x,z_n(x))w_n(x), w_n(x)\rangle - \langle RA_1 w_n(x), w_n(x)\rangle$, for $x \in \Omega$. Taking $v_n = z_n/\|z_n\|$ in $\partial B_1(0)$ and using that $H^0 = H^0(A)$ is finite dimensional, we may assume that $v_n(x) \to v(x) \in \partial B_1(0) \cap H^0$. Hence, by the unique continuation property, we must have

$$|z_n(x)| \to \infty,\ \text{for a. e. } x \in \Omega. \tag{5.6}$$

Considering $N > 2$ and invoking the Sobolev Embedding Theorem, we may assume that there exists $w \in H$ such that

$$\begin{gathered} w_n \rightharpoonup w, \quad \text{weakly in } H, \\ w_n(x) \to w(x), \quad \text{for a. e. } x \in \Omega, \\ |w_n(x)| \le h_q(x) \in L^q(\Omega), 1 \le q < 2N/(N-2), \quad \text{for a. e. } x \in \Omega. \end{gathered} \tag{5.7}$$

Observing that $(|w_n(x)|^2) \subset L^{N/(N-2)}(\Omega)$ is bounded and that $w_n(x) \to w(x)$, a. e. on Ω, we may also suppose (see [17]) that $|w_n(x)|^2 \rightharpoonup |w(x)|^2$, weakly in $L^{N/(N-2)}(\Omega)$. Thus, since $C(x) \in L^{\frac{N}{2}}(\Omega)$, we get

$$\lim_{n\to\infty} \int_\Omega C(x)|w_n(x)|^2\,dx = \int_\Omega C(x)|w(x)|^2\,dx. \tag{5.8}$$

Now, we use (5.7) and (F_3) one more time to find $h \in L^1(\Omega)$ such that

$$g_n(x) - C(x)|w_n(x)|^2 \ge h(x), \text{ for a. e. } x \in \Omega.$$

Hence, invoking (5.6), (5.7), (F_3), the above inequality and Fatou's Lemma, we have

$$\liminf_{n\to\infty} \int_\Omega \left(g_n(x) - C(x)|w_n(x)|^2\right)\,dx \ge -\int_\Omega C(x)|w(x)|^2\,dx.$$

But, this last relation and (5.8) contradict (5.5). The Lemma 5.6 is proved. $\diamondsuit$

Proposition 5.7 *Suppose $F \in C^2(\bar\Omega \times \mathbb{R}^2, \mathbb{R})$ satisfies $(F_2) - (F_3)$, with (F_4) holding when $n(A_0) \neq 0$. If $i(A_1) > i(A_0) + 1$, then the functional I given by (5.1) satisfies the conditions (I_4) and (I_5).*

Proof: First, we invoke the Lemma 5.1 and the Remark 5.2 to find $k_1 \in \mathbb{N}$ such that, for every $k \ge k_1$, we have

$$\begin{gathered} i_k(L_A) = i(A) + k = \dim(H_k^-(A)), \\ i_k(L_{A_1}) = i(A_1) + k = \dim(H_k^-(A_1)), \\ i_k(L_{A_0}) = i(A_0) + k = \dim(X_k^1). \end{gathered} \tag{5.9}$$

Observing that the conditions (F_1) and (F_3) imply that $L_A \le L_{A_1}$, from $i(A_1) > i(A_0) + 1$ and (5.9), we get that $\dim(H_k^-(A)) \ge \dim(X_k^1) + 1$, for $k \ge k_1$. Thus, I satisfies the condition (I_4).

We now verify that the condition (I_5) is satisfied by I: applying the Lemma 5.1 one more time and taking k_1 larger if necessary, we may suppose that

$$\int_\Omega \langle RA_1 w, w\rangle\,dx \le \frac{1}{2}\|w\|^2, \ \forall\ w \in H_{k_1}^\perp \cap H. \tag{5.10}$$

We now consider the subspace $V = H_{k_1}^-(A_1)$ of H_{k_1} and take $0 < \beta < 1/4$ so that

$$\langle L_{A_1} v, v\rangle \le -2\beta\|v\|^2, \ \forall\ v \in V. \tag{5.11}$$

Applying Lemma 5.6, we find $M > 0, \alpha > 0$ such that (5.4) holds for β given above.

Given $k > k_1$, we set $Y_k = \operatorname{span}\{e_{-(k_1+1)}, \dots, e_{-k}\}$ and $\hat{H}_k = V \oplus Y_k$. Taking $w \in \hat{H}_k \cap \partial B_1(0)$, we write $w = v + y$, with $v \in V$, $y \in Y_k$, and we use (5.4), (5.10) and (5.11), to obtain

$$\langle D^2 I_k(z)w, w\rangle \leq -\beta\|v\|^2 - \left(\frac{1}{2} - 2\beta\right)\|y\|^2,$$

for every $z \in C_\alpha$ with $\|z\| > M$. Since, by (5.9), $\dim(\hat{H}_k) = \dim(V) + k - k_1 = i(A_1) + k > i(A_0) + 1 + k = \dim X_k^1 + 1$, we conclude that I satisfies (I_5) for every $0 < \mu < \{-\beta, \frac{1}{2} - 2\beta\}$. That concludes the proof of Proposition 5.7. ♢

Proofs of Theorems 1.1 and 1.2: As observed above the functional I given by (5.1) satisfies (I_1). The Corollary 5.4, the Lemma 5.5 and the Proposition 5.6 imply that I satisfies $(I_2) - (I_5)$. Since H_k is an invariant subspace of L_A and I satisfies (I_1) and (I_2), we also have that the $(PSB)^*$ condition is satisfied by I. Invoking the Theorem 2.2, we obtain that I has a critical point other than zero. That concludes the proofs of Theorems 1.1 and 1.2. ♢

Proof of Theorem 1.3: We just observe that under the hypotheses $(\hat{F}_4)$ and $i(A_1) > i(A_0) + n(A_0) + 1$, we may prove the corresponding versions of Lemma 5.3 and Proposition 5.7 by taking $X^1 = H^-(A_0) \oplus H^0(A_0)$ and $X^2 = H^+(A_0)$, respectively. ♢

Remark 5.8 *We note that versions of the Theorems 1.1, 1.2 and 1.3 can be proved when we have* $\limsup_{|z|\to\infty} D^2F(x,z)$ *bounded from above by* RA_1*. On that case, the results should be given in function of the relative numbers of negative eigenvalues of the problem (1.2) associated to the matrices* A_0 *and* A_1*.*

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