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Nonlinearities in a second order ODE *

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Abstract

In this paper we study the semilinear second order ordinary differential equation

$$u'' + r(t)u' + g(t,u) = f(t)\,.$$

Under a growth condition on g, we prove the existence and uniqueness for the Dirichlet problem and establish conditions for the existence of periodic solutions.

1 Introduction

The two-point boundary-value problem for a semilinear second order ODE

$$u'' + ru' + g(t,u) = 0, \quad u(0) = u_0, \quad u(T) = u_T$$

has been studied by many authors. In his pioneering work, Picard [7] proved the existence of a solution by an application of the well known method of successive approximations under a Lipschitz condition on g and a smallness condition on T. Sharper results were obtained by Hamel [2] in the special case of a forced pendulum equation (see also [4], [5]). The existence of periodic solutions for this equation was first considered by Duffing [1] in 1918. In the absence of friction (i.e. $r = 0$), variational methods have been applied by Lichtenstein [3], who considered the functional

$$I(u) = \int_0^T \frac{(u')^2}{2} - G(t,u(t))dt,$$

where $G(t,x) = \int_0^x g(t,s)ds$. Finally, we want to mention the topological approach introduced in 1905 by Severini [8] who used a shooting method. He also presented and gave a survey of results obtained using Leray-Schauder techniques and degree theory. For further results, see [6].

In this work, we prove the existence and uniqueness of a solution to the Dirichlet problem under a growth condition on g. Then, we apply this result for finding periodic solutions.

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Let $S : H^2(0,T) \to L^2(0,T)$ be the semilinear operator given by

$$Su = u'' + ru' + g(t,u)\,.$$

Assume that the function g satisfies the growth condition

$$\frac{g(t,u)-g(t,v)}{u-v} \le \frac{c_p}{p(t)} \quad \text{for } t\in[0,T] \text{ and } u,v\in\mathbb{R} \quad (u\neq v)\,, \tag{1.1}$$

where $p \in C^1([0,T])$ is strictly positive, $r_0 := pr - p' \in H^1(0,T)$ is non-decreasing, and $c_p < \lambda_p$ with λ_p the first eigenvalue of the problem

$$-(pu')' = \lambda_p u, \quad u(0) = u(T) = 0\,.$$

To state a general existence and uniqueness result for the Dirichlet problem associated to our equation, we need the following apriori bounds.

Lemma 1.1 *Assume that g satisfies* (1.1) *and let $u, v \in H^2(0,T)$ with* $\mathrm{Tr}(u) = \mathrm{Tr}(v)$. *Then*

$$\|p(Su - Sv)\|_2 \ge (\lambda_p - c_p)\|u - v\|_2$$

and

$$\|p(Su - Sv)\|_2 \ge \frac{\lambda_p - c_p}{\sqrt{\lambda_p}}\Big(\int_0^T p(u'-v')^2\Big)^{1/2}$$

Proof. A simple computation shows that

$$\|p(Su-Sv)\|_2\|u-v\|_2 \ge \int_0^T p(u'-v')^2 - \int_0^T r_0(u-v)(u'-v') - c_p\|u-v\|_2^2$$

and because $-\int_0^T r_0(u-v)(u'-v') = \frac{1}{2}\int_0^T r_0'(u-v)^2 \ge 0$, the result follows since $\|u-v\|_2^2 \le \frac{1}{\lambda_p}\int_0^T p(u'-v')^2$. ♢

Remarks i) For simplicity and by the previous lemma, we may denote by k_1 the best constant such that $\|u-v\|_{1,2} \le k_1\|p(Su-Sv)\|_2$ for $u, v \in H^2(0,T)$ with $\mathrm{Tr}(u) = \mathrm{Tr}(v)$.
ii) In particular, if $r \in H^1(0,T)$ is non-decreasing, the result holds for $p \equiv 1$ and $c_1 < \lambda_1 = \left(\frac{\pi}{T}\right)^2$.

Theorem 1.2 *Let g satisfy* (1.1). *Then the Dirichlet problem*

$$\begin{aligned} Su &= f(t) \quad \text{in } (0,T) \\ u(0) &= u_0, \quad u(T) = u_T \end{aligned} \tag{1.2}$$

is uniquely solvable in $H^2(0,T)$ for any $f \in L^2(0,T)$ and arbitrary boundary data.

Proof. Without loss of generality, we may suppose that $p \equiv 1$. For $0 \leq \sigma \leq 1$ we consider the operator S_σ given by $S_\sigma u := u'' + ru' + \sigma g(t,u)$. We remark that if k_σ is the constant of lemma 1.1 for S_σ, then $k_\sigma \leq k_1$.

From the theory of linear operators, for fixed $\overline{u} \in H^1(0,T)$ we may define $u = K\overline{u}$ as the unique solution of the problem

$$\begin{aligned} S_0 u = f(t) - g(t,\overline{u}) \quad &\text{in } (0,T) \\ u(0) = u_0, \quad u(T) = u_T\,. \end{aligned}$$

Continuity of $K : H^1(0,T) \to H^1(0,T)$ follows immediately from the inequality

$$\|K\overline{u} - K\overline{v}\|_{1,2} \leq k_1 \|S_0(K\overline{u}) - S_0(K\overline{v})\|_2 = k_1 \|g(\cdot,\overline{u}) - g(\cdot,\overline{v})\|_2$$

and the fact that $\|g(\cdot,\overline{u}) - g(\cdot,\overline{v})\|_2 \to 0$ for $\overline{u} \to \overline{v}$ in $H^1(0,T) \hookrightarrow C([0,T])$. Moreover, if $\varphi(t) = \frac{u_T - u_0}{T} t + u_0$ we have that

$$\|K\overline{u} - \varphi\|_{1,2} \leq k_1 \|f - g(\cdot,\overline{u}) - S_0\varphi\|_2 \leq C$$

for some constant $C = C(R)$. Moreover, as

$$\|(K\overline{u})''\|_2 = \|f - g(\cdot,\overline{u}) - r(K\overline{u})'\|_2$$

it follows that $K(B_R)$ is H^2-bounded. Thus, by the compactness of the imbedding $H^2(0,T) \hookrightarrow H^1(0,T)$ we conclude that K is compact.

Let us assume that $u = \sigma K u$ for some $\sigma \in (0,1]$. Then $u'' + ru' + \sigma g(t,u) = \sigma f$, and

$$\|u - \sigma\varphi\|_{1,2} \leq k_1 \|S_\sigma u - S_\sigma(\sigma\varphi)\|_2 = k_1 \|\sigma f - S_\sigma(\sigma\varphi)\|_2$$

This proves that the set $\{u : u = \sigma K u\}$ is uniformly bounded, and by Leray-Schauder theorem K has a fixed point. Uniqueness of the solution follows from lemma 1.1. ♢

As a simple consequence, we have an existence result for the general Dirichlet problem

$$\begin{aligned} Su = f(t,u,u') \quad &\text{in } (0,T) \\ u(0) = u_0, \quad u(T) = u_T \end{aligned} \tag{1.3}$$

Corollary 1.3 *Let f be continuous and g satisfy* (1.1). *Assume that the growing condition*

$$|f(t,u,x)| \leq c|(u,x)| + d \tag{1.4}$$

holds for some constant $c < \frac{1}{k_1\|p\|_\infty}$. Then (1.3) *is solvable in $H^2(0,T)$.*

Proof. By (1.4) and the previous theorem, the operator $K : H^1(0,T) \to H^1(0,T)$ given by $K\overline{u} = u$, with u the unique solution of

$$Su = f(t, \overline{u}, \overline{u}') \quad \text{in } (0,T)$$
$$u(0) = u_0, \quad u(T) = u_T$$

is well defined and compact. Moreover, as

$$\|K\overline{u} - \varphi\|_{1,2} \leq k_1 \|p(S(K\overline{u}) - S\varphi)\|_2 \leq k_1 \|p\|_\infty (\|S\varphi)\|_2 + c\|\overline{u}\|_{1,2} + d)$$

then $K(B_R) \subset B_R$ for R large and the result follows from Schauder Theorem.

2 Solutions to the periodic problem

In this section we'll apply the previous results to the periodic problem

$$\begin{aligned} Su = f(t) \quad &\text{in } (0,T) \\ u(0) = u(T), \quad &u'(0) = u'(T) \end{aligned} \tag{2.1}$$

It is well known that the forced pendulum equation $u'' + b\sin(u) = f$ admits periodic solutions for constant b if f is periodic and *orthogonal to constants*. We'll show in the general case that in the presence of friction this orthogonality condition can be reinterpreted in terms of a certain $p_1 > 0$. More precisely, we'll show that in some cases -including the generalized pendulum equation- (2.1) is not solvable for any f such that $\langle p_1, f \rangle$ is large enough.

Lemma 2.1 *For any $c \in \mathbb{R}$ there exists a unique p_c such that $p_c(0) = p_c(T) = c$ and $p_c' - rp_c$ is constant. Furthermore, $p_c = cp_1$, and p_1 is strictly positive.*

Proof. From the equation $p_c' - rp_c = k_c$ we obtain that

$$p_c = \left(c + k_c \int_0^t e^{-\int_0^s r} ds\right) e^{\int_0^t r}$$

and from the condition $p_c(0) = p_c(T) = c$ we conclude that

$$k_c = c\frac{1 - e^{\int_0^T r}}{\int_0^T e^{\int_s^T r} ds} = ck_1$$

Thus, $p_c = cp_1$. Moreover, if $k_1 \geq 0$ it's immediate that $p_1 > 0$, and if $k_1 < 0$, assuming that p_1 vanishes there exists $t_0 \in (0,T)$ such that $p_1(t_0) = 0 \leq p_1'(t_0)$. Then $k_1 = p_1'(t_0) \geq 0$, a contradiction. ♢

Using the preceding lemma we'll see that periodic solutions of $Su = f$ satisfy an orthogonality condition. Indeed, from

$$u'' + ru' + g(t,u) = f$$

we obtain

$$(p_1u')' - k_1u' + p_1g(t,u) = p_1f\,.$$

By the equality $p_1u'\Big|_0^T = u\Big|_0^T = 0$ we have

$$\int_0^T p_1g(t,u) = \int_0^T p_1f\,.$$

Corollary 2.2 *With the previous notation, let us assume that $g(t,u) \le g_{max}$ for any $t \in [0,T]$, $u \in \mathbb{R}$ and some constant $g_{max} \in \mathbb{R}$ (respectively, $g(t,u) \ge g_{min}$ for any $t \in [0,T]$, $u \in \mathbb{R}$ and some constant $g_{min} \in \mathbb{R}$). Then* (2.1) *is not solvable for any $f \in L^2(0,T)$ such that $\langle p_1, f\rangle > g_{max}\|p_1\|_1$ (resp. $\langle p_1, f\rangle < g_{min}\|p_1\|_1$).*

Now we'll give some existence results for (2.1), assuming that g satisfies (1.1). Our method is based in the existence and uniqueness result given by Theorem 1.2: indeed, for fixed $s \in \mathbb{R}$ we may define u_s as the unique solution of the problem

$$\begin{aligned} Su &= f(t) \quad \text{in } (0,T) \\ u(0) &= u(T) = s \end{aligned}$$

Lemma 2.3 *The mapping $s \to u_s$ is continuous for the H^1-norm.*

Proof. For $s \to s_0$ and $w_s = u_s - u_{s_0}$ we have

$$\begin{aligned} 0 &= \int_0^T p(Su_s - Su_{s_0})w_s \\ &\le pw_s'w_s\Big|_0^T - \int_0^T p(w_s')^2 + \frac{r_0w_s^2}{2}\Big|_0^T - \int_0^T r_0'\frac{w_s^2}{2} + c_p\int_0^T w_s^2 \end{aligned}$$

Because $\int_0^T r_0'\frac{w_s^2}{2} \ge 0$, we conclude that

$$0 \le (1 - \frac{c_p}{\lambda_p})\int_0^T p(w_s')^2 \le pw_s'w_s\Big|_0^T + r_0\frac{w_s^2}{2}\Big|_0^T.$$

Since $w_s(0) = w_s(T) = s - s_0 \to 0$ it suffices to prove that $\|w_s\|_{1,\infty}$ is bounded. As $\|u_s - s\|_{1,2} \le k_1\|p(f - g(\cdot,s))\|_2$, we deduce that w_s is H^1-bounded. Moreover, from the equality $u_s'' = f - ru_s' - g(t,u_s)$ we obtain that $\|w_s\|_{2,2}$ is bounded, and from the imbedding $H^2(0,T) \hookrightarrow C^1([0,T])$ the proof is complete. ◇

From the previous remarks, the solvability of (2.1) is equivalent to the solvability of the equation $\psi(s) = \int_0^T p_1f$, where $\psi : \mathbb{R} \to \mathbb{R}$ is given by $\psi(s) = \int_0^T p_1g(t,u_s)$. Continuity of ψ follows immediately from the previous lemma, and hence (2.1) will admit a solution if and only if there exist $s^{\pm}$ such that

$$\psi(s^+) \ge \langle p_1, f\rangle \ge \psi(s^-)$$

Remark. Writing $u_s(t) - s = \int_0^t p^{-1/2} p^{1/2} u_s'$ we obtain that

$$\|u_s - s\|_\infty \leq \delta_p \|p(f - g(\cdot, s))\|_2$$

for $\delta_p = \left(\int_0^T \frac{1}{p}\right)^{1/2} \frac{\sqrt{\lambda_p}}{\lambda_p - c_p}$.

Thus, if we consider the condition

$$\|pg(\cdot, s)\|_2 \leq c|s| + d \quad \text{with } c\delta_p < 1 \tag{2.2}$$

then $u_s(t) \in \mathcal{J}_s^\varepsilon$ for any $t \in [0, T]$, where $\mathcal{J}_s^\varepsilon$ is the interval centered in s with radius $\delta_p(c|s| + d) + \varepsilon$, $\varepsilon = \delta_p \|pf\|_2$. As a simple consequence we have the following

Theorem 2.4 *Let g satisfy* (1.1)-(2.2), *and assume that there exist $s^\pm$ such that*

$$g|_{[0,T]\times\mathcal{J}^\varepsilon_{s^+}} \geq \frac{\int_0^T p_1 f}{\|p_1\|_1} \geq g|_{[0,T]\times\mathcal{J}^\varepsilon_{s^-}}$$

for $\varepsilon = \delta_p \|pf\|_2$. Then (2.1) *admits a solution u_s for some s between s^- and s^+.*

In particular, if there exist $s^\pm$ such that

$$g|_{[0,T]\times\mathcal{J}^\varepsilon_{s^+}} \geq 0 \geq g|_{[0,T]\times\mathcal{J}^\varepsilon_{s^-}}$$

then (2.1) *admits a solution u_s for some s between s^- and s^+ for any $f \perp p_1$ such that $\delta_p \|pf\|_2 \leq \varepsilon$.*

Proof. As $u_s^\pm([0,T]) \subset \mathcal{J}^\varepsilon_{s^\pm}$, we obtain:

$$\int_0^T p_1 g(t, u_{s^+}) \geq \int_0^T p_1 f \geq \int_0^T p_1 g(t, u_{s^-})$$

and the result holds. ♢

Using the fact that $|s| - \delta_p(c|s| + d) \to +\infty$ we deduce the following existence results:

Corollary 2.5 *Let g satisfy* (1.1)-(2.2), *and assume, for some $M > 0$ that*

$$g(t, x) sg(x) \geq 0 \text{ for } \quad |x| \geq M$$

or

$$g(t, x) sg(x) \leq 0 \text{ for } \quad |x| \geq M$$

Then (2.1) *is solvable for any $f \perp p_1$.*

Corollary 2.6 *Let g satisfy* (1.1)-(2.2), *and assume that*

$$\lim_{|x|\to+\infty} g(t, x) sg(x) = +\infty \quad \text{or} \quad \lim_{|x|\to+\infty} g(t, x) sg(x) = -\infty$$

uniformly on t. Then (2.1) *is solvable for any f.*

Proof. Under the first assumption, there exists M such that

$$g(t,x)sg(x) \geq \frac{|\int_0^T p_1 f|}{\|p_1\|_1} \quad \text{for } |x| \geq M$$

Hence, taking $s > 0$ such that $s - \delta_p(cs + d + \|pf\|_2) \geq M$ we have

$$\int_0^T p_1 g(t, u_s) \geq |\int_0^T p_1 f| \geq \int_0^T p_1 f\,.$$

In the same way, for $s < 0$ with $s + \delta_p(-cs + d + \|pf\|_2) \leq -M$ we obtain $\int_0^T p_1 g(t,u_s) \leq \int_0^T p_1 f$ and the proof is complete. The case $g(t,x)sg(x) \to -\infty$ is analogous. ◇

Remark. In the previous corollaries (2.5)-(2.6), we also have that all the solutions belong to a compact arc of $H^1(0,T)$, namely $\{u_s : -S \leq s \leq S\}$ with

$$S = \frac{M + \delta_p(d + \|pf\|_2|)}{1 - \delta_p c}\,.$$

We may also apply theorem (2.4) to the forced pendulum equation with friction

$$u'' + ru' + b\sin(u) = f\,. \tag{2.3}$$

We first remark that in this case condition (1.1) reads

$$|b(t)| \leq \frac{c_p}{p(t)} \quad \text{for any } t \in [0,T] \tag{2.4}$$

for some $p > 0$ with $pr - p'$ nondecreasing and $c_p < \lambda_p$.

Theorem 2.7 *With the previous notation, let us assume that*
i) b *satisfies* (2.4) *and does not vanish in* $(0,T)$.
ii) $\|p(f \pm b)\|_2 \leq \frac{c}{\delta_p}$ *for some* $c < \frac{\pi}{2}$.
iii) $|\int_0^T p_1 f| \leq cos(c)\|p_1 b\|_1$
Then there exist $s_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $s_2 \in [\frac{\pi}{2}, \frac{3}{2}\pi]$ *such that* $u_{s_i} + 2k\pi$ *is a periodic solution of* (2.3) *for any integer* k.

Proof. From the previous computations for $s = \frac{\pi}{2} + k\pi$ we obtain that

$$\|u_s - s\|_\infty \leq \delta_p \|p(f - (-1)^k b\|_2 \leq c < \frac{\pi}{2}$$

As $\sin u_s = (-1)^k \cos(u_s - s)$, taking k such that $(-1)^k b > 0$ we conclude that

$$\int_0^T p_1 b \sin u_s = \int_0^T p_1 |b| \cos(u_s - s) \geq cos(c)\|p_1 b\|_1 \geq \int_0^T p_1 f$$

In the same way, for $s = \frac{\pi}{2} + (k \pm 1)\pi$

$$\int_0^T p_1 b \sin u_s = -\int_0^T p_1 |b| \cos(u_s - s) \leq -cos(c)\|p_1 b\|_1 \leq \int_0^T p_1 f$$

and the result holds. $\diamondsuit$

Remark. In particular, condition iii) is fulfilled if f is orthogonal to p_1.

If we assume that $\|pf\|_2 \leq \frac{\pi}{2\delta_p}$ we also obtain existence under slightly different conditions.

Theorem 2.8 *With the previous notation, let us assume that*
i) b satisfies (2.4) *and does not vanish in* $(0,T)$
ii) $\|pf\|_2 \leq \frac{\pi}{2\delta_p}$, $\|p(f - |b|)\|_2 < \frac{c}{\delta_p}$ *for some* $c < \frac{\pi}{2}$.
iii) $\sin(\delta_p \|pf\|_2) \leq \frac{\int_0^T p_1 f}{\|p_1 b\|_1} \leq \cos(\delta_p \|p(f - |b|)\|_2)$.
Then, if $b > 0$ (resp. $b < 0$) there exist $s_1 \in [0, \frac{\pi}{2}]$, $s_2 \in [\frac{\pi}{2}, \pi]$ *(resp.* $s_1 \in [-\frac{\pi}{2}, 0]$, $s_2 \in [\pi, \frac{3}{2}\pi]$*) such that* $u_{s_i} + 2k\pi$ *is a periodic solution of* (2.3) *for any integer* k.
Moreover, if we replace ii) and iii) by
ii') $\|pf\|_2 \leq \frac{\pi}{2\delta_p}$, $\|p(f + |b|)\|_2 < \frac{c}{\delta_p}$ *for some* $c < \frac{\pi}{2}$.
iii') $\sin(\delta_p \|pf\|_2) \leq \frac{-\int_0^T p_1 f}{\|p_1 b\|_1} \leq \cos(\delta_p \|p(f + |b|)\|_2)$
then if $b < 0$ (resp. $b > 0$) there exist $s_1 \in [0, \frac{\pi}{2}]$, $s_2 \in [\frac{\pi}{2}, \pi]$ *(resp.* $s_1 \in [-\frac{\pi}{2}, 0]$, $s_2 \in [\pi, \frac{3}{2}\pi]$*) such that* $u_{s_i} + 2k\pi$ *is a periodic solution of* (2.3) *for any integer* k.

Proof. It follows like in the previous theorem, using the fact that if $s = k\pi$ then $\|u_s - s\|_\infty \leq \delta_p \|pf\|_2$, and

$$|\int_0^T p_1 b \sin u_s| \leq \|p_1 b\|_1 \sin(\delta_p \|pf\|_2)\,.$$

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