Nonlinear Differential Equations,
Electron. J. Diff. Eqns., Conf. 05, 2000, pp. 285–298
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Long-time asymptotics for the damped Boussinesq equation in a disk *

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Abstract

For the damped Boussinesq equation the first initial-boundary value problem is considered in a unit disk. Its strong solution is constructed in the form of a series in the small parameter present in the initial conditions. The global-in-time solvability follows from the construction. The first-order long-time asymptotics is calculated with the uniform in space estimate of the remainder.

1 Introduction

In 1872 J. Boussinesq [2] derived an equation describing the propagation of small amplitude, long waves on the surface of shallow water. He was the first to give a scientific explanation of the existence of solitary waves, or solitons, discovered in 1834 by Scott Russell (see [4]). The classical Boussinesq equation can be wrtitten as

$$u_{tt} = -\alpha u_{xxxx} + u_{xx} + \beta (u^2)_{xx}, \tag{1.1}$$

where $u(x,t)$ is an elevation of the free surface of fluid and the constant coefficients α and β depend on the depth of fluid and the characteristic speed of long waves. Equation (1.1) with $\alpha < 0$, as derived by Boussinesq, is called the "bad" Boussinesq equation, while (1.1) with $\alpha > 0$ received the name of the "good" Boussinesq one. The latter is linearly stable and governs small nonlinear oscillations of an elastic beam.

In spite of the fact that (1.1) was deduced earlier than the Korteweg-de Vries one, the mathematical theory for it is not as complete as for the latter [1]. Both of them model nonlinear wave propagation, but the Boussinesq equation is of the second order in time and desribes waves travelling in both directions.

Equation (1.1) takes into account dispersion and nonlinearity, but in real processes viscosity also plays an important rôle [1]. Therefore, it is interesting to consider the equation

$$u_{tt} - 2bu_{txx} = -\alpha u_{xxxx} + u_{xx} + \beta (u^2)_{xx}, \tag{1.2}$$

* *Mathematics Subject Classifications:* 35K55, 35B40.
Key words: damped Boussinesq equation, boundary-value problem, long-time asymptotics.

Published October 25, 2000.

where the second term on the left-hand side is responsible for strong internal damping [3]. Here α and b are positive constants, and β is constant in $\mathbb{R}$. In the papers [7, 8] Cauchy and spatially periodic problems have been studied for Eq. (1.2), and the long-time asymptotics of their global in time solutions have been obtained. In [9, 10] the method applied has been developed further and adapted for solving spatially 1-D boundary value problems. In [11] a radially symmetric boundary value problem for (1.2) in a unit disk has been examined with "small" initial conditions, homogeneous boundary ones, and periodicity conditions in the angle. The study of the long-time behavior of its solutions can be considered a direct continuation of [9], where the first boundary value problem for (1.2) on an interval has been considered. Passing from an interval to a circle immediatley leads to the effect of the "loss of smoothness", i.e., the increase of smoothness of the intial data does not lead to the improvement of the regularity of the solution. This is a result of the combined influence of the nonlinearity and the circular geometry. The main tool for solving the problem in a disk is the Fourier-Bessel series, and the major difficulty in its application consists in comparatively poor convergence of this series. This comparison is made with the Fourier series used for solving the 1-D problem in [9] .

In the present paper we shall consider the first initial-boundary value problem for the generalization of (1.2) in a disk in the general spatially 2-D case. We shall show how the 2-D circular geometry permits one to improve the smoothness of the constructed solution via imposing more periodicity conditions. The "loss of smoothness" still takes place, but it is only partial.

2 Statement of the problem and auxiliary results

Using polar coordinates we can pose the problem as follows:

$$\begin{gathered} u_{tt} - 2b\Delta u_t = -\alpha\Delta^2 u + \Delta u + \beta\Delta(u^2), \quad (r,\theta)\in\Omega,\ t>0, \\ u(r,\theta,0) = \varepsilon^2\varphi(r,\theta), \quad u_t(r,\theta,0) = \varepsilon^2\psi(r,\theta), \quad (r,\theta)\in\Omega, \\ u\mid_{\partial\Omega} = \Delta u\mid_{\partial\Omega} = 0, \quad t>0, \end{gathered} \tag{2.1}$$

where $|u(0,\theta,t)| < +\infty$, $u(r,\theta,t)$ is periodic in θ with a period 2π, α, b, ε are positive constants, β is a constant in $\mathbb{R}$, $\varphi(r,\theta)$ and $\psi(r,\theta)$ are real-valued functions,

$$\Delta = (1/r)\partial_r(r\partial_r) + (1/r^2)\partial_\theta^2, \quad \text{and} \quad \Omega = \{(r,\theta):\ |r|<1,\ \theta\in[-\pi,\pi)\}.$$

Denote by $L_{2,r}(\Omega)$ the real space $L_2(\Omega)$ with the weight r endowed with the scalar product $(\cdot,\cdot)_{r,0}$ and the corresponding norm $\|\cdot\|_{r,0}$. For studying the problem (2.1) we shall use the expansions in the series of eigenfunctions of the Laplace operator in Ω. For a function $f(r,\theta)\in L_{2,r}(\Omega)$ this expansion is

$$f(r,\theta) = \textstyle\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\widehat{f}_{mn}\chi_{mn}(r,\theta), \tag{2.2}$$

$$\widehat{f}_{mn} = \tfrac{(f,\chi_{mn})_{r,0}}{\|\chi_{mn}\|^2_{r,0}}, \quad \chi_{mn}(r,\theta) = J_m(\lambda_{mn}r)e^{im\theta}, \quad m\in\mathbf{Z},\, n\in\mathbf{N},$$

where $J_m(z)$ are Bessel functions of index m, λ_{mn} are its positive zeros numbered in increasing order, $n = 1, 2, ...$ is the number of the zero.

Denoting by $\|\cdot\|_r$ the norm in the weighted space $L_{2,r}(0,1)$ we can write that for sufficiently large positive λ [6]

$$\frac{C_1}{\lambda} \leq \|J_m(\lambda r)\|_r^2 \leq \frac{C_2}{\lambda}. \tag{2.3}$$

For bounded m large positive zeros of $J_m(z)$ have the following asymptotic expansion uniform in m (McMahon's expansion, see [5, p. 247]):

$$\lambda_{mn} = \mu_{mn} + O(\frac{1}{\mu_{mn}})), \ \mu_{mn} = (m + 2n - \frac{1}{2})\frac{\pi}{2}, \ n \to +\infty. \tag{2.4}$$

In the sequel we shall need the weighted Sobolev spaces $H_r^s(\Omega)$, $s \in \mathbb{R}$, which differ from $H^s(\Omega) \equiv W_2^s(\Omega)$ in that the weighted space $L_{2,r}(\Omega)$ is used instead of $L_2(\Omega)$. It is convenient to introduce the norm in $H_r^s(\Omega)$ by the formula

$$\|f\|_{r,s}^2 = \sum_{m,n} \lambda_{mn}^{2s} |\widehat{f}_{mn}|^2 \|\chi_{mn}\|_{r,0}^2.$$

We shall also use the Banach space $C^k([0,+\infty), H_r^s(\Omega))$ equipped with the norm

$$\|u\|_{C^k} = \sum_{j=0}^{k} \sup_{t \geq 0} \|\partial_t^j u(t)\|_{s,r}.$$

Let $f(x,\omega)$ be defined on $[0,1] \times [a,d]$, $-\infty < a, d < +\infty$. We shall denote by $V_0^1(f(x,\omega))$ the total variation of the function $f(x,\omega)$ in $x \in [0,1]$. Consider the integral

$$I_m(\lambda,\omega) = \int_0^1 x f(x,\omega) J_m(\lambda x) dx, \ m \geq 0, \ \lambda > 0, \ \omega \in [a,d].$$

The following lemma is the extension of the proposition given in [12, p. 595] to the case when the integrand depends on a parameter.

Lemma 1 *Assume that for each fixed $\omega \in [a,d]$ the function $\sqrt{x} f(x,\omega)$ has a bounded total variation in $x \in [0,1]$ which is absolutely integrable in $\omega \in [a,d]$, i.e.,*

$$V_0^1(\sqrt{x} f(x,\omega)) = V_f(\omega) \in L_1(a,d).$$

Moreover, let

$$\lim_{x \to 0^+} \sqrt{x} f(x,\omega) = F(\omega) \in L_1(a,d).$$

Then for all $m \geq 0$, $\lambda > 0$, and $\omega \in [a,d]$

$$|I_m(\lambda,\omega)| \leq \frac{C_\omega}{\lambda^{3/2}},$$

where C_ω is independent of m and λ, and $C_\omega \in L_1(a,d)$.

The next proposition gives a tool for increasing the decay of the integral $I_m(\lambda,\omega)$ in λ.

Lemma 2 *Suppose that $f(x,\omega)$ has partial derivatives in $x \in (0,1)$ through second order, $f(0,\omega) = \partial_x f(0,\omega) = 0$ (in case $m = 0$ only $\partial_x f(0,\omega) = 0$), and $f(1,\omega) = \partial_x f(1,\omega) = 0$. Moreover, assume that for any fixed $\omega \in [a,d]$ the function $\sqrt{x}\partial_x^2 f(x,\omega)$ has a bounded total variation in $x \in [0,1]$ which is absolutely integrable in $\omega \in [a,d]$,, i.e.,*

$$V_0^1(\sqrt{x}\partial_x^2 f(x,\omega)) = V_2(\omega) \in L_1(a,d),$$

and

$$\lim_{x\to 0^+} \sqrt{x}\partial_x^2 f(x,\omega) = F_2(\omega) \in L_1(a,d).$$

Then for $m \geq 0$ and $\lambda > 0$,

$$|I_m(\lambda,\omega)| \leq \frac{C_\omega (m+1)^2}{\lambda^{7/2}},$$

where C_ω is independent of m and λ and $C_\omega \in L_1(a,d)$.

Proof Integrating two times by parts in $I_m(\lambda,\omega)$, expanding the integrand around $x_0 = 0$ by Taylor's formula, and applying Lemma 1 to the result we deduce the necessary estimate. $\diamondsuit$

For treating the series expansion coefficients of the nonlinearity $(u^2)^\wedge_{mn}(t)$ we shall need the estimates of the coefficients

$$\begin{aligned} a_{mnpqls} &= g_{mnpqls}/\|J_m(\lambda_{mn}r)\|_r^2, \\ g_{mnpqls} &= \textstyle\int_0^1 rJ_m(\lambda_{mn}r)J_p(\lambda_{pq}r)J_l(\lambda_{ls}r)dr, \end{aligned} \tag{2.5}$$

with integers m, p, $l \geq 0$; n, q , $s \geq 1$.

Lemma 3 *The inequalities*

$$|a_{mnpqls}| \leq C\sqrt{\tfrac{\lambda_{mn}}{\lambda_{pq}\lambda_{ls}}}, \tag{2.6}$$

$$|a_{mnpqls}| \leq \tfrac{C}{\sqrt{\lambda_{mn}}}\left[\sqrt{\tfrac{\lambda_{pq}}{\lambda_{ls}}} + \sqrt{\tfrac{\lambda_{ls}}{\lambda_{pq}}} + m\right], \tag{2.7}$$

hold for constants independent of m, n, p, q, l, s.

Proof Using (2.3) and estimating each of the Bessel functions in the integrand of (2.5) by means of the inequality (see [6, 12])

$$|J_\nu(z)| \leq \frac{C}{\sqrt{z}},\ z > 0, \tag{2.8}$$

we establish (2.6). For proving (2.7) we first integrate by parts in (2.5) and then estimate the Bessel functions by means of (2.8).

3 The main results

We shall present below the results concerning the global in time solvability of the problem (2.1), construction of its solutions, and their long-time asymptotic behavior. First, we formulate some assumptions on a sufficiently smooth function $f(r,\theta)$ with $(r,\theta) \in \Omega$.

Assumptions A

- $\partial_\theta^k f(r,-\pi) = \partial_\theta^k f(r,\pi)$ for $k = 0,1,2$.
- $f(r,\theta)$ satisfies the hypotheses of Lemma 2 with $m = 0$, i.e., $\partial_r f(0,\theta) = f(1,\theta) = \partial_r f(1,\theta) = 0$, $\lim_{r\to 0^+} \sqrt{r}\partial_r^2 f(r,\theta) = F_{2,0}(\theta)$ is in $L_1(-\pi,\pi)$, $V_0^1(\sqrt{r}\partial_r^2 f(r,\theta)) = V_{2,0}(\theta)$ is in $L_1(-\pi,\pi)$.
- $\partial_\theta^3 f(r,\theta)$ satisfies the hypotheses of Lemma 2 in the general case, i.e.,

$$\partial_\theta^3 f(0,\theta) = \partial_r \partial_\theta^3 f(0,\theta) = \partial_\theta^3 f(1,\theta) = \partial_r \partial_\theta^3 f(1,\theta) = 0,$$

  $\lim_{r\to 0^+} \sqrt{r}\partial_r^2 \partial_\theta^3 f(r,\theta) = F_{2,3}(\theta)$ is in $L_1(-\pi,\pi)$, $V_0^1(\sqrt{r}\partial_r^2\partial_\theta^3 f(r,\theta)) = V_{2,3}(\theta)$ is in $L_1(-\pi,\pi)$.

Assumptions B

- $f(r,-\pi) = f(r,\pi)$; $f(r,\theta)$ and $\partial_\theta f(r,\theta)$ satisfy the hypotheses of Lemma 1, i.e., $\lim_{r\to 0^+} \sqrt{r} f(r,\theta) = \Psi(\theta)$ is in $L_1(-\pi,\pi)$, $V_0^1(\sqrt{r} f(r,\theta)) = V_{0,0}(\theta)$ is in $L_1(-\pi,\pi)$, $\lim_{r\to 0^+} \sqrt{r}\partial_\theta f(r,\theta) = F_{0,1}(\theta)$ is in $L_1(-\pi,\pi)$, $V_0^1(\sqrt{r}\partial_\theta f(r,\theta)) = V_{0,1}(\theta)$ is in $L_1(-\pi,\pi)$.

Theorem 1 *If $\alpha > b^2$, $\varphi(r,\theta)$ satisfies Assumption A, and $\psi(r,\theta)$ satisfies Assumptions B respectively, then there is ε_0 positive such that for $0 < \varepsilon \le \varepsilon_0$ and $s < 5/2$, Problem (2.1) has a strong solution in*

$$C^2([0,+\infty),\, H_r^{s-4}(\Omega)) \cap C^1([0,+\infty), H_r^{s-2}(\Omega)) \cap C^0([0,+\infty), H_r^s(\Omega))$$

with $\Delta u \in C^1([0,+\infty), H_r^{s-4}(\Omega)) \cap C^0([0,+\infty), H_r^{s-2}(\Omega))$ and $\Delta(u^2)$, $\Delta^2 u \in C^0([0,+\infty), H_r^{s-4}(\Omega))$. If $1/2 < s < 5/2$, then the solution is unique.

Moreover, u and ∇u are continuous and bounded in $\overline{\Omega} \times [0,+\infty)$, and the solution can be represented as

$$u(r,\theta,t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(r,\theta,t), \tag{3.1}$$

where the functions $u^{(N)}(r,\theta,t)$ will be defined in the proof (see (4.9) and (4.11)) and a bar over the letter denotes a closed domain. This series converges absolutely and uniformly with respect to $(r,\theta) \in \overline{\Omega}$, $t \ge 0$, $\varepsilon \in [0,\varepsilon_0]$, and ∇u can be calculated termwise.

Remark 3.1 When $\alpha > b^2$, there exists an infinite number of damped oscillations. This case is the most interesting one from both the mathematical and the physical points of view. When $0 < \alpha < b^2$, the linear stability criteria are also satisfied, but aperiodic processes play the main rôle.

Remark 3.2 It is not difficult to give an example of the initial functions $\varphi(r,\theta)$ and $\psi(r,\theta)$ satisfying Assumptions A and B. Using separation of variables we obtain

$$\begin{gathered}
\varphi(r,\theta) = R(r)\Theta(\theta), \quad \text{with } R^{(k)}(0) = R^{(k)}(1),\ k = 0,\ 1; \\
\lim_{r\to 0^+} \sqrt{r}R''(r) = c_1 < \infty, \quad V_0^1(\sqrt{r}R''(r)) = c_2 < \infty; \\
\Theta^{(k)}(-\pi) = \Theta^{(k)}(\pi),\ k = 0,1,2; \quad \Theta'''(\theta) \in L_1(-\pi,\pi); \\
\psi(r,\theta) = R_1(r)\Theta_1(\theta), \quad \text{with } \Theta_1(-\pi) = \Theta_1(\pi), \quad \Theta_1'(\theta) \in L_1(-\pi,\pi); \\
\lim_{r\to 0^+} \sqrt{r}R_1(r) = c_3 < \infty \quad V_0^1(\sqrt{r}R_1(r)) = c_4 < \infty.
\end{gathered}$$

Theorem 2 *Under the assumptions of Theorem 1, the following asymptotic expansion is valid as $t \to +\infty$:*

$$u(r,\theta,t) = \exp(-b\lambda_{01}^2 t)\{[A_\varepsilon \cos(\sigma_{01}t) + B_\varepsilon \sin(\sigma_{01}t)]J_0(\lambda_{01}r) + O(\exp(-b\lambda_{01}^2 t))\}, \tag{3.2}$$

where $\sigma_{01} = \lambda_{01}\sqrt{(\alpha-b^2)\lambda_{01}^2+1}$, the coefficients A_ε, $B_\varepsilon \sim c\varepsilon^2$ are defined in the proof (see (5.1) and (5.2)), and the estimate of the residual term is uniform with respect to $(r,\theta) \in \overline{\Omega}$ and $\varepsilon \in [0,\varepsilon_0]$.

4 Proof of Theorem 1: global in time solvability and construction of solutions

To satisfy the boundary conditions we seek a solution of (2.1) in the form

$$u(r,\theta,t) = \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} \widehat{u}_{mn}(t)\chi_{mn}(r,\theta), \quad \widehat{u}_{mn}(t) = \frac{(u,\chi_{mn})_{r,0}(t)}{\|\chi_{mn}\|_{r,0}^2}. \tag{4.1}$$

Using the fact that $\widehat{u}_{mn}(t) = (-1)^m\overline{\widehat{u}_{-m,n}(t)}$, $m \geq 0$, $n \geq 1$, we can rewrite this expansion as

$$\begin{aligned}
u(r,\theta,t) &= \sum_{n=1}^{\infty} \widehat{u}_{0n}(t)J_0(\lambda_{0n}r) + \sum_{m,n=1}^{\infty} J_m(\lambda_{mn}r)[\widehat{u}_{mn}(t)e^{im\theta} + \overline{\widehat{u}_{mn}(t)}e^{-im\theta}] \\
&= \sum_{m,n}^{*} \widehat{u}_{mn}(t)J_m(\lambda_{mn}r)e^{im\theta}.
\end{aligned} \tag{4.2}$$

Expanding $\Delta(u^2)$ as a series of this type and using the formula $(\Delta(u^2))^\wedge_{mn} = -\lambda_{mn}^2(u^2)^\wedge_{mn}$, we substitute (4.1) into (2.1) and obtain the following Cauchy problem for $\widehat{u}_{mn}(t)$, $m \in \mathbf{Z}$, $n \in \mathbf{N}$:

$$\widehat{u}''_{mn}(t) + 2b\lambda_{mn}^2 \widehat{u}'_{mn}(t) + (\alpha\lambda_{mn}^4 + \lambda_{mn}^2)\widehat{u}_{mn}(t) = -\beta\lambda_{mn}^2 (u^2)^{\wedge}_{mn}(t),\ t > 0,$$

$$\widehat{u}_{mn}(0) = \varepsilon^2 \widehat{\varphi}_{mn},\ \widehat{u}'_{mn}(0) = \varepsilon^2 \widehat{\psi}_{mn}, \tag{4.3}$$

where $\widehat{\varphi}_{mn}$ and $\widehat{\psi}_{mn}$ are the coefficients of the (4.1)-type expansions of $\varphi(r,\theta)$ and $\psi(r,\theta)$.

We need to establish the following estimates as $m \geq 0$, $n \geq 1$:

$$|\widehat{\varphi}_{mn}| \leq \frac{C}{\lambda_{mn}^{5/2}(m+1)}, \tag{4.4}$$

$$|\widehat{\psi}_{mn}| \leq \frac{C}{\lambda_{mn}^{1/2}(m+1)}. \tag{4.5}$$

For (4.4) with $m = 0$, we use Lemma 2 with $m = 0$ and (2.3). To obtain (4.4) with $m \geq 1$ we integrate three times by parts in θ in the integral representation of $\widehat{\varphi}_{mn}$, use the periodicity conditions for $\partial_\theta^k \varphi(r,\theta)$, $k = 0, 1, 2$, and apply Lemma 2. The estimate (4.5) is derived in an analogous way. The only difference consists in applying Lemma 1 for deducing (4.5) with $m = 0$.

Now we should calculate the coefficients $(u^2)^{\wedge}_{mn}(t)$ in the right-hand side of (4.3). Multiplying the series representations (4.2) and calculating the integrals in θ we get for $m \geq 0$, $n \geq 1$

$$\begin{aligned}
(u^2)^{\wedge}_{mn}(t) &= \frac{1}{\|\chi_{mn}\|_{r,0}^2} \Big(\sum_{p,q} \widehat{u}_{pq}(t)\chi_{pq} \cdot \sum_{l,s} \widehat{u}_{ls}(t)\chi_{ls}, \chi_{mn} \Big)_{r,0} \\
&= \sum_{p,l\geq 0; q,s\geq 1; p+l=m} a_{mnpqls} \widehat{u}_{pq}(t)\widehat{u}_{ls}(t) \\
&\quad + \sum_{p,q,l,s\geq 1; l-p=m} a_{mnpqls} \overline{\widehat{u}_{pq}(t)}\widehat{u}_{ls}(t) \\
&\quad + \sum_{p,q,l,s\geq 1; p-l=m} a_{mnpqls} \widehat{u}_{pq}(t)\overline{\widehat{u}_{ls}(t)},
\end{aligned} \tag{4.6}$$

where a_{mnpqls} are defined by (2.5).

Setting $\widehat{\Phi}_{mn} = \varepsilon\widehat{\varphi}_{mn}$, $\widehat{\Psi}_{mn} = \varepsilon\widehat{\psi}_{mn}$, (it is convenient to keep ε in these coefficients in order to simplify some estimates) and $\sigma_{mn} = \lambda_{mn}\sqrt{(\alpha - b^2)\lambda_{mn}^2 + 1}$ we integrate (4.3) in t and get

$$\begin{aligned}
\widehat{u}_{mn}(t) &= \varepsilon \exp(-b\lambda_{mn}^2 t) \\
&\quad \times \Big\{ \Big[\cos(\sigma_{mn}t) + \frac{b\lambda_{mn}^2}{\sigma_{mn}} \sin(\sigma_{mn}t)\Big]\widehat{\Phi}_{mn} + \frac{\sin(\sigma_{mn}t)}{\sigma_{mn}} \widehat{\Psi}_{mn} \Big\} \\
&\quad - \frac{\beta\lambda_{mn}^2}{\sigma_{mn}} \int_0^t \exp[-b\lambda_{mn}^2(t-\tau)] \sin[\sigma_{mn}(t-\tau)] (u^2)^{\wedge}_{mn}(\tau) d\tau.
\end{aligned} \tag{4.7}$$

For solving this nonlinear integral equation we shall use the perturbation theory. Representing $\widehat{u}_{mn}(t)$ as a formal series in ε

$$\widehat{u}_{mn}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} \widehat{v}_{mn}^{(N)}(t) \tag{4.8}$$

we substitute (4.8) into (4.7), compare the coefficients of equal powers of ε, and obtain the following recurrence formulas for $m \geq 0$, $n \geq 1$, $N \geq 0$, $t > 0$

$$\widehat{v}_{mn}^{(0)}(t) = \exp(-b\lambda_{mn}^2 t)\left\{[\cos(\sigma_{mn}t) + \frac{b\lambda_{mn}^2}{\sigma_{mn}}\sin(\sigma_{mn}t)]\widehat{\Phi}_{mn} + \frac{\sin(\sigma_{mn}t)}{\sigma_{mn}}\widehat{\Psi}_{mn}\right\}, \tag{4.9}$$

$$\begin{aligned}
\widehat{v}_{mn}^{(N)}(t) \quad = \quad & -\frac{\beta\lambda_{mn}^2}{\sigma_{mn}}\int_0^t \exp[-b\lambda_{mn}^2(t-\tau)]\sin[\sigma_{mn}(t-\tau)] \\
& \times\Big\{ \sum_{p,l\geq 0;\ q,s\geq 1; p+l=m} a_{mnpqls} \sum_{j=1}^{N} \widehat{v}_{pq}^{(j-1)}(\tau)\widehat{v}_{ls}^{(N-j)}(\tau) \\
& + \sum_{p,q,l,s\geq 1; l-p=m} a_{mnpqls} \sum_{j=1}^{N} \overline{\widehat{v}_{pq}^{(j-1)}(\tau)}\widehat{v}_{ls}^{(N-j)}(\tau) \\
& + \sum_{p,q,l,s\geq 1; p-l=m} a_{mnpqls} \sum_{j=1}^{N} \widehat{v}_{pq}^{(j-1)}(\tau)\overline{\widehat{v}_{ls}^{(N-j)}(\tau)}\Big\} d\tau, \quad N \geq 1\,.
\end{aligned}$$

The following estimates hold for $m \geq 0,\ n \geq 1,\ N \geq 0,\ t > 0$:

$$|\widehat{v}_{mn}^{(N)}(t)| \leq c^N (N+1)^{-2} \lambda_{mn}^{-5/2} (m+1)^{-1} \exp(-b\lambda_{01}^2 t), \tag{4.10}$$

where $c = c(b, \beta) = c_1|\beta|/b$. These estimates are proved by induction on the number N with the help of (4.9).

To obtain the representation (3.1) we perform the interchange of summation in the series (4.2) with the coefficients defined by (4.8) and get

$$u(r, \theta, t) = \sum_{N=0}^{\infty} \varepsilon^{N+1} u^{(N)}(r, \theta, t), \tag{4.11}$$

where

$$u^{(N)}(r, \theta, t) = \sum_{m,n}{}^{*}\ \widehat{v}_{mn}^{(N)}(t)\chi_{mn}(r, \theta)$$

This interchange is possible due to the absolute and uniform convergence of the series in $(r, \theta) \in \overline{\Omega}$, $t \geq 0$, $\varepsilon \in [0, \varepsilon_0]$. Here $\varepsilon_0 < 1/c$.

For proving that the formally constructed function (4.11) is really a solution of (2.1) in the required functional space we deduce the following estimates for $k = 0, 1, 2$; $m \geq 0$, $n \geq 1$, $t > 0$:

$$|\partial_t^k \widehat{u}_{mn}(t)| \leq c\lambda_{mn}^{2k-5/2}(m+1)^{-1}\exp(-b\lambda_{01}^2 t). \tag{4.12}$$

Taking into consideration (2.3), (2.4), and (4.10) we conclude that the series

$$\sum_{m,n} \lambda_{mn}^{2s} |\widehat{u}_{mn}(t)|^2 \|J_m(\lambda_{mn} r)\|_r^2$$

converges uniformly with respect to $t \geq 0$ for $s < 5/2$ (we check the convergence of the iterated series $\sum_{m=0}^{\infty} \sum_{n=1}^{\infty}$ using first (2.4) for fixed m and then applying Fubini-Tonelli's theorem). This allows us to establish that $u \in C^0([0,+\infty), H_r^s(\Omega))$, $s < 5/2$. It follows from (4.12) with $k = 0$ that the series (4.11) converges absolutely and uniformly with respect to $(r,\theta) \in \overline{\Omega}$, $t \geq 0$, and $\varepsilon \in [0, \varepsilon_0]$. Therefore, $u(r,\theta,t)$ is continuous and bounded in this domain.

As regards u_t and Δu, their series expansion coefficients have the same estimates (4.12) with $k = 1$. Analysing the convergence of the series

$$\|\Delta u\|_{r,s}^2 = \sum_{m,n} \lambda_{mn}^{2s} |(\Delta u)_{mn}^{\wedge}(t)|^2 \|J_m(\lambda_{mn} r)\|_r^2$$

by means of (2.3), (2.4) we conclude that u_t, $\Delta u \in C^0([0,+\infty), H_r^{s-2}(\Omega))$, $s < 5/2$. The series expansion coefficients of u_{tt}, Δu_t, $\Delta^2 u$, and $\Delta(u^2)$ have the same estimates (4.12)with $k = 2$, therefore all these functions belong to the space $C^0([0,+\infty), H_r^{s-4}(\Omega))$ with $s < 5/2$.

Uniqueness of solutions We argue by contradiction. Assume that there exist two solutions $u^{(1)}$ and $u^{(2)}$ of the problem (2.1) from the class stated in the theorem. Setting $w = u^{(1)} - u^{(2)}$ we expand w into the series of the type of (4.2) and get

$$w(r,\theta,t) = \sum_{m,n}{}^{*} \widehat{w}_{mn}(t) \chi_{mn}(r,\theta),$$

where the coefficients $\widehat{w}_{mn}(t)$ satisfy the integral equation

$$\begin{aligned} \widehat{w}_{mn}(t) = -\frac{\beta \lambda_{mn}^2}{\sigma_{mn}} \int_0^t \exp[-b\lambda_{mn}^2 (t-\tau)] \sin[\sigma_{mn}(t-\tau)] \\ \times \{[(u^{(1)})^2]_{mn}^{\wedge}(\tau) - [(u^{(2)})^2]_{mn}^{\wedge}(\tau)\} d\tau. \end{aligned} \tag{4.13}$$

The difference of squares in the integrand can be represented as

$$\begin{aligned} &\sum_{p,q,l,s} a_{mnpqls} [\widehat{u}_{pq}^{(1)}(t) \widehat{u}_{ls}^{(1)}(t) - \widehat{u}_{pq}^{(2)}(t) \widehat{u}_{ls}^{(2)}(t)] \\ &= \sum_{p,q,l,s} a_{mnpqls} [\widehat{u}_{pq}^{(1)}(t) \widehat{w}_{ls}(t) + \widehat{u}_{ls}^{(2)}(t) \widehat{w}_{pq}(t)]. \end{aligned}$$

In fact, we have convolutions with respect to the "angular" indeces in the above written sum (see (4.11)). We estimate only the sum with the additional condition $p + l = m$. The other terms can be treated analogously. Fixing some small

$\delta > 0$ and $\kappa > \delta > 0$ and using (2.4), (4.3), and the Cauchy-Schwartz inequality we can write that

$$
\begin{aligned}
&\Big| \sum_{p,q,l,s;p+l=m} a_{mnpqls}\widehat{u}^{(1)}_{pq}(t)\widehat{w}_{ls}(t)\Big| \\
&\leq \ C\sqrt{\lambda_{mn}}\sum_{q,l,s}\frac{|\widehat{u}^{(1)}_{m-l,q}(t)|}{\lambda^{1/2}_{m-l,q}}\frac{|\widehat{w}_{ls}(t)|}{\lambda^{1/2}_{ls}} \\
&\leq \ C\sqrt{\lambda_{mn}}\sum_{q,l,s}\frac{q^{(1+\delta)/2}}{\lambda^{\kappa}_{ls}\lambda^{2}_{m-l,q}}\cdot\frac{|\widehat{w}_{ls}(t)|\lambda^{\kappa}_{ls}}{q^{(1+\delta)/2}\lambda^{1/2}_{ls}} \\
&\leq \ C\sqrt{\lambda_{mn}}\Big(\sum_{q,l,s}\frac{q^{1+\delta}}{\lambda^{4}_{m-l,q}\lambda^{2\kappa}_{ls}}\Big)^{1/2}\Big(\sum_{q}\frac{1}{q^{1+\delta}}\sum_{l,s}\lambda^{2\kappa}_{ls}|\widehat{w}_{ls}(t)|^2\|J_l(\lambda_{ls}r)\|^2_r\Big)^{1/2} \\
&\leq \ C\sqrt{\lambda_{mn}}\|w(t)\|_{r,\kappa}.
\end{aligned}
$$

Here the convergence of the triple series takes place for $\kappa > 1/2$. The sum containing $\widehat{u}^{(2)}_{ls}(t)$ can be estimated in an analogous way.

Estimating both sides of (4.13) we obtain

$$
|\widehat{w}_{mn}(t)|^2 \leq C\lambda_{mn}\Big(\int_0^t \exp[-b\lambda^2_{mn}(t-\tau)]\|w(\tau)\|_{\kappa}d\tau\Big)^2.
$$

Multiplying both sides of the last inequality by $\lambda^{2\kappa}_{mn}\|\chi_{mn}\|^2$, summing in m, n, and using (2.3) we deduce that for some $h > 0$ and $t \in [0, h]$

$$
\|w(t)\|^2_{\kappa} \leq C\Xi(t)\big(\sup_{t\in[0,h]}\|w(t)\|^2_{\kappa}\big),
$$

where

$$
\begin{aligned}
\Xi(t) &= \sum_{m,n}\lambda^{2\kappa}_{mn}\|\chi_{mn}\|^2_{\kappa}\Big(\int_0^t \exp[-b\lambda^2_{mn}(t-\tau)]d\tau\Big)^2 \\
&= \sum_{m,n}\lambda^{2\kappa}_{mn}\|\chi_{mn}\|^2_{\kappa}\left[\frac{1-\exp(-b\lambda^2_{mn}t)}{b\lambda^2_{mn}}\right]^2.
\end{aligned}
$$

For $\kappa < 1$ this series converges absolutely and uniformly with respect to $t \geq 0$. Thus, $\Xi(t)$ is a nondecreasing continuous function on $[0, h]$ and $\Xi(0) = 0$. Therefore,

$$
\big(\sup_{t\in[0,h]}\|w(t)\|^2_{\kappa}\big)^2 \leq C\Xi(t)\big(\sup_{t\in[0,h]}\|w(t)\|^2_{\kappa}\big)^2 \leq C(h)\big(\sup_{t\in[0,h]}\|w(t)\|^2_{\kappa}\big)^2,
$$

where $C(h) = C\Xi(h)$. We can make the constant $C(h)$ less than one by the appropriate choice of h. This contradiction allows one to obtain uniqueness for $t \in [0, h]$.

Then we continue this process on the intervals $[T_1, T_2]$, $[T_2, T_3]$, $\dots$, $[T_k, T_{k+1}]$, $\dots$. with $T_k = kh$ and $k \to \infty$. Since

$$\int_{T_k}^{t} \exp[-b\lambda_{mn}^2(t-\tau)]d\tau = \frac{1-\exp[-b\lambda_{mn}^2(t-T_k)]}{\lambda_{mn}^2}$$

we deduce that for $t \in [T_k, T_{k+1}]$

$$\left(\sup t \in [T_k, T_{k+1}]\|w(t)\|_\kappa^2\right)^2 \le C\Xi(t-T_k)\left(\sup t \in [T_k, T_{k+1}]\|w(t)\|_\kappa^2\right)^2.$$

Setting $t = T_k + \eta$, $\eta \in [0, h]$, so that $\Xi(t - T_k) = \Xi(\eta)$, and observing that the condition $C\Xi(\eta)$ has been already satisfied, we establish the uniqueness for all $t \ge 0$ and $1/2 < \kappa < 1$. We note that $\|w(t)\|_{r,\kappa_1} \le c\|w(t)\|_{r,\kappa_2}$ for $\kappa_1 \le \kappa_2$ and all $t \ge 0$. Consequently, $w(t) \in H_r^{\kappa_2}(\Omega) \subseteq H_r^{\kappa_1}(\Omega)$ for $t \ge 0$. Therefore, the uniqueness takes place for $1/2 < \kappa < 5/2$, where the solution still exists. This completes the proof of Theorem 1.

5 Proof of Theorem 2: long-time asymptotics

To obtain the asymptotic expansion of the solution, we single out the term $\widehat{u}_{01}(t)J_0(\lambda_{01}r)$ in (4.2) and obtain a subtle asymptotic estimate of $\widehat{u}_{01}(t)$. Then we estimate the remaining series $R_1(r,t) = \sum_{n=2}^{\infty}\widehat{u}_{0n}(t)J_0(\lambda_{0n}r)$ and $R_2(r,\theta,t) = \sum_{m,n=1}^{\infty} J_m(\lambda_{mn}r)[\widehat{u}_{mn}(t)e^{im\theta} + \overline{\widehat{u}_{mn}(t)}e^{-im\theta}]$.

According to (4.8), (4.9) we have

$$\widehat{u}_{01}(t) = \sum_{N=0}^{\infty} \varepsilon^{N+1}\widehat{v}_{01}^{(N)}(t).$$

Adding and subtracting the integrals from t to ∞ in the integral representations of $\widehat{v}_{01}^{(N)}(t)$, $N \ge 1$, we write

$$\begin{gathered}
\widehat{v}_{01}^{(0)}(t) = \exp(-b\lambda_{01}^2 t)[A_\varepsilon^{(0)}\cos(\sigma_{01}t) + B_\varepsilon^{(0)}\sin(\sigma_{01}t),\\
\widehat{v}_{01}^{(N)}(t) = \exp(-b\lambda_{01}^2 t)\{[A_\varepsilon^{(N)} + R_A^{(N)}(t)]\cos(\sigma_{01}t) + [B_\varepsilon^{(N)} + R_B^{(N)}(t)]\sin(\sigma_{01}t)\},\\
A_\varepsilon^{(0)} = \varepsilon\widehat{\varphi}_{01}, \quad B_\varepsilon^{(0)} = \tfrac{\varepsilon}{\sigma_{01}}(b\lambda_{01}^2\widehat{\varphi}_{01} + \widehat{\psi}_{01}),\\
A_\varepsilon^{(N)} = \tfrac{\beta\lambda_{01}^2}{\sigma_{01}}\int_0^\infty \exp(b\lambda_{01}^2\tau)\sin(\sigma_{01}\tau)Q_{01}^{(N)}(\widehat{v}(\tau))d\tau,\\
B_\varepsilon^{(N)} = -\tfrac{\beta\lambda_{01}^2}{\sigma_{01}}\int_0^\infty \exp(b\lambda_{01}^2\tau)\cos(\sigma_{01}\tau)Q_{01}^{(N)}(\widehat{v}(\tau))d\tau,\\
R_A^{(N)}(t) = \tfrac{\beta\lambda_{01}^2}{\sigma_{01}}\int_t^\infty \exp(b\lambda_{01}^2\tau)\sin(\sigma_{01}\tau)Q_{01}^{(N)}(\widehat{v}(\tau))d\tau,\\
R_B^{(N)}(t) = -\tfrac{\beta\lambda_{01}^2}{\sigma_{01}}\int_t^\infty \exp(b\lambda_{01}^2\tau)\cos(\sigma_{01}\tau)Q_{01}^{(N)}(\widehat{v}(\tau))d\tau,
\end{gathered} \tag{5.1}$$

$$Q_{01}^{(N)}(\widehat{v}(t)) \quad = \quad \sum_{q,s=1}^{\infty} a_{010q0s}\sum_{j=1}^{N}\widehat{v}_{0q}^{(j-1)}(t)\widehat{v}_{0q}^{(N-j)}(t)$$

$$+\sum_{q,l,s=1}^{\infty} a_{01lqls} \sum_{j=1}^{N} \overline{\widehat{v}_{0q}^{(j-1)}(t)}\widehat{v}_{ls}^{(N-j)}(t)$$
$$+\sum_{q,l,s=1}^{\infty} a_{01lqls} \sum_{j=1}^{N} \widehat{v}_{lq}^{(j-1)}(t)\overline{\widehat{v}_{ls}^{(N-j)}(t)},$$

where $\widehat{v}_{mn}^{(s)}(t)$ and $0 \le s \le N-1$ are defined by (4.9). Using (2.7) we deduce that for $t > 0$, $N \ge 1$

$$|R_{A,B}^{(N)}(t)| \le c^N \exp(-b\lambda_{01}^2 t).$$

Thus, we obtain

$$\widehat{u}_{01}(t) = \exp(-b\lambda_{01}^2 t)\{[A_\varepsilon \cos(\sigma_{01}t) + B_\varepsilon \sin(\sigma_{01}t)] + O(\exp(-b\lambda_{01}^2 t))\},$$
$$A_\varepsilon = \textstyle\sum_{N=0}^{\infty} \varepsilon^{N+1} A_\varepsilon^{(N)}, \quad B_\varepsilon = \sum_{N=0}^{\infty} \varepsilon^{N+1} B_\varepsilon^{(N)}, \tag{5.2}$$

where $A_\varepsilon^{(N)}$ and $B_\varepsilon^{(N)}$ are defined by (5.1) and the series converges absolutely and uniformly with respect to $\varepsilon \in [0, \varepsilon_0]$.

Finally, we can represent the solution as

$$u(r,\theta,t) = \widehat{u}_{01}(t)J_0(\lambda_{01}r) + R_1(r,t) + R_2(r,\theta,t), \tag{5.3}$$

where $R_{1,2}$ have the following estimates:

$$|R_{1,2}| \le c\exp(-2b\lambda_{01}^2 t). \tag{5.4}$$

Combining (5.2)-(5.4) we deduce (3.3).

6 Conclusion

In the radially symmetric case considered in [11] we have encountered the effect of the “loss of smoothness”. In the general spatially 2-D case studied above this is no longer true. Indeed, the sums in (4.6) include the convolutions with respect to the “angular” indeces. The “purely radial part”

$$\Re_n(t) = \sum_{q,s=1}^{\infty} a_{0n0q0s}\widehat{u}_{0q}(t)\widehat{u}_{0s}(t)$$

which formed the Fourier-Bessel coefficients of the expansion of u^2 in the radially symmetric case of [11] is also present in the series expansion coefficient of $(u^2)^\wedge_{0n}$ in (4.6), namely:

$$(u^2)^\wedge_{0n}(t) = \Re_n(t) + \sum_{q,l,s=1}^{\infty} a_{0nlqls}[\widehat{u}_{lq}(t)\overline{\widehat{u}_{ls}(t)} + \overline{\widehat{u}_{lq}(t)}\widehat{u}_{ls}(t)],$$

but the convergence of the series (4.2) is mainly determined by the decay of $(u^2)^\wedge_{mn}$ for large m and n. It explains the possibility of improving the smoothness of the solution through the "angular" index m, i.e., by means of imposing more periodicity conditions on the initial data. However, there are no convolutions with respect to the "radial" indeces in (4.6), therefore, some "loss of smoothness" still takes place.

In conclusion, we would like to trace the influence of geometry comparing the three boundary value problems, namely: (i) spatially 1-D case on an interval [9], (ii) spatially 2-D radially symmetric problem in a disk [11], and (iii) the general spatially 2-D problem studied above. No "loss of smoothness" takes place in (i), complete "loss of smoothness" occurs in *(ii)*, and "partial loss of smoothness" can be observed in (iii). The major term of the long-time asymptotics in the case (iii) coincides with that of (ii). In order to see the difference it is necessary to calculate the following terms. We would like to emphasize that the above mentioned effects occur as a result of the combined influence of the geometry and the nonlinearity of the equation.

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