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A mixed semilinear parabolic problem from combustion theory *

Claudia Lederman, Juan Luis Vazquez, & Noemi Wolanski

Abstract

We prove existence, uniqueness, and regularity of the solution to a mixed initial boundary-value problem. The equation is semilinear uniformly parabolic with principal part in divergence form, in a non-cylindrical space-time domain. Here we extend our results in [12] to a more general domain. As in [12], we assume only mild regularity on the coefficients, on the non-cylindrical part of the lateral boundary (where the Dirichlet data are given), and on the Dirichlet data.

This problem is of interest in combustion theory, where the non-cylindrical part of the lateral boundary may be considered as an approximation of a flame front. In particular, the results in this paper are used in [11] to prove the uniqueness of a "limit" solution to the combustion problem in a two-phase situation.

1 Introduction

In this paper we prove existence, uniqueness, and regularity of the solution to the mixed initial boundary-value problem

$$\sum_{i,j} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) + \sum_i b_i \frac{\partial u}{\partial x_i} + c\,u - u_t = \beta(x,t,u) \quad \text{in } \mathcal{R}$$
$$\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_j}\eta_i = 0 \quad \text{on } \partial_N\mathcal{R}$$
$$u = \phi \quad \text{on } \partial_D\mathcal{R},$$

where $\mathcal{R} \subset \mathbb{R}^N \times (0,T)$ is a bounded non-cylindrical space-time domain, $\partial_N\mathcal{R}$ is an open subset of the parabolic boundary, $\partial_p\mathcal{R}$, and $\partial_D\mathcal{R} = \partial_p\mathcal{R} \setminus \partial_N\mathcal{R}$. This

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is a semilinear uniformly parabolic equation with principal part in divergence form, in a non-cylindrical space-time domain. We look for a weak solution $u \in C(\overline{\mathcal{R}})$, with $\nabla u \in C(\overline{\mathcal{R}})$. Here we extend our results in [12] to a more general domain.

The non-cylindrical part of $\partial_p \mathcal{R}$ is $\partial_D \mathcal{R} \cap \{t > 0\}$. As in [12], we assume only mild regularity on the coefficients, and on $\partial_D \mathcal{R} \cap \{t > 0\}$. We also assume a minimum smoothness on the Dirichlet datum ϕ.

This problem is of interest in combustion theory. In that situation, the non-cylindrical part of the lateral boundary may be considered as an approximation of a flame front. The second order part of the equation is the Laplace operator. In particular, the results in this paper are used in [11] to prove the uniqueness of a "limit" solution to the combustion problem in a two phase situation. We point out that in [10] –where we proved the uniqueness of a "limit" solution to the combustion problem in a one phase situation– both the results in [12] and in the present paper can be applied. However, in [11] –which is a two phase situation– the results in [12] do not apply and we need to use the more general results we are presenting here.

In the combustion context of [10] and [11] the initial datum $\phi(x, 0)$ is only globally Hölder continuous with Hölder continuous spatial gradient near the initial flame front $\overline{\partial_D \mathcal{R} \cap \{t > 0\}} \cap \{t = 0\}$. The solution u must satisfy that $\nabla u \in C(\overline{\mathcal{R}} \cap \{t > 0\})$ and ∇u must be continuous up to time $t = 0$ near the flame front $\overline{\partial_D \mathcal{R} \cap \{t > 0\}}$. With that regularity of the datum, standard Schauder or Sobolev type results cannot be applied, even if we had a cylindrical space-time domain or $\partial_N \mathcal{R} = \emptyset$. In order to get our results, both in [12] and here, we reduce the problem posed in a non-cylindrical space-time domain to a similar problem in a domain which is a space-time cylinder. Once this is done, the main point is the proof of the regularity of ∇u up to the boundary with mild regularity assumptions on the data.

We point out that we prove the existence of a weak solution $u \in C^{\gamma, \frac{\gamma}{2}}(\overline{\mathcal{R}})$ with $\nabla u \in C(\mathcal{R} \cup \partial_N \mathcal{R})$ assuming that ϕ is only Hölder continuous. Further continuity of ∇u is obtained in every neighborhood of a point in $\partial_D \mathcal{R}$ where ϕ is smooth enough.

We remark that there is a vast body of literature on mixed boundary-value problems for parabolic equations (see, for instance, [2, 3, 4, 8, 13]). However, the results we present here, and those in [12], cannot be derived from those papers.

The paper is organized as follows: In Section 2 we introduce the notation and hypotheses to be used throughout the paper and, in particular, we define the non-cylindrical space-time domain we are going to work with. As a previous step to the study of the mixed semilinear parabolic problem, we prove in Section 3 results on existence, uniqueness and regularity, as well as a priori estimates, for the corresponding linear problem. Section 4 is devoted to the proof of the main result in this paper, i.e. Theorem 4.1, which is an existence, uniqueness and regularity result for the mixed semilinear problem. Finally, we show in Section 5 how the results in this paper are used to prove the uniqueness of a "limit" solution to the combustion problem.

2 Notation and hypotheses

Throughout this paper the spatial dimension is denoted by N, and the following notation is used:

The symbol ∇ will denote the corresponding operator in the space variables; the symbol ∂_p applied to a domain will denote parabolic boundary.

For an integer $m \geq 0$, $0 < \alpha < 1$, and a space-time cylinder $Q = \Omega \times (0,T) \subset \mathbb{R}^{N+1}$, $C^{m+\alpha, \frac{m+\alpha}{2}}(Q)$ will denote the parabolic Hölder space which is denoted by $H^{m+\alpha, \frac{m+\alpha}{2}}(Q)$ in [9].

For $\mathcal{D} \subset \mathbb{R}^{N+1}$ a general domain, $C^{m+\alpha, \frac{m+\alpha}{2}}(\mathcal{D})$ will denote the space of functions in $C^{m+\alpha, \frac{m+\alpha}{2}}(Q)$ for every space-time cylinder $Q \subset \mathcal{D}$.

For $\mathcal{D}$ bounded, we will say that $u \in C^{m+\alpha, \frac{m+\alpha}{2}}(\overline{\mathcal{D}})$ if there exists a domain $\mathcal{D}'$ with $\overline{\mathcal{D}} \subset \mathcal{D}'$ and a function $u' \in C^{m+\alpha, \frac{m+\alpha}{2}}(\mathcal{D}')$ such that $u = u'$ in $\overline{\mathcal{D}}$. And we will denote by $C^{\text{dini}}(\overline{\mathcal{D}})$ the set of functions which are continuous in $\overline{\mathcal{D}}$ and such that their modulus of continuity $\omega(r)$ with respect to the parabolic norm $\|(x,t)\| = |x| + |t|^{\frac{1}{2}}$ satisfy the Dini condition

$$\int_0^1 \frac{\omega(r)}{r}\, dr < \infty.$$

Throughout the paper we will let $\Omega = \mathbb{R} \times \Sigma$ and $\Sigma \subset \mathbb{R}^{N-1}$ be a bounded Lipschitz domain with interior unit normal η'. We will denote by $\eta = (0, \eta')$ the interior unit normal to $\partial\Omega$. We will denote points in $\overline{\Omega}$ by $x = (x_1, x')$ with $x_1 \in \mathbb{R}$ and $x' \in \overline{\Sigma}$.

On the other hand p, q will be Lipschitz continuous functions in $\overline{\Sigma} \times [0,T]$, and we will denote

$$\mathcal{D} := \{(x,t) \in \Omega \times (0,T) \,/\, p(x',t) < x_1 < q(x',t)\}.$$

We will assume, in addition, that there exists a constant $\mu_0 > 0$ such that $q(x',t) - p(x',t) \geq \mu_0$ in $\overline{\Sigma} \times [0,T]$.

We define, as usual, $\partial_p \mathcal{D} := \overline{\partial \mathcal{D} \setminus \{t = T\}}$ and let

$$\partial_N \mathcal{D} := \{(x,t) \in \partial_p \mathcal{D} \,/\, x' \in \partial\Sigma,\ 0 < t \leq T,\ p(x',t) < x_1 < q(x',t)\},$$
$$\partial_D \mathcal{D} := \partial_p \mathcal{D} \setminus \partial_N \mathcal{D},$$
$$\partial_S \mathcal{D} := \{(x,t) \in \partial_p \mathcal{D} \,/\, x_1 = p(x',t) \ \text{ or } \ x_1 = q(x',t)\}.$$

For $R_0 < \mu_0$, we define

$$\mathcal{D}_{R_0} := \{(x,t) \in \Omega \times (0,T) \,/\, p(x',t) < x_1 < p(x',t) + R_0\}.$$

3 The linear problem

We will prove in this paper an existence, uniqueness, and regularity result, as well as a-priori estimates, for a mixed initial-boundary value problem associated to a uniformly parabolic equation with principal part in divergence form, with a nonlinear forcing term. This will be done on the non-cylindrical space-time domain $\mathcal{D}$ defined in the previous section. To do so, we devote this section to the study of the corresponding linear problem. In Proposition 3.1 we prove existence and uniqueness for the linear problem, with a minimum smoothness of the data. In Proposition 3.2 we prove a regularity result for the linear problem. We first prove the following existence and uniqueness result for the linear problem:

Proposition 3.1 *Let $\mathcal{D}$, $\partial_N\mathcal{D}$, $\partial_D\mathcal{D}$ and $\partial_S\mathcal{D}$ be as above. For $i,j = 1,\cdots,N$, let a_{ij}, b_i, c, $g \in L^\infty(\mathcal{D})$. Assume that $a_{ij}(x,t)\xi_i\xi_j \geq \lambda|\xi|^2$ for some $\lambda > 0$ and every $\xi \in \mathbb{R}^N$, $(x,t) \in \mathcal{D}$. Let $\phi \in C^{\alpha,\frac{\alpha}{2}}(\overline{\mathcal{D}})$. Then there exists a unique function $u \in C(\overline{\mathcal{D}})$, with $\nabla u \in L^2_{\text{loc}}(\overline{\mathcal{D}} \setminus \partial_S\mathcal{D})$, such that u is a weak solution to the following mixed initial boundary-value problem*

$$\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_i b_i \frac{\partial u}{\partial x_i} + c\,u - u_t = g \quad \text{in } \mathcal{D} \tag{1}$$

$$\sum_{i,j} a_{ij}\frac{\partial u}{\partial x_j}\eta_i = 0 \quad \text{on } \partial_N\mathcal{D} \tag{2}$$

$$u = \phi \quad \text{on } \partial_D\mathcal{D}. \tag{3}$$

Moreover, there exist $0 < \gamma \leq \alpha$ and $C > 0$, depending only on α, T, λ, $\|a_{ij}\|_{L^\infty(\mathcal{D})}$, $\|b_i\|_{L^\infty(\mathcal{D})}$, $\|c\|_{L^\infty(\mathcal{D})}$, $\|\phi\|_{C^{\alpha,\frac{\alpha}{2}}(\partial_D\mathcal{D})}$, $\|g\|_{L^\infty(\mathcal{D})}$, the domain Σ and the functions p and q, such that $u \in C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}})$ and

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}})} \leq C. \tag{4}$$

Now let $\psi_1(x',t) = \phi(p(x',t),x',t)$, $\psi_2(x',t) = \phi(q(x',t),x',t)$ and assume, in addition, that $\psi_{i\,t} \in L^2(\Sigma\times(0,T))$, $\nabla_{x'}\psi_i \in L^2(\Sigma\times(0,T))$ for $i = 1,2$. Then $\nabla u \in L^2(\mathcal{D})$.

Proof: Let ψ_1, ψ_2 be as in the statement. We will first prove the proposition with the extra assumption that $\psi_{i\,t} \in L^2(\Sigma\times(0,T))$, $\nabla_{x'}\psi_i \in L^2(\Sigma\times(0,T))$ for $i = 1,2$.

We straighten up both lateral boundaries by taking a new coordinate system. In fact, we let $y = H(x,t)$ be defined by

$$y_1 = \frac{x_1 - p(x',t)}{q(x',t) - p(x',t)}\,, \tag{5}$$

$$y_i = x_i \quad \text{for } i > 1. \tag{6}$$

Then, for $(y,t) \in Q := (0,1)\times\Sigma\times(0,T)$, we let $\overline{u}(y,t) = u(x,t)$. Then, $u \in C(\overline{\mathcal{D}})$, with $\nabla u \in L^2(\mathcal{D})$, is a weak solution to (1)–(3) if and only if $\overline{u} \in C(\overline{Q})$, with $\nabla\overline{u} \in L^2(Q)$, is a weak solution to

$$\overline{\mathcal{L}}\overline{u} := \sum_{i,j}\frac{\partial}{\partial y_i}\left(\bar{a}_{ij}\frac{\partial\overline{u}}{\partial y_j}\right) + \sum_i \bar{b}_i\frac{\partial\overline{u}}{\partial y_i} + \bar{c}\,\overline{u} - \overline{u}_t = \bar{g} \quad \text{in } Q, \tag{7}$$

$$\sum_{i,j}\bar{a}_{ij}\frac{\partial\overline{u}}{\partial y_j}\eta_i = 0 \quad \text{on } \partial_N Q := (0,1)\times\partial\Sigma\times(0,T], \tag{8}$$

$$\bar{u} = \bar{\phi} \quad \text{on } \partial_D Q := \partial_p Q \setminus \partial_N Q, \tag{9}$$

where $\overline{g}(y,t)=g(x,t)$, $\overline{\phi}(y,t)=\phi(x,t)$, $\bar{c}(y,t)=c(x,t)$,

$$\begin{aligned}\bar{a}_{ij}(y,t) &= \sum_{k,l} a_{kl}(x,t)\frac{\partial H_i}{\partial x_k}(x)\frac{\partial H_j}{\partial x_l}(x),\\ \bar{b}_i(y,t) &= \sum_j b_j(x,t)\frac{\partial H_i}{\partial x_j}(x,t)-\frac{\partial H_1}{\partial t}(x,t)\delta_{1i}+\sum_j \bar{a}_{ji}(y,t)\frac{q_{x_j}-p_{x_j}}{q-p}(x',t).\end{aligned}$$

Note that the equation for $\overline{u}$ has bounded coefficients and right hand side. On the other hand, it is uniformly parabolic (with parabolicity constant depending only on λ and the functions p and q).

The existence and uniqueness of a function $\overline{u}\in C\big([0,T];L^2((0,1)\times\Sigma)\big)$ with $\nabla\overline{u}\in L^2(Q)$, which is a weak solution to (7)–(9), can be obtained, for instance, from Theorem X.9 in [1], by proceeding as in Proposition 1.1 in [12].

Let us prove that there exist $0<\gamma'\le\alpha$ and $C>0$ depending only on the L^∞ norm of the coefficients of the equation in (7)–(9), the constants α, λ, T, the domain Σ, $\|\overline{\phi}\|_{C^{\alpha,\frac{\alpha}{2}}(\partial_D Q)}$, $\|\overline{g}\|_{L^\infty(Q)}$, $\|\overline{u}\|_{L^\infty(Q)}$ and the functions p and q, such that

$$\|\overline{u}\|_{C^{\gamma',\frac{\gamma'}{2}}(\overline{Q})}\le C. \tag{10}$$

To prove (10) we first take the set $Q_\delta:=(0,1)\times\Sigma_\delta\times(0,T)$ where $\Sigma_\delta:=\{y'\in\Sigma\,/\,\operatorname{dist}(y',\partial\Sigma)>\delta\}$ and $\delta>0$ is small to be fixed later. Then we can get estimate (10) in $\overline{Q_\delta}$ by applying Theorem 10.1, Chap. III in [9].

Next, let $(y_0,t_0)\in[0,1]\times\partial\Sigma\times[0,T]$. We will straighten up $\partial_N Q$. For that purpose we denote $y_0=(y_{0_1},y_0')$, with $y_{0_1}\in[0,1]$ and $y_0'\in\partial\Sigma$ and we take $\mathcal{O}\subset\mathbb{R}^{N-1}$ a neighborhood of y_0' such that $[0,1]\times(\partial\Sigma\cap\mathcal{O})$ is parameterized in the variables $(z_1,\cdots,z_{N-1})$ by

$$y_1=z_1,\quad 0\le z_1\le 1, \tag{11}$$
$$y'=\sigma'(z_2,\cdots,z_{N-1}),\quad (z_2,\cdots,z_{N-1})\in\mathcal{N}\subset\mathbb{R}^{N-2}. \tag{12}$$

Here $\mathcal{N}$ is the ball in $\mathbb{R}^{N-2}$ with center in the origin and radius $r=1$. Since Σ is a Lipschitz domain we may assume that, in the neighborhood $\mathcal{O}$, σ' is the graph of a Lipschitz function G in the direction y_N and that every point $y\in[0,1]\times\mathcal{O}$ can be written in a unique way as $y=h^{-1}(z)$, where

$$\begin{gathered}y_1=z_1\\ y_i=z_i,\quad 2\le i\le N-1,\\ y_N=z_N+G(z_2,\ldots,z_{N-1}),\end{gathered} \tag{13}$$

with $z\in[0,1]\times\mathcal{N}\times\{|z_N|<2\delta\}$, for some $\delta>0$, and h a Lipschitz invertible function with non-vanishing Jacobian in $[0,1]\times\overline{\mathcal{O}}$ and $h([0,1]\times\mathcal{O})=[0,1]\times\mathcal{N}\times\{|z_N|<2\delta\}$ and $h([0,1]\times(\mathcal{O}\cap\Sigma))=[0,1]\times\mathcal{N}\times\{0<z_N<2\delta\}$.

Here δ can be chosen independent of (y_0,t_0), and will remain fixed from now on. Let $\overline{\overline{u}}(z,t):=\overline{u}(y,t)$ for $z_N\ge 0$ and $\tilde{Q}_+:=\{(z,t)\in\tilde{Q}\,/\,z_N>0\}$, where

$$\tilde{Q}:=(0,1)\times\mathcal{N}\times\{|z_N|<2\delta\}\times(0,T).$$

Then $\overline{\overline{u}} \in C\left([0,T]; L^2((0,1)\times\mathcal{N}\times\{0<z_N<2\delta\})\right)$, with $\nabla\overline{\overline{u}} \in L^2(\tilde{Q}_+)$, is a weak solution in $\tilde{Q}_+$ to

$$\overline{\overline{\mathcal{L}}}\,\overline{\overline{u}} := \sum_{i,\,j}\frac{\partial}{\partial z_i}\left(\bar{\bar{a}}_{ij}\frac{\partial\overline{\overline{u}}}{\partial z_j}\right) + \sum_j \bar{\bar{b}}_j\frac{\partial\overline{\overline{u}}}{\partial z_j} + \bar{\bar{c}}\overline{\overline{u}} - \overline{\overline{u}}_t = \bar{\bar{g}} \tag{14}$$

which is a uniformly parabolic equation with principal part in divergence form with bounded coefficients and free term. Here

$$\bar{\bar{a}}_{ij}(z,t) = \sum_{k,\,l}\bar{a}_{kl}(y,t)\frac{\partial h_i}{\partial y_k}(y)\frac{\partial h_j}{\partial y_l}(y).$$

We extend $\overline{\overline{u}}$ to $\{z_N<0\}$ by reflection. This is, we define for $z_N<0$

$$\overline{\overline{u}}(z,t) = \overline{\overline{u}}(z_1,z_2,\cdots,-z_N,t).$$

In this way, $\overline{\overline{u}} \in C\left([0,T]; L^2((0,1)\times\mathcal{N}\times\{|z_N|<2\delta\})\right)$, with $\nabla\overline{\overline{u}} \in L^2(\tilde{Q})$, becomes a weak solution in the domain $\tilde{Q}$ of equation (14), where we have, for $z_N<0$,

$$\bar{\bar{a}}_{ij}(z,t) = \begin{cases} \bar{\bar{a}}_{ij}(z_1,\cdots,z_{N-1},-z_N,t) & \text{if } i<N, j<N, \text{ or } i=j=N, \\ -\bar{\bar{a}}_{ij}(z_1,\cdots,z_{N-1},-z_N,t) & \text{if } i=N, j<N, \text{ or } i<N, j=N, \end{cases}$$

$$\bar{\bar{b}}_j(z,t) = \begin{cases} \bar{\bar{b}}_j(z_1,\cdots,z_{N-1},-z_N,t) & \text{if } j<N, \\ -\bar{\bar{b}}_j(z_1,\cdots,z_{N-1},-z_N,t) & \text{if } j=N, \end{cases}$$

and

$$\bar{\bar{c}}(z,t) = \bar{\bar{c}}(z_1,\cdots,z_{N-1},-z_N,t), \quad \bar{\bar{g}}(z,t) = \bar{\bar{g}}(z_1,\cdots,z_{N-1},-z_N,t).$$

Thus, $\overline{\overline{u}}$ is a weak solution in $\tilde{Q}$ of a uniformly parabolic equation with principal part in divergence form with bounded coefficients and free term.

We apply again Thm. 10.1, Chap. III in [9] to conclude that there exist $0<\gamma'\le\alpha$ and $C>0$ such that

$$\|\overline{\overline{u}}\|_{C^{\gamma',\frac{\gamma'}{2}}(\bar{\tilde{Q}}_{\frac{1}{2}})} \le C.$$

Here $\tilde{Q}_{\frac{1}{2}} = (0,1)\times\mathcal{N}_{\frac{1}{2}}\times\{|z_N|<\delta\}\times(0,T)$, where $\mathcal{N}_{\frac{1}{2}}$ is the ball in $\mathbb{R}^{N-2}$ with center in the origin and radius $r=1/2$.

The constants γ' and C depend only on α, λ, T, Σ, the functions p and q, the L^∞ norm of the coefficients of the equation in (14), and free term in $\tilde{Q}$, $\|\bar{\bar{u}}\|_{L^\infty(\tilde{Q})}$, $\|\overline{\overline{\phi}}\|_{C^{\alpha,\frac{\alpha}{2}}(\bar{\tilde{Q}}_+\cap(\{t=0\}\cup\{z_1=0\}\cup\{z_1=1\}))}$. Here $\overline{\overline{\phi}}(z,t) := \overline{\phi}(y,t)$ for $(y,t)\in\partial_D Q$. Therefore (10) holds.

Since $\|u\|_{L^\infty(\mathcal{D})}$ is bounded by a constant depending only on T, $\|\phi\|_{L^\infty(\partial_D\mathcal{D})}$, $\|c\|_{L^\infty(\mathcal{D})}$ and $\|g\|_{L^\infty(\mathcal{D})}$, we conclude that (4) holds.

Finally, the proof of the results in the statement without the extra assumption that $\psi_{i_t}\in L^2(\Sigma\times(0,T))$, $\nabla_{x'}\psi_i\in L^2(\Sigma\times(0,T))$ for $i=1,2$, as well as the proof of the uniqueness of solution follow as in Proposition 1.1 in [12].

We next prove a regularity result for the linear problem.

Proposition 3.2 *Let $\mathcal{D}$, $\partial_N\mathcal{D}$, $\partial_D\mathcal{D}$, a_{ij}, b_i, c, g, ϕ, ψ_i as in Proposition 3.1. Let $u \in C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}})$, with $\nabla u \in L^2(\mathcal{D})$, be the unique weak solution to (1)–(3). Now assume that $\Sigma \in C^3$, $p, q \in C^1(\overline{\Sigma} \times [0,T])$, $\nabla_{x'}p, \nabla_{x'}q \in C^{\text{dini}}(\overline{\Sigma} \times [0,T])$, and that $\nabla_{x'}p(x',t)\cdot\eta' = 0$ and $\nabla_{x'}q(x',t)\cdot\eta' = 0$ on $\partial\Sigma \times (0,T)$. Assume also that $a_{ij} \in C^{\text{dini}}(\overline{\mathcal{D}})$ and $a_{ij} = \delta_{ij}$ on $\partial_N\mathcal{D}$. Then, $\nabla u \in C(\mathcal{D} \cup \partial_N\mathcal{D})$.*

If, in addition, $\psi_1(x',t) \in C^1(\overline{\Sigma} \times (0,T])$, with $\nabla_{x'}\psi_1 \in C^{\text{dini}}(\overline{\Sigma} \times (0,T])$ and $\frac{\partial\psi_1}{\partial\eta'} = 0$ on $\partial\Sigma \times (0,T)$, there holds that ∇u is continuous in $\overline{\mathcal{D}} \cap \{x_1 < q(x',t)\} \cap \{t > 0\}$.

If, moreover, $\psi_1(x',t) \in C^1(\overline{\Sigma} \times [0,T])$, with $\nabla_{x'}\psi_1 \in C^{\text{dini}}(\overline{\Sigma} \times [0,T])$, and $\nabla\phi \in C^{\text{dini}}(\overline{\mathcal{D}}_{R_0} \cap \{t = 0\})$, with $\frac{\partial\phi}{\partial\eta} = 0$ on $\overline{\partial_N\mathcal{D}_{R_0}} \cap \{t = 0\}$, there holds that $\nabla u \in C(\overline{\mathcal{D}}_{R_0/2})$ and there exist a constant $C > 0$ and an increasing function $\omega(r)$, with $\omega(0^+) = 0$, such that

$$\|\nabla u\|_{L^\infty(\mathcal{D}_{R_0/2})} \le C, \tag{15}$$

$$|\nabla u(x,t) - \nabla u(y,s)| \le \omega(|x-y| + |t-s|^{1/2}),\ (x,t),\ (y,s) \in \overline{\mathcal{D}}_{R_0/2}. \tag{16}$$

With the same regularity of ψ_1 and no regularity assumptions on $\phi(x,0)$, for every $\tau > 0$, (15)–(16) holds in $\mathcal{D}_{R_0/2} \cap \{t \ge \tau\}$ with C and ω independent of $\phi(x,0)$ but depending on τ.

Analogously, if $\psi_2(x',t) \in C^1(\overline{\Sigma} \times (0,T])$, with $\nabla_{x'}\psi_2 \in C^{\text{dini}}(\overline{\Sigma} \times (0,T])$ and $\frac{\partial\psi_2}{\partial\eta'} = 0$ on $\partial\Sigma \times (0,T)$, and with no regularity assumptions on ψ_1 and on $\phi(x,0)$, there holds that ∇u is continuous in $\overline{\mathcal{D}} \cap \{x_1 > p(x',t)\} \cap \{t > 0\}$.

Also, if $\psi_i \in C^1(\overline{\Sigma} \times [0,T])$, with $\nabla_{x'}\psi_i \in C^{\text{dini}}(\overline{\Sigma} \times [0,T])$, and $\frac{\partial\psi_i}{\partial\eta'} = 0$ on $\partial\Sigma \times (0,T)$ for $i = 1,2$ and $\nabla\phi \in C^{\text{dini}}(\overline{\mathcal{D}} \cap \{t = 0\})$ with $\frac{\partial\phi}{\partial\eta} = 0$ on $\overline{\partial_N\mathcal{D}} \cap \{t = 0\}$, there holds that $\nabla u \in C(\overline{\mathcal{D}})$.

If $a_{ij} \in C^{1+\mu,\frac{1+\mu}{2}}(\mathcal{D})$, b_i, c, $g \in C^{\mu,\frac{\mu}{2}}(\mathcal{D})$, u is a classical solution in the sense that $u \in C^{2+\mu,1+\frac{\mu}{2}}(\mathcal{D})$.

Proof: In this proof we use the same notation as in the proof of Proposition 3.1. Since we have assumed that Σ is a C^3 domain, we may take as σ' in (11)-(12) a C^3 regular parameterization. Also η', the interior unit normal to Σ, is a C^2 function of the point $y' \in \partial\Sigma$. Then, instead of taking $y = h^{-1}(z)$ as in (13), we take $y = h^{-1}(z)$ in the following way:

$$y_1 = z_1 \tag{17}$$

$$y' = \sigma'(z_2, \cdots, z_{N-1}) + \eta'(\sigma'(z_2, \cdots, z_{N-1}))\, z_N, \tag{18}$$

so now h is, in addition, a C^2 function.

Let us now assume that $a_{ij} \in C^{\text{dini}}(\overline{\mathcal{D}})$ and $a_{ij} = \delta_{ij}$ on $\partial_N\mathcal{D}$. In order to prove that $\nabla u \in C(\mathcal{D} \cup \partial_N\mathcal{D})$ we consider a point $(y_0, t_0) \in (0,1) \times \partial\Sigma \times [0,T]$ and the corresponding function $\overline{\overline{u}}(z,t)$ which is defined and continuous in $\overline{\tilde{Q}}$ with $\nabla\overline{\overline{u}} \in L^2(\tilde{Q})$. Also, $\overline{\overline{u}}$ is a weak solution to (14) in $\tilde{Q}$.

Let us see that the principal coefficients in (14), $\bar{\bar{a}}_{ij}$, belong to $C^{\text{dini}}(\overline{\tilde{Q}})$. In fact, using that $\nabla_{x'}p, \nabla_{x'}q \in C^{\text{dini}}(\overline{\Sigma} \times [0,T])$, we get that $\bar{\bar{a}}_{ij}$ are Dini continuous in $\{z_N \geq 0\} \cup \{z_N \leq 0\}$. Then, we only need to verify that $\bar{\bar{a}}_{iN}(z_1, \cdots, z_{N-1}, 0, t) = 0$ for $i < N$. We observe that, for $y' \in \partial\Sigma$,

$$\bar{a}_{ij}(y,t) = \delta_{ij} \quad i,j > 1,$$
$$\bar{a}_{1j}(y,t) = \frac{p_{y_j}(y_1 - 1) - q_{y_j} y_1}{q - p}(y', t) \quad j > 1.$$

Therefore,

$$\begin{aligned} \bar{\bar{a}}_{iN}(z_1, \cdots, z_{N-1}, 0, t) &= \nabla h_i \cdot \nabla h_N + \frac{\partial h_N}{\partial y_1} \left(-\frac{\partial h_i}{\partial y_1} + \sum_{k \geq 1} \bar{a}_{k1} \frac{\partial h_i}{\partial y_k} \right) \\ &\quad + \frac{\partial h_i}{\partial y_1} \left(\sum_{k > 1} \left(\frac{p_{y_k}(y_1 - 1) - q_{y_k} y_1}{q - p} \right) \frac{\partial h_N}{\partial y_k} \right). \end{aligned}$$

From the fact that $h\big(z_1, \sigma'(z_2, \cdots, z_{N-1}) + \eta'(\sigma'(z_2, \cdots, z_{N-1}))z_N\big) = z$ we deduce that, on $(0,1) \times \partial\Sigma$, ∇h_i is tangent to $(0,1) \times \partial\Sigma$ for $i < N$ and $\nabla h_N = \eta$. Therefore, $\nabla h_i \cdot \nabla h_N = 0$ on $(0,1) \times \partial\Sigma$ for $i < N$. Since $\eta_1 = 0$, there holds that $\frac{\partial h_N}{\partial y_1} = 0$. Finally, we use the fact that $\nabla_{y'} p \cdot \eta' = 0$ and $\nabla_{y'} q \cdot \eta' = 0$ on $\partial\Sigma \times (0,T)$ and conclude that we have $\bar{\bar{a}}_{iN} = 0$ on $\{z_N = 0\}$ for $i < N$. We can now apply Theorem 1.3.1 in [5] in $\tilde{Q}$ to deduce that $\nabla\bar{\bar{u}}$ is continuous in $(0,1) \times \mathcal{N} \times \{|z_N| < 2\delta\} \times (0,T]$.

On the other hand, a direct application of Theorem 1.3.1 in [5] to u gives the continuity of ∇u in $\mathcal{D}$.

The rest of the proof follows as that of Proposition 1.2 in [12]. Namely, under further assumptions on the Dirichlet data ψ_i and/or ϕ we obtain the continuity of ∇u up to the corresponding subset of the Dirichlet boundary by suitably applying the results in [5].

From classical Schauder estimates we deduce that, when $a_{ij} \in C^{1+\mu, \frac{1+\mu}{2}}(\mathcal{D})$ and $b_i, c, g \in C^{\mu, \frac{\mu}{2}}(\mathcal{D})$, there holds that $u \in C^{2+\mu, 1+\frac{\mu}{2}}(\mathcal{D})$. This completes the present proof.

4 The semilinear problem

In this section we prove the main result in the paper, Theorem 4.1, which is an existence, uniqueness and regularity result for the mixed semilinear problem.

Theorem 4.1 *Let $\mathcal{D}$, $\partial_N\mathcal{D}$, $\partial_D\mathcal{D}$, $\partial_S\mathcal{D}$, a_{ij}, b_i, c and ϕ as in Proposition 3.1. Let $\beta(x,t,u) \in L^\infty(\mathcal{D} \times \mathbb{R})$ be such that $\beta(x,t,\cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ uniformly for $(x,t) \in \mathcal{D}$. There exists a unique function $u \in C^{\gamma, \frac{\gamma}{2}}(\overline{\mathcal{D}})$ for some $0 < \gamma \leq \alpha$, with $\nabla u \in L^2_{\text{loc}}(\overline{\mathcal{D}} \setminus \partial_S\mathcal{D})$, such that u is a weak solution to the following problem*

$$\textstyle\sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij} \frac{\partial u}{\partial x_j}\right) + \sum_i b_i \frac{\partial u}{\partial x_i} + c\, u - u_t = \beta(x,t,u) \quad \textit{in } \mathcal{D} \qquad (19)$$

$$\sum_{i,j} a_{ij} \frac{\partial u}{\partial x_j} \eta_i = 0 \quad \text{on } \partial_N \mathcal{D} \tag{20}$$

$$u = \phi \quad \text{on } \partial_D \mathcal{D}. \tag{21}$$

Now let $\psi_1(x', t) = \phi(p(x', t), x', t)$, $\psi_2(x', t) = \phi(q(x', t), x', t)$ and assume, in addition, that $\psi_{i_t} \in L^2(\Sigma \times (0, T))$, $\nabla_{x'} \psi_i \in L^2(\Sigma \times (0, T))$ for $i = 1, 2$. Then $\nabla u \in L^2(\mathcal{D})$.

Moreover, Propositions 3.1 and 3.2 apply to u. In particular, further assumptions on Σ, p, q, the coefficients a_{ij} and on the Dirichlet data ψ_i and/or ϕ give regularity results for ∇u up to the corresponding subset of the Dirichlet boundary, $\partial_D \mathcal{D}$.

If, in addition, $\beta(\cdot, \cdot, u) \in C^{\mu, \frac{\mu}{2}}(\mathcal{D})$ uniformly for u in compact subsets of $\mathbb{R}$, $a_{ij} \in C^{1+\mu, \frac{1+\mu}{2}}(\mathcal{D})$, b_i, $c \in C^{\mu, \frac{\mu}{2}}(\mathcal{D})$, there holds that $u \in C^{2+\mu, 1+\frac{\mu}{2}}(\mathcal{D})$.

Proof: The proof of existence and uniqueness of the solution is analogous to that of Theorem 1.1 in [12] and follows by using Schauder's fixed point Theorem and the result of Proposition 3.1. We include the proof for the sake of completeness.

Let $B = \|\beta\|_{L^\infty}$, and let γ and C be the constants given by Proposition 3.1 when ϕ is fixed and $\|g\|_{L^\infty(\mathcal{D})} \leq B$. Let $0 < \nu < \gamma$ and let $K > 0$ be such that $\|\cdot\|_{C^{\nu, \frac{\nu}{2}}(\overline{\mathcal{D}})} \leq K \|\cdot\|_{C^{\gamma, \frac{\gamma}{2}}(\overline{\mathcal{D}})}$. Let us consider the set

$$\mathcal{B} = \{v \in C^{\nu, \frac{\nu}{2}}(\overline{\mathcal{D}}) \,/\, \|v\|_{C^{\nu, \frac{\nu}{2}}(\overline{\mathcal{D}})} \leq K\, C\}.$$

Let $\mathcal{T}$ be defined on $\mathcal{B}$ by $\mathcal{T} v := u$ where u is the unique solution given by Proposition 3.1 when $g(x, t) = \beta(x, t, v(x, t))$. Then

$$\|\mathcal{T} v\|_{C^{\nu, \frac{\nu}{2}}(\overline{\mathcal{D}})} \leq K \|\mathcal{T} v\|_{C^{\gamma, \frac{\gamma}{2}}(\overline{\mathcal{D}})} \leq K\, C.$$

Therefore, $\mathcal{T}$ maps $\mathcal{B}$ continuously into a compact subset of $\mathcal{B}$. So that $\mathcal{T}$ has a fixed point u which clearly is a solution to (19)–(21).

To prove uniqueness, we let u_1 and u_2 be solutions to (19)–(21). Then $w = u_1 - u_2$ is a solution to (1)–(3) with a different coefficient c (which depends on u_1 and u_2), and with $g = \phi = 0$. By Proposition 3.1, $w = 0$.

If, in addition, $\beta(\cdot, \cdot, u) \in C^{\mu, \frac{\mu}{2}}(\mathcal{D})$ uniformly for u in compact subsets of $\mathbb{R}$, there holds that u is a solution of (1)–(3) with $g \in C^{\gamma', \frac{\gamma'}{2}}(\mathcal{D})$, with $\gamma' = \min\{\mu, \gamma\}$. Then, if $a_{ij} \in C^{1+\mu, \frac{1+\mu}{2}}(\mathcal{D})$, b_i, $c \in C^{\mu, \frac{\mu}{2}}(\mathcal{D})$, there holds that $u \in C^{2+\gamma', 1+\frac{\gamma'}{2}}(\mathcal{D})$, so that $g \in C^{\mu, \frac{\mu}{2}}(\mathcal{D})$ and we deduce that $u \in C^{2+\mu, 1+\frac{\mu}{2}}(\mathcal{D})$.

5 The combustion problem

The purpose of this section is to show how the results in this paper apply in [11] to a problem in combustion theory. In [11] the following two phase free boundary problem is considered: find a function $u(x, t)$, defined in $\mathcal{D} \subset \mathbb{R}^N \times (0, T)$, satisfying that

$$\Delta u + \sum a_i(x, t)\, u_{x_i} - u_t = 0 \quad \text{in } \{u > 0\} \cup \{u \leq 0\}^\circ, \tag{22}$$

$$u = 0\,, \quad |\nabla u^+|^2 - |\nabla u^-|^2 = 2M \quad \text{on } \partial\{u > 0\}, \tag{23}$$

where $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$, M is a positive constant and a_i are bounded. We will refer to this free boundary problem as Problem $\mathcal{P}$.

This free boundary problem arises in several contexts (cf. [14]). The most important motivation to date has come from combustion theory, where it appears as a limit situation in the description of the propagation of premixed equi-diffusional deflagration flames. In this case, u is the limit, as $\varepsilon \to 0$, of solutions u^ε to

$$\Delta u^\varepsilon + \sum a_i(x,t)\, u^\varepsilon_{x_i} - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon), \tag{24}$$

with $\varepsilon > 0$, $\beta_\varepsilon \geq 0$, $\beta_\varepsilon(s) = \frac{1}{\varepsilon}\beta(\frac{s}{\varepsilon})$, support β=[0,1] and $\int \beta(s)\, ds = M$. We call this equation $\mathcal{P}_\varepsilon$.

Problem $\mathcal{P}$ admits *classical* solutions only for good data and for small times. Different generalized concepts of solution have been proposed, among them the concepts of *limit* solution (that is, $u = \lim u^\varepsilon$) and *viscosity* solution, cf. [7], [6], resp. The purpose of [11] is to investigate conditions under which the three concepts agree and produce a unique solution.

The results in [11] can be summarized as saying that –under appropriate conditions– *if a classical solution of problem $\mathcal{P}$ exists, then it is at the same time the unique classical solution, the unique limit solution and also the unique viscosity solution.*

The results of [11] extend those in [10], where similar conclusions are obtained for the one phase version of this problem (i.e., under the assumption that $u \geq 0$).

One of the main results in [11] is Theorem 6.1, which gives simultaneously the uniqueness of *classical* and *limit* solution. The main tool in the proof of this theorem is the following basic result of [11]:

Theorem 5.1 *(Theorem 5.1 in [11]) Let $\Sigma \subset \mathbb{R}^{N-1}$ a bounded C^3 domain, $\Omega = (0,d) \times \Sigma$, $Q = \Omega \times (0,T)$, $\partial_N Q = (0,d) \times \partial\Sigma \times (0,T]$. Let w be a classical subsolution to $\mathcal{P}$ in Q, with $\frac{\partial w}{\partial \eta} = 0$ on $\partial_N Q$. Assume, in addition, that there exists $\delta_0 > 0$ such that*

$$|\nabla w^+|^2 - |\nabla w^-|^2 = 2M + \delta_0 \qquad \text{on } Q \cap \partial\{w > 0\}.$$

Then, there exists a family $v^\varepsilon \in C(\overline{Q})$, with $\nabla v^\varepsilon \in L^2_{loc}(\overline{Q})$, of weak subsolutions to $\mathcal{P}_\varepsilon$ in Q, with $\frac{\partial v^\varepsilon}{\partial \eta} = 0$ on $\partial_N Q$, such that, as $\varepsilon \to 0$, $v^\varepsilon \to w$ uniformly in $\overline{Q}$.

For the precise hypotheses and definitions, and detailed proofs of these results, we refer the reader to [11].

The results of the present paper are needed in Theorem 5.1 for the construction of the family v^ε which is constructed as follows:

Let A be the constant in Lemma 4.1 of [11] and let $\varepsilon > 0$ be small.

Let $p_\varepsilon, q_\varepsilon \in C^1(\overline{\Sigma} \times [0,T])$ with $\nabla_{x'} p_\varepsilon, \nabla_{x'} q_\varepsilon \in C^{\alpha,\frac{\alpha}{2}}(\overline{\Sigma} \times [0,T])$ be such that $\{w > A\varepsilon\}$ is given by $x_1 < p_\varepsilon(x',t)$ and $\{w < -A\varepsilon\}$ is given by $x_1 > q_\varepsilon(x',t)$.

Let the domain be

$$\mathcal{D}^\varepsilon = \{(x,t) \in Q \,/\, p_\varepsilon(x',t) < x_1 < q_\varepsilon(x',t)\}.$$

Let w^ε be the solution to $\mathcal{P}_\varepsilon$ in $\mathcal{D}^\varepsilon$ with boundary conditions

$$w^\varepsilon(x,t) = \begin{cases} A\varepsilon & \text{on } x_1 = p_\varepsilon(x',t), \\ -A\varepsilon & \text{on } x_1 = q_\varepsilon(x',t), \end{cases}$$
$$\frac{\partial w^\varepsilon}{\partial \eta} = 0 \quad \text{on } \partial_N \mathcal{D}^\varepsilon := \partial \mathcal{D}^\varepsilon \cap \partial_N Q,$$

and initial datum $w_0^\varepsilon \in C^\alpha(\overline{\mathcal{D}}^\varepsilon \cap \{t=0\})$ (for the choice of the function w_0^ε we refer to [11]).

Then, the family v^ε is defined by

$$v^\varepsilon = \begin{cases} w & \text{in } \{|w| \geq A\varepsilon\}, \\ w^\varepsilon & \text{in } \mathcal{D}^\varepsilon. \end{cases}$$

Now we point out where the results in the present paper apply. From Theorem 4.1, there exists a unique solution $w^\varepsilon \in C^{\gamma,\frac{\gamma}{2}}(\overline{\mathcal{D}}^\varepsilon)$ with $\nabla w^\varepsilon \in C(\overline{\mathcal{D}}^\varepsilon \cap \{t > 0\}) \cap L^2(\mathcal{D}^\varepsilon)$. Moreover, since $w_0^\varepsilon \in C^{1+\alpha}$ in a subset of $\overline{\mathcal{D}}^\varepsilon \cap \{t=0\}$, further continuity of ∇w^ε can be derived from Proposition 3.2.

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Claudia Lederman
Departamento de Matemática, Facultad de Ciencias Exactas
Universidad de Buenos Aires
(1428) Buenos Aires - Argentina
e-mail: clederma@dm.uba.ar

Juan Luis Vazquez
Departamento de Matemáticas, Universidad Autónoma de Madrid
28049 Madrid - Spain
e-mail: juanluis.vazquez@uam.es

Noemi Wolanski
Departamento de Matemática, Facultad de Ciencias Exactas
Universidad de Buenos Aires
(1428) Buenos Aires - Argentina
e-mail: wolanski@dm.uba.ar