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A one-dimensional nonlinear degenerate elliptic equation *

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Abstract

We study the one-dimensional version of the Euler-Lagrange equation associated to finding the best constant in the Caffarelli-Kohn-Nirenberg inequalities. We give a complete description of all non-negative solutions which exist in a suitable weighted Sobolev space $\mathcal{D}_a^{1,2}(\Omega)$. Using these results we are able to extend the parameter range for the inequalities in higher dimensions when we consider radial functions only, and gain some useful information about the radial solutions in the N-dimensional case.

1 Introduction

The motivation of this paper are the following inequalities due to Caffarelli, Kohn and Nirenberg (see [3]): for some positive constants $C_{a,b}$

$$\left(\int_{\mathbb{R}^N} |x|^{-bp}|u|^p \, dx\right)^{2/p} \le C_{a,b} \int_{\mathbb{R}^N} |x|^{-2a}|\nabla u|^2 \, dx \tag{1}$$

holds for all $u \in C_0^\infty(\mathbb{R}^N)$, if and only if

$$\text{for } N \ge 2: \quad -\infty < a < \frac{N-2}{2}, \quad a \le b \le a+1, \quad \text{and } p = \frac{2N}{N-2(1+a-b)}, \tag{2}$$

(with the case $N = 2$ excluding $a = b$), and

$$\text{for } N = 1: \quad -\infty < a < -\frac{1}{2}, \quad a+\frac{1}{2} < b \le a+1, \quad \text{and } p = \frac{2}{-1+2(b-a)}. \tag{3}$$

Let $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$, with respect to the norm $||\cdot||_a$ induced by the inner product

$$(u, v) = \int_{\mathbb{R}^N} |x|^{-2a}\nabla u \cdot \nabla v \, dx. \tag{4}$$

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Then we see that (1) holds for $u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)$. Define the best embedding constants

$$S(a,b) = \inf_{u \in \mathcal{D}_a^{1,2}(\mathbb{R}^N)\setminus\{0\}} E_{a,b}(u), \tag{5}$$

where

$$E_{a,b}(u) = \frac{\displaystyle\int_{\mathbb{R}^N} |x|^{-2a} |\nabla u|^2 \, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |x|^{-bp} |u|^p \, dx\right)^{2/p}}. \tag{6}$$

The extremal functions for $S(a,b)$ are ground state solutions of the Euler equation

$$-div(|x|^{-2a}\nabla u) = |x|^{-bp}u^{p-1}, \quad u \geq 0, \quad \text{in } \mathbb{R}^N. \tag{7}$$

This equation is a prototype of more general nonlinear degenerate elliptic equations from some physical phenomena related to equilibrium of anisotropic continuous media which possibly are somewhere perfect insulators and somewhere are perfect conductors (e.g., [8]).

Note that the classical Sobolev inequality ($a = b = 0$) and the Hardy inequality ($a = 0,\ b = 1$) are special cases of (1), (see also generalizations in [14] by Lin). These inequalities have been studied by many authors. In [1], [16], the best constant and the minimizers for the Sobolev inequality ($a = b = 0$) were given by Aubin, and Talenti. In [13], Lieb considered the case $a = 0$, $0 < b < 1$ and gave the best constants and explicit minimizers. In [7], Chou and Chu considered the case $a \geq 0$ and gave the best constants and explicit minimizers. Also, Lions in [15] (for $a = 0$) , and Wang and Willem in [17] (for $a > 0$), have established the compactness of all minimizing sequences up to dilations. The symmetry of the minimizers has been studied in [13] and [7]. In fact, *all* nonnegative solutions in $\mathcal{D}_a^{1,2}(\mathbb{R}^N)$ for the corresponding Euler equation (7) are *radial solutions* and explicitly given ([1], [16], [13], [7]). This was established in [7], using a generalization of the moving plane method (e.g., [9], [2], [6]). The case $a < 0$ has been studied recently in [11], [4], [19]; in [5] we have studied the case $a < 0$ and discovered some new phenomena including symmetry breaking of the ground state solutions for certain values of the parameters a and b.

In this paper we concentrate on the case $N = 1$. Under conditions (3), equation (7) becomes

$$-(|x|^{-2a}u')' = |x|^{-bp}u^{p-1},\ u \geq 0, \text{ in } \Omega \subset \mathbb{R}, \tag{8}$$

with $\Omega = \mathbb{R}$, and more generally, we consider this equation in an open interval $\Omega \subset \mathbb{R}$ (possibly unbounded). We are interested in solutions in

$$\mathcal{D}_a^{1,2}(\Omega) := \overline{C_0^\infty(\Omega)}^{||\cdot||_a}.$$

This problem is of *critical case* in the sense that there is a family of dilations with *two* parameters that leave the problems invariant (see (11) in Section 2). This feature distinguishes the case $N = 1$ from the case $N \geq 2$, for the latter case a one-parameter family of dilations exists. Nevertheless, we are able to give a complete and detailed solution for the structure of the solutions of (8) for both the ground states and the bound states, as well as some interesting qualitative properties of solutions. First, we give the following which is a corollary of our main results.

Theorem 1.1 *Let a, b, p satisfy (3) with $b < a+1$. Then (8) has a ground state solution in $\mathcal{D}_a^{1,2}(\Omega)$ if and only if $0 \in \overline{\Omega}$ and Ω is unbounded, or $0 \notin \overline{\Omega}$ and Ω is bounded.*

Remark 1.2 *Due to the degeneracy, the ground state solutions given here may not be continuous at 0 and can be identically zero in a subinterval of Ω. In fact, when $\Omega = (-c, \infty)$ with $0 < c \leq \infty$, the ground state solution is positive in $[0, \infty)$ and equal to zero in $(-c, 0)$. This will be clear from the proof (see also Remarks 2.4, 4.2).*

On the other hand, as applications of the results for $N = 1$, especially of the results for $\Omega = \mathbb{R}_+$ (the half real line), we can expand the parameter range of the validity for inequalities (1) for $N \geq 3$ when we consider radially symmetric functions only. This generalizes a result due to Lieb ([13]) where he considered the case $a = 0$. More precisely, for $N \geq 1$ define $C_{0,R}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) \mid u \text{ is radial}\}$ and

$$\mathcal{D}_{a,R}^{1,2}(\mathbb{R}^N) = \overline{C_{0,R}^\infty(\mathbb{R}^N)}^{||\cdot||_a}$$

Theorem 1.3 *Inequality (1) holds for any $u \in \mathcal{D}_{a,R}^{1,2}(\mathbb{R}^N)$ if and only if*

$$a < \frac{N-2}{2}, \quad a - \frac{N-2}{2} < b \leq a+1 \quad \textit{and} \quad p = \frac{2N}{N-2(1+a-b)}.$$

Remark 1.4 *(i) The best constants in the above embeddings can be explicitly given (see (22)). (ii) We note that Theorem 1.3 shows that for $N \geq 3$, inequalities (1) hold for radial functions in a range substantially larger than (2) for a and b. For the case $a = 0$, this fact was noted in [10], and it was considered again by Lieb in [13] with a direct proof rather than reduction to the one dimensional case.*

The paper is organized as follows. In Section 2, we shall first recall some of our results from [5] and outline the methods there. In Section 3 we indicate how the methods can be used to study (8) on the interval $(0, \infty)$. Theorem 1.3 will also be proved in Section 3, involving studies of equation (8) on the interval $(0, \infty)$. The study can be carried out using ideas from our earlier work [5]. Finally in Section 4, we consider equation (8) in an interval other than the whole real line or the half real line. Theorem 1.1 follows from these results.

2 The whole real line

First, we need the following ODE

$$-v_{tt} + \lambda^2 v = v^{p-1}, \quad v > 0, \quad \text{in } \mathbb{R} \tag{9}$$

with $p > 2$. The only positive solutions which are in $H^1(\mathbb{R})$, are translates of

$$v(t) = \left(\frac{\lambda^2 p}{2}\right)^{1/(p-2)} \left(\cosh\left(\frac{p-2}{2}\lambda t\right)\right)^{-2/(p-2)}. \tag{10}$$

We describe briefly the results we obtained in [5], for the case $N = 1$. Any nonzero solution of (7), $u \in \mathcal{D}_a^{1,2}(\mathbb{R})$ is a critical point for the energy

$$E_{a,b}(u) = \frac{\displaystyle\int_{\mathbb{R}} |x|^{-2a}|u'|^2\,dx}{\left(\displaystyle\int_{\mathbb{R}} |x|^{-bp}|u|^p\,dx\right)^{2/p}}.$$

There is a two-sided dilation invariance of (8): for $(\tau_+, \tau_-) \in (0, \infty)^2$

$$u(x) \to u_{\tau_+,\tau_-}(x) = \begin{cases} \tau_+^{-\frac{1+2a}{2}} u(\tau_+ x) & x > 0 \\ \tau_-^{-\frac{1+2a}{2}} u(\tau_- x) & x < 0. \end{cases} \tag{11}$$

That is, if u is a solution of (8) then so is u_{τ_+,τ_-}. More generally, the energy functional $E_{a,b}(u)$ is invariant under these two-sided dilations.

To reveal the relation with equation (9), to a function $u \in \mathcal{D}_a^{1,2}(\mathbb{R})$, we associate a $\mathbb{R}^2$-valued function $\mathbf{w}(t) = (w_1(t_1), w_2(t_2))$ for $(t_1, t_2) \in \mathcal{C}$ with $\mathcal{C}$ being the union of two real lines $\mathbb{R} \cup \mathbb{R}$, where

$$\begin{aligned} u(x) &= (-x)^{(1+2a)/2} w_1(-\ln(-x)), && \text{for } x < 0, \\ u(x) &= x^{(1+2a)/2} w_2(-\ln x), && \text{for } x > 0, \end{aligned} \tag{12}$$

and $t_1 = -\ln(-x)$ for $x < 0$, and $t_2 = -\ln x$ for $x > 0$. Under the transformation (12) we have a Hilbert space isomorphism between $\mathcal{D}_a^{1,2}(\mathbb{R})$ and $H^1(\mathcal{C}, \mathbb{R}^2)$ (see [5] for details) and equation (8) is equivalent to the system of autonomous equations

$$-\mathbf{w}_{tt} + \left(\frac{1+2a}{2}\right)^2 \mathbf{w} = \nabla W(\mathbf{w}), \tag{13}$$

where $W(\mathbf{w}) = (|w_1|^p + |w_2|^p)/p$. Note that each of the two equations is the same as (9), where $\lambda = \frac{1+2a}{2}$.

Critical points of $E_{a,b}(u)$ on $\mathcal{D}_a^{1,2}(\mathbb{R})$ now correspond to critical points of a new energy functional on $H^1(\mathcal{C}, \mathbb{R}^2)$

$$F_{a,b}(\mathbf{w}) = \frac{\displaystyle\int_{\mathbb{R}} |\mathbf{w}_t|^2 + \left(\frac{1+2a}{2}\right)^2 |\mathbf{w}|^2\,dt}{\left(\displaystyle\int_{\mathbb{R}} pW(\mathbf{w})\,dt\right)^{2/p}}, \quad \mathbf{w} \in H^1(\mathcal{C}, \mathbb{R}^2).$$

Each of the two ODE's of (13) has the zero solution, and the only (positive) homoclinic solutions are translates of

$$v(t) = \left(\frac{(1+2a)^2}{4(1-2(1+a-b))}\right)^{\frac{1-2(1+a-b)}{4(1+a-b)}} \left(\cosh \frac{(1+2a)(1+a-b)}{1-2(1+a-b)} t\right)^{-\frac{1-2(1+a-b)}{2(1+a-b)}}. \tag{14}$$

The minimizers of $F_{a,b}(\mathbf{w})$ are achieved by $\mathbf{w}$, for which one of the two components w_1 or w_2 is identically zero and the other is a translate of $v(t)$ given above. Using v and $F_{a,b}$ we have

$$\begin{aligned} S(a,b) &= \frac{(-1-2a)^{2(b-a)}}{2^{2(1+a-b)}(-1+2(b-a))^{-1+2(b-a)}(1+a-b)^{2(1+a-b)}} \\ &\quad \times \left(\frac{\Gamma^2\left(\frac{1}{2(1+a-b)}\right)}{\Gamma\left(\frac{1}{1+a-b}\right)}\right)^{2(1+a-b)}. \end{aligned} \tag{15}$$

We observe that as $b \searrow a + \frac{1}{2}$, we obtain $S(a,b) \to -1-2a$. Note that when both w_1 and w_2 are nonzero and are (possibly different) translates of $v(t)$ in (14) we get the energy $F_{a,b}(\mathbf{w})$ to be higher

$$R(a,b) = 2^{2(1+a-b)} S(a,b),$$

which is the least energy in the radial class. On this energy level, there is a two parameter family of positive solutions, according to the two parameters that control by how much w_1 and w_2 are translated from (14).

Note that the two-sided dilations (11) correspond to the translations invariance of $\mathcal{C}$ for (13). Correspondingly, $u(x)$ defined in (12) is a two parameter family of solutions for (8), which possibly after a dilation given in (11) is radial in $\mathbb{R}$. Explicitly, $u_{\tau_+,\tau_-}(x)$ is equal to

$$\begin{cases} \tau_+^{\frac{1+2a}{2}} \left(\frac{(1+2a)^2}{1-2(1+a-b)}\right)^{\frac{1-2(1+a-b)}{4(1+a-b)}} \dfrac{x^{1+2a}}{[1+(\tau_+ x)^{\frac{2(1+2a)(1+a-b)}{1-2(1+a-b)}}]^{\frac{1-2(1+a-b)}{2(1+a-b)}}} & x>0 \\ \tau_-^{\frac{1+2a}{2}} \left(\frac{(1+2a)^2}{1-2(1+a-b)}\right)^{\frac{1-2(1+a-b)}{4(1+a-b)}} \dfrac{(-x)^{1+2a}}{[1+(-\tau_- x)^{\frac{2(1+2a)(1+a-b)}{1-2(1+a-b)}}]^{\frac{1-2(1+a-b)}{2(1+a-b)}}} & x<0. \end{cases} \tag{16}$$

Summarizing these, we have the following theorems from [5].

Theorem 2.1 *(Best constants and existence of ground states) Let a, b, p satisfy (3) with $b < a+1$. Then $S(a,b)$ is explicitly given in (15), and up to a dilation of the form (11) it is achieved at a function of the form (12) with either $w_1 = 0$ and w_2 given by (14) or vice versa. Consequently, the ground states for $S(a,b)$ are always nonradial.*

Theorem 2.2 *(Bound state solutions) Let a, b, p satisfy (3) with $b < a+1$. Then the only bound state solutions of (8) besides the ground state solutions are given by (16).*

Remark 2.3 *In [5] we also proved the nonexistence of extremal functions when $b = a+1$ for which $S(a, a+1) = \left(\frac{1+2a}{2}\right)^2$, as well as the asymptotic property of $S(a,b)$ as $b \to \left(a+\frac{1}{2}\right)^+$. Note that all solutions of (8), possibly after a dilation given in (11), satisfy the modified inversion symmetry $u(x) = |x|^{1+2a} u\left(\frac{x}{|x|^2}\right)$. This was also discovered in [5] for bound state solutions when $N \geq 2$.*

Remark 2.4 *Standard elliptic regularity arguments break down for problem (8) at $x = 0$ due to the existence of the weights. In fact, solutions of (8) need not even be continuous at $x = 0$. We observe that the ground states are never continuous and that the only bound states other than the ground states are continuous if and only if $\tau_+ = \tau_-$ in (16). By a direct computation we can see easily that the solutions that are continuous belong to $C^1(\mathbb{R})$ if and only if $b < \frac{(1+2a)^2}{4a}$.*

3 Half line domain

Recall from Section 1, for an open interval $\Omega \subset \mathbb{R}$ (possibly unbounded), we let

$$\mathcal{D}_a^{1,2}(\Omega) = \overline{C_0^\infty(\Omega)}^{||\cdot||_a}.$$

We follow the same idea from [5], as used in Section 2. Under the transformation

$$u(x) = x^{(1+2a)/2} v(-\ln x), \tag{17}$$

we still have a Hilbert space isomorphism between $\mathcal{D}_a^{1,2}(\Omega)$ and $H_0^1(\tilde{\Omega})$ where $\tilde{\Omega} \subset \mathcal{C}$ is the image of Ω. Especially when $\Omega = (0,\infty)$, $\tilde{\Omega}$ is $\mathbb{R}$, one of the two components of $\mathcal{C}$. In this section we look at the problem

$$-(|x|^{-2a}u')' = |x|^{-bp}u^{p-1}, \; u \geq 0, \; \; u \in \mathcal{D}_a^{1,2}(\Omega) \tag{18}$$

with $\Omega = (0,\infty)$. Under the above transformation, equation (18) becomes (9) with $\lambda = \frac{1+2a}{2}$. According to (10), after transformed back to Ω, the only solutions of (18) are

$$u_\tau(x) = \tau^{\frac{1+2a}{2}} \left(\frac{(1+2a)^2}{1-2(1+a-b)}\right)^{\frac{1-2(1+a-b)}{4(1+a-b)}} \frac{x^{1+2a}}{\left(1+(\tau x)^{\frac{2(1+2a)(1+a-b)}{1-2(1+a-b)}}\right)^{\frac{1-2(1+a-b)}{2(1+a-b)}}}. \tag{19}$$

Let $E_{a,b}(u,\Omega)$ be the restriction of $E_{a,b}(u)$ on $\mathcal{D}_a^{1,2}(\Omega)$. We have the following result.

Theorem 3.1 *Let a, b, p satisfy (3) with $b < a+1$. Then the best constant $\inf_{\mathcal{D}_a^{1,2}(\Omega)\setminus\{0\}} E_{a,b}(u,\Omega)$ is achieved by a family of functions given by (19). Moreover, all nontrivial bound state solutions of (18) are given by (19).*

As a consequence of the above result we can prove the inequality in Theorem 1.3 for radial functions in any space dimension.

Proof of Theorem 1.3: We work in the class of radial functions $u(|x|) = u(r)$ defined on $\mathbb{R}^N$. Denote by ω_{N-1} the area of the unit $N-1$ dimensional sphere in $\mathbb{R}^N$. Then

$$\int_{\mathbb{R}^N} |x|^{-2a}|\nabla u(x)|^2\,dx = \omega_{N-1}\int_0^\infty r^{N-1-2a}u'(r)^2\ dr, \tag{20}$$

and

$$\int_{\mathbb{R}^N} |x|^{-bp}|u(x)|^p\,dx = \omega_{N-1}\int_0^\infty r^{N-1-bp}|u(r)|^p\ dr. \tag{21}$$

If we denote

$$\bar a = a - \frac{N-1}{2},\ \text{and}\ \bar b = b - \frac{N-1}{2} + \frac{(1+a-b)(N-1)}{N},$$

then

$$p = \frac{2}{-1+2(\bar b - \bar a)}.$$

With a proof similar to Lemma 2.1 in [5], we have

$$\mathcal{D}^{1,2}_{a,R}(\mathbb{R}^N) = \overline{C^\infty_{0,R}(\mathbb{R}^N\setminus\{0\})}^{||\cdot||_a},$$

where $C^\infty_{0,R}(\mathbb{R}^N\setminus\{0\})$ is the space of radial, smooth functions with compact support in $\mathbb{R}^N\setminus\{0\}$. Since inequalities (1) hold for $\bar a + \frac{1}{2} < \bar b \le \bar a + 1$ for functions with compact support in $\mathbb{R}$, they also hold for functions with compact support in $(0,\infty)$. Hence, for b in the interval $\left(a - \frac{N-2}{2}, a+1\right]$, inequalities (1) still hold in the class of radial functions. In fact, with a, b, and p satisfying (2), if we denote by $R(a,b)$ the best constant for the embedding $\mathcal{D}^{1,2}_{a,R}(\mathbb{R}^N)$ into the weighted $L^p_b(\mathbb{R}^N)$, we have

$$R(a,b) = \omega_{N-1}^{\frac{p-2}{p}} S(\bar a, \bar b), \tag{22}$$

where $S(\bar a, \bar b)$ is given in (15).

When looking for solutions of (7) in $\mathcal{D}^{1,2}_{a,R}(\mathbb{R}^N)$, the results in $(0,\infty)$ solve the problem by symmetry reduction. Equation (7) becomes

$$-r^{-2a}u'' - (N-1-2a)r^{-2a-1}u' = r^{-bp}u^{p-1}. \tag{23}$$

Multiplying the reduced equation by r^{N-1}, we obtain

$$-(r^{N-1-2a}u')' = r^{N-1-bp}u^{p-1}. \tag{24}$$

Therefore, if $-\frac{N-2}{2} < b - a < 1$, the only positive radial solutions are those in (19) corresponding to $\bar a$ and $\bar b$. $\square$

4 Domains other than $\mathbb{R}$ or $\mathbb{R}_+$

In this section we look at problem (18), with Ω different from $\mathbb{R}$ or $\mathbb{R}_+$. We divide the section in three cases according to the position of zero, relative to the interval Ω.

A. We consider $0 \notin \overline{\Omega}$.

(i) The first subcase is Ω is bounded. From (12), $\tilde{\Omega}$ is a bounded interval in one of the two components of $\mathcal{C}$. It is well-known that (9) on such an interval with Dirichlet boundary condition, has a unique positive solution (see e.g. [18], [12]).

(ii) In the second subcase, Ω is unbounded which corresponds to $\tilde{\Omega}$ being of the form $(-\infty, m)$ or (m, ∞) for a finite m. We use a simple Pohazaev type argument to show that problem (18) has no solution in $\mathcal{D}_a^{1,2}(\Omega)$. For convenience, assume $\Omega \subset (0, \infty)$. Performing the transformation

$$u(x) = x^{(1+2a)/2} v(-\ln x), \tag{25}$$

(denote $t = -\ln x$), equation (18) is equivalent to

$$-v_{tt} + \left(\frac{1+2a}{2}\right)^2 v = v^{p-1}, \quad v(t) > 0 \text{ on } (-\infty, m), \text{ and } v(m) = 0,$$

where m is some real number. Assuming there is a solution $v(t)$, we multiply by $v_t(t)$ and integrate from $s \in (-\infty, m)$ to m. We get,

$$v_t^2(m) = v_t^2(s) - \left(\frac{1+2a}{2}\right)^2 v^2(s) + \frac{2}{p} v^p(s).$$

Integrate the equality above again, from t to m, to obtain

$$(m-t) v_t^2(m) = \int_t^m v_t^2(s) - \left(\frac{1+2a}{2}\right)^2 v^2(s) + \frac{2}{p} v^p(s)\, ds. \tag{26}$$

Since $u \in \mathcal{D}_a^{1,2}(\Omega)$ we have that $v \in H_0^1(-\infty, m)$ and also in $L^p(-\infty, m)$. In equality (26), let $t \to -\infty$. The right hand side is bounded and this implies $v_t(m) = 0$, hence v is identically zero, which provides the necessary contradiction.

B. The second case is when zero is one of the endpoints of Ω.

(i) If Ω is bounded, then after the transformation (25) we have the same situation as in A(ii), hence there is no solution.

(ii) If Ω is unbounded, we are in the case of the half line treated in Section 3.

C. Finally, the case when $0 \in \Omega$.

(i) If Ω is bounded, we have no solution by the argument in A(ii), carried out on each of the two components of $\Omega \setminus \{0\}$.

(ii) If Ω is unbounded, and not equal to $\mathbb{R}$, then $\Omega \setminus \{0\}$ has one component equal to the half line and the other one being a bounded segment. This translates into $\tilde{\Omega}$ being the union of a copy of $\mathbb{R}$ and an unbounded interval of $\mathbb{R}$. By using a combination of arguments above we conclude that the only nontrivial solution is (19) on the half line, and zero on the bounded segment.

Summarizing the above we have the following complete solution for (8) as we have treated the cases of the whole real line and the half real line in Sections 2 and 3 respectively.

Theorem 4.1 *Let a, b, p satisfy (3) with $b < a + 1$.*

(i) Assume Ω is not equal to $\mathbb{R}$ or $\mathbb{R}_+$ and $0 \in \overline{\Omega}$. If Ω is bounded, (18) has no solutions at all. If Ω is unbounded (18) has a unique solution (up to a dilation) which is the ground state, ann it is zero on the bounded component of $\Omega \setminus \{0\}$.

(ii). Assume $0 \notin \overline{\Omega}$. If Ω is bounded, (18) has a unique solution which is the ground state. If Ω is unbounded (18) has no solution at all.

Now, Theorem 1.1 follows from Theorems 2.1, 3.1, and 4.1.

Remark 4.2 *We comment here that a result in the spirit of Theorem 1.1 was given for the case $b = 0, -1 < a < -\frac{1}{2}$ in [4] (p.386, Theorem 4.1) in which it was claimed that (18) has a ground state solution in $\mathcal{D}_a^{1,2}(\Omega)$ if and only if $\Omega = \mathbb{R}$, $\mathbb{R}_\pm$, or Ω is bounded with $0 \notin \overline{\Omega}$. Here we have shown that (18) also has ground state solutions in $\mathcal{D}_a^{1,2}(\Omega)$ when $\Omega = (-c, \infty)$ and $(-\infty, c)$ for any $c \geq 0$. The reason is that due to the degeneracy the maximum principle does not hold in intervals that contain 0. Therefore there are nonzero nonnegative solutions which are identically zero in a subinterval.*

Remark 4.3 *For completeness, we remark that for $b = a + 1$ we have $p = 2$, and (18) becomes a linear problem:*

$$-(|x|^{-2a}u')' = |x|^{-2(a+1)}u, \ u \geq 0, \ \text{ in } \Omega = (\alpha, \beta) \subset \mathbb{R}, \tag{27}$$

Using transformation (12) and analyzing the resulting equation we can easily conclude that (27) has a nonzero solution if and only if $-\frac{3}{2} < a < -\frac{1}{2}$, $0 \notin \Omega$ and $\frac{\beta}{\alpha} = exp(\pm\frac{2\pi}{\sqrt{4-(1+2a)^2}})$ where $\pm$ depends on the sign of β. We leave the details to reader.

References

[1] T. Aubin, *Problèmes isopérimétriques de Sobolev,* J. Diff. Geom., **11**, (1976), 573 - 598.

[2] L. A. Caffarelli, B. Gidas, J. Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth,* Comm. Pure Appl. Math., **42**, (1989), 271 - 297.

[3] L. A. Caffarelli, R. Kohn, L. Nirenberg, *First order interpolation inequalities with weights,* Compositio Mathematica, **53**, (1984), 259 - 275.

[4] P. Caldiroli, R. Musina, *On the existence of extremal functions for a weighted Sobolev embedding with critical exponent,* Cal. Var. and PDEs, **8**, (1999), 365 - 387.

[5] F. Catrina, Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg Inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal functions,* Comm. Pure Appl. Math., to appear.

[6] W. Chen, C. Li, *Classification of solutions of some nonlinear elliptic equations,* Duke Math. J., **63**, (1991), 615 - 622.

[7] K. S. Chou, C. W. Chu, *On the best constant for a weighted Sobolev-Hardy inequality,* J. London Math. Soc., **2**, (1993), 137 - 151.

[8] R. Dautray, J.-L. Lions, *Mathematical analysis and numerical methods for science and technology*, Vol. 1, Springer-Verlag, Berlin, 1985.

[9] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* R^n, Adv. Math., Suppl. Studies **7A**, (1981), Academic Press, New York, 369 - 402.

[10] V. Glaser. A. Martin, H. Grosse, W. Thirring, *A family of optimal conditions for the absence of bound states in a potential*, Studies in Mathematical Physics, Princeton University Press, (1976), 169 - 194.

[11] T. Horiuchi, *Best constant in weighted Sobolev inequality with weights being powers of distance from the origin*, J. Inequal. and Appl. **1**, (1997), 275 - 292.

[12] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* R^n*,* Arch. Rat. Mech. Anal., **105**, (1989), 243 - 266.

[13] E. H. Lieb, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities,* Annals of Math., **118**, (1983), 349 - 374.

[14] C. S. Lin, *Interpolation inequalities with weights,* Comm. PDEs, **11**, (1986), 1515 - 1538.

[15] P.-L. Lions, *Concentration compactness principle in the calculus of variations. The limit case. Part 1,* Rev. Mat. Ibero., **1.1**, (1985), 145 - 201.

[16] G. Talenti, *Best constant in Sobolev inequality,* Ann. Mat. Pura Appl., **110**, (1976), 353 - 372.

[17] Z.-Q. Wang, M. Willem, *Singular minimization problems,* J. Diff. Equ., **161**, (2000), 307 - 320.

[18] M. Willem, *Minimax Theorems,* Birkhäuser, Boston, (1996).

[19] M. Willem, *A decomposition lemma and critical minimization problems*, preprint.

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