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Multichain-type solutions for Hamiltonian systems *

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Dedicated to Alan Lazer
on his 60th birthday

Abstract

The existence of basic and more complicated multichain heteroclinic solutions is established for a class of forced slowly oscillating Hamiltonian systems. Constrained minimization arguments are the key tool in obtaining the results.

1 Introduction

Consider the Hamiltonian system

$$\ddot{q} + V_q(t,q) = 0\,, \tag{1.1}$$

where $q = (q_1, \dots, q_n) \in \mathbb{R}^n$ and V satisfies

(V_1) $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$, is 1-periodic in t and 1-periodic in q_i, $1 \le i \le n$;

(V_2) $V(t,0) = 0 > V(t,x)$ with $x \in \mathbb{R}^n \backslash \mathbb{Z}^n$.

This system was studied by Strobel [18] who proved the following:

(a) For each $\xi \in \mathbb{Z}^n$, there is an $\eta \in \mathbb{Z}^n \backslash \{\xi\}$ and a solution Q of (1.1) heteroclinic from ξ to η, i.e. $Q(-\infty) = \xi$ and $Q(\infty) = \eta$

(b) For each $\xi \neq \eta \in \mathbb{Z}^n$, there is a heteroclinic chain of solutions of (1.1) joining ξ and η, i.e. there exist $\xi_0 = \xi$, $\xi_1, \dots, \xi_k = \eta$ and solutions Q_i of (1.1) heteroclinic from ξ_{i-1} to ξ_i, $1 \le i \le k$.

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Earlier versions of (a) and (b) when $V = V(x)$ were obtained in [14, 16, 8]. More recently, Bertotti and Montecchiari [7] have treated (1.1) where $V(t,x) = a(t)W(x)$ with W satisfying (V_1)–(V_2) and a almost periodic in t. They also find infinitely many heteroclinic solutions of (1.1) but without a nondegeneracy condition as in [18], they cannot make as precise existence statements as [18].

In his setting, under a further nondegeneracy condition involving the functions Q_i in (b), Strobel proved that in fact there exist infinitely many solutions of (1.1) heteroclinic from ξ to η which are near the chain $Q_1, \dots, Q_k$ and are distinguished by the amount of time they spend near $Q_1(\infty), \dots, Q_{k-1}(\infty)$.

In this paper, results related to [18] will be proved for two classes of potentials that are of a more restricted form than $V(t,x)$, namely $a(t)W(x)$. However $a(t)$ is not necessarily periodic in t and unlike [18], no nondegeneracy conditions will be required. The function W satisfies the time independent version of (V_1)–(V_2):

(W_1) $W \in C^2(\mathbb{R}^n, \mathbb{R})$ is 1-periodic in q_i, and $1 \le i \le n$;

(W_2) $W(0) = 0 > W(x)$ with $x \in \mathbb{R}^n \backslash \mathbb{Z}^n$.

For the first class of potentials, roughly speaking, $a(t)$ is nearly constant near a sequence of its local maxima and minima which are sufficiently far apart. This will be made precise in §2. For example if $a(t)$ is 1-periodic, continuous, positive, and non-constant, for all small $\epsilon > 0$, $a(\epsilon t)W(x)$ will be an allowable potential. A second class of potentials are of the form $(\alpha_1(\epsilon t) + \alpha_2(t))W(x)$ where α_1, α_2 are e.g. each like the a just described.

Bolotin and MacKay [10] have recently studied multichain type solutions for a class of slowly oscillating problems in a setting that is more general than ours in some ways but less general in particular in t dependence. Their approach involves a mixture of analytical and minimization arguments. In very recent work, Alessio, Bertotti, and Montecchiari [4] studied a generalization of [7] and also showed that by perturbing such a situation by a term of the form $\alpha(\epsilon t)W(x)$ with α almost periodic and ϵ small, they get solutions of multichain type. Although there is some intersection with this paper, the point of view taken here is quite different from that of [4]. For other related results in a small perturbation setting, see Ambrosetti and Badiale [1], Ambrosetti and Berti [3], Berti [5], Berti and Bolle [6].

It is also worth noting that there has been a considerable amount of work in a PDE setting on standing wave solutions for nonlinear Schrödinger equations which have slowly oscillating spacially dependent potentials. See e.g. Floer and Weinstein [12], Oh [13], Thandi [19], del Pino and Felmer [11], and Ambrosetti, Badiale, and Cingolani [2] to mention a few.

In §2, the existence of basic heteroclinic solutions will be established. The existence of heteroclinics near finite chains of basic solutions will be given in §3. Some simple observations then yield the case of solutions of infinite chain type. The proofs involve elementary minimization and comparison arguments.

2 Basic heteroclinic solutions

In this section, the existence of basic heteroclinic orbits will be established. To begin, let

$$\mathcal{A} = \{a \in C(\mathbb{R},\mathbb{R}) \mid 0 < \underline{a} \leq a(t) \leq \overline{a} < \infty\}$$

where $\underline{a} < \overline{a}$. Our first goal is to find a solution of (1.1) heteroclinic from 0 to some $\xi \in \mathbb{R}^n \backslash \{0\}$. Choose $r > 0$ which is small compared to $1 \equiv \inf\{|\xi_i - \xi_j| \mid \xi_i \neq \xi_j \in \mathbb{Z}^n\}$, i.e. $r \ll 1$. A further condition will be imposed on r later. Let $B_r(z)$ denote an open ball of radius r about $z \in \mathbb{R}^n$. Let $b_1 < b_2 - 1$. A heteroclinic solution of (1.1) will be obtained such that the transition between the end states occurs mainly in $[b_1, b_2]$. Define

$$\begin{aligned}\Gamma = \Gamma(b_1, b_2) \quad = \quad & \big\{q \in W^{1,2}_{\text{loc}}(\mathbb{R},\mathbb{R}^n) : q(t) \in \overline{B}_r(0), t \leq b_1, \\ & \text{and } q(t) \in \overline{B}_r(\xi) \text{ for some } \xi \in \mathbb{Z}^n \backslash \{0\},\ t \geq b_2\big\}.\end{aligned}$$

Set

$$L(q) = \frac{1}{2}|\dot{q}|^2 - a(t)W(q),$$

the Lagrangian for (1.1), and define the associated functional

$$I(q) = \int_{\mathbb{R}} L(q)dt.$$

Finally define

$$c = c(b_1, b_2) = \inf_{q \in \Gamma} I(q)\,. \tag{2.1}$$

Proposition 2.1 *If $a \in \mathcal{A}$ and W satisfies (W_1)–(W_2), there is a $Q \in \Gamma(b_1, b_2)$ such that $I(q) = c(b_1, b_2)$.*

Proof: Let (q_m) be a minimizing sequence for (2.1). Then the form of I and Γ imply (q_m) is bounded in $W^{1,2}_{\text{loc}}$ and converges weakly in $W^{1,2}_{\text{loc}}$ and strongly in L^{∞}_{loc} to $Q \in \Gamma(b_1, b_2)$. Moreover standard weak lower semicontinuity arguments imply $I(Q) = c(b_1, b_2)$.

Remark 2.2 (i) As in [14], $Q(-\infty) = 0$ and $Q(\infty) \in \mathbb{Z}^n \backslash \{0\}$.

(ii) Standard regularity arguments show Q is a solution of (1.1) for $t \in (b_1, b_2)$ and also for those values of $t \leq b_1$, $t \geq b_2$ when $Q(t) \notin \partial B_r(0)$, $Q(t) \notin \partial B_r(Q(\infty))$ respectively.

It remains to choose a subfamily $\mathcal{A}^* \subset \mathcal{A}$ for which $a \in \mathcal{A}^*$ implies Q is a solution of (1.1). First a few observations about Q are necessary. Suppose $Q(\infty) = \xi$.

Lemma 2.3 *For $0 < \rho < r$, there is an $\omega = \omega(\rho) > 0$ and $t_1 = t_1(\rho) \in [b_1 - \omega, b_1]$ such that $Q(t_1) \in \overline{B}_\rho(0)$. Moreover ω can be chosen independently of $a \in \mathcal{A}$.*

Proof: Since $Q(-\infty) = 0$, $Q(t) \in B_\rho(0)$ for t near $-\infty$. The point is to find ω independently of $a \in \mathcal{A}$. Let $\eta \in \mathbb{Z}^n \backslash \{0\}$ and define

$$R(t) = \begin{cases} 0 & \text{if } t \le b_1, \\ (t - b_1)\eta & \text{if } b_1 \le t \le b_1 + 1, \\ \eta & \text{if } t \ge b_1 + 1. \end{cases} \tag{2.2}$$

Then R belongs to Γ, so

$$c = I(Q) \le I(R) \equiv M\,. \tag{2.3}$$

Set

$$\beta(\rho) = \inf_{|x - \mathbb{Z}^n| \ge \rho} -W(x)\,. \tag{2.4}$$

By (W_1)–(W_2), $\beta(\rho) > 0$. If $|Q(t)| > \rho$ in $[b_1 - \omega, b_1]$, by (2.3)–(2.4),

$$M \ge I(\varphi) \ge \int_{b_1 - \omega}^{b_1} -a(t)W(Q)dt \ge \underline{a}\beta(\rho)\omega\,.$$

Thus the Lemma holds for any $\omega > M(\underline{a}\beta(\rho))^{-1}$.

Corollary 2.4 *There is a $t_2 = t_2(\rho) \in [b_2, b_2 + \omega]$ such that $Q(b_2) \in \overline{B}_\rho(\xi)$.*

Proof: As in Lemma 2.3. After obtaining t_1, we define

$$P(t) = \begin{cases} 0 & \text{if } t \le t_1 - 1, \\ (t - (t_1 - 1))Q(t_1) & \text{if } t_1 - 1 \le t \le t_1, \\ Q(t) & \text{if } t \ge t_1. \end{cases} \tag{2.5}$$

Then $P \in \Gamma(b_1, b_2)$ so $I(Q) \le I(P)$ and in particular by (2.5),

$$\int_{-\infty}^{t_1} L(Q)dt \le \int_{-\infty}^{t_1} L(P)dt = \int_{t_1 - 1}^{t_1} L(P)dt \equiv \varphi(\rho) \tag{2.6}$$

and the definition of $\varphi(\rho)$ shows $\varphi(\rho) \to 0$ as $\rho \to 0$. Similarly, it can be assumed that

$$\int_{t_2}^{\infty} L(Q)dt \le \varphi(\rho).$$

Lemma 2.5 *For $\rho \ll r$, $Q(t) \in B_{r/2}(0)$ for $t \le t_1$ and $Q(t) \in B_{r/2}(\xi)$ for $t \ge t_2$.*

Proof: The first assertion will be proved. If it is not valid, $Q(s) \in \partial B_{r/2}(0)$ for some $s < t_1$. By Lemma 2.3, $Q(t_1) \in \overline{B}_\rho(0)$. For $\rho \ll r$, the cost of Q going from $\partial B_{r/2}(0)$ to $\partial B_\rho(0)$, as measured by I, exceeds that of going from 0 to $\partial B_\rho(0)$ as in (2.5)–(2.6). Since Q minimizes I in Γ, the Lemma follows.

Lemma 2.6 *There is an $s_1 \in [b_1, b_1 + \omega]$ and $s_2 \in [b_2 - \omega, b_2]$ such that $Q(s_1)$, $Q(s_2) \in \overline{B}_\rho(0) \cup \overline{B}_\rho(\xi)$.*

Proof: The proof of Lemma 2.3 shows there exists s_i with $Q(s_i) \in \overline{B}_\rho(x_i)$ for some $x_i \in \mathbb{Z}^n$, $i = 1, 2$. Thus the possibility that $x_i \notin \{0, \xi\}$ must be excluded. Since $Q(-\infty) = 0$, $Q(b_2) \in \overline{B}_r(\xi)$, $r \ll 1$, and Q minimizes I in Γ_1, simple comparison arguments in the spirit of Lemma 2.5 show $x_i = 0$ or ξ, $i = 1, 2$.

To show that Q is a solution of (1.1), further conditions will have to be imposed on $a \in \mathcal{A}$:

- (a_1) there is a $T > 0$ and a sequence of points $(m_i)_{i\in\mathbb{Z}} \subset \mathbb{R}$ such that $m_{i+1} - m_i \geq T$
- (a_2) there is a $\gamma > 0$ and $\theta_i \in (2\omega, m_i - m_{i-1} - 2\omega)$, such that for all $i \in \mathbb{Z}$, where
  - (i) $a(t) - a(s) \geq \gamma$, $t \in [m_i - \omega, m_i + \omega]$, $s \in [m_i - \theta_i - \omega, m_i - \theta_i + \omega]$.
  - (ii) $a(t) - a(s) \geq \gamma$, $t \in [m_i - \omega, m_i + \omega]$, $s \in [m_i + \theta_{i+1} - \omega, m_i + \theta_{i+1} + \omega]$.

Define

$$\mathcal{A}^* = \{a \in \mathcal{A} : (\mathrm{a}_1) \text{ and } (\mathrm{a}_2) \text{ hold}\}.$$

Conditions (a_1)–(a_2) are satisfied if e.g. a is T periodic in t, with T appropriately large, $m_{i+1} = m_i + T$, $a(m_i) = \max a$, $\theta_{i+1} = \theta_i + T$, $a(m_i + \theta_i) = \min a$, $\gamma = \frac{1}{2}(a(m_i) - a(m_i + \theta_i))$ and a oscillates slowly so (a_2) holds. More generally, it suffices that a remains near its maximum and minimum on a large time interval. In particular, as mentioned in the Introduction, these conditions will be satisfied if $a(t) = b(\epsilon t)$ with b positive, continuous, 1-periodic in t, and $\not\equiv$ constant, and ϵ sufficiently small. Suppose further

$$\varphi(\rho) < \frac{\gamma}{32M}\left(\underline{a}/\overline{a}\right)\beta(r). \tag{2.7}$$

Choosing $(b_1, b_2) = (m_i, m_{i+1})$, we have

Theorem 2.7 *Suppose* (W_1)*–*(W_2) *hold,* ρ *and* r *satisfy* $\rho \ll r \ll 1$ *and (2.7), and* $a \in \mathcal{A}^*$. *Then for each* $i \in \mathbb{Z}$, *(1.1) has a solution* $Q = Q_i \in \Gamma(m_i, m_{i+1})$ *with* $I(Q_i) = c(m_i, m_{i+1})$.

Proof: Since it does not effect the argument, for notational simplicity, we set $i = 1$. By Remark 2.2 and Lemma 2.5, Q is a solution of (1.1) except possibly for $t \in (t_1, m_1] \cup [m_2, t_2)$. Suppose e.g. $Q(t) \in \partial B_r(0)$ for some $t \in (t_1, m_1]$. Then the cost analysis of Lemma 2.5 shows, $Q(s_1) \in \overline{B}_\rho(\xi)$, $Q(t) \in B_{\frac{r}{2}}(\xi)$ for $t \geq s_1$, and

$$\int_{s_1}^{\infty} L(Q)dt \leq \varphi(\rho).$$

Therefore, $Q^*(t) = Q(t-\tau) \in \Gamma$ for any $\tau \in [0, m_2 - s_1]$. Since $\theta_2 < m_2 - m_1 - 2\omega$ and $s_1 < m_1 + \omega$, $\theta_2 < m_2 - s_1 - \omega < m_2 - s_1$ so taking $\tau = \theta_2$ shows

$$0 \geq I(Q) - I(Q^*) = -\int_{\mathbb{R}} (a(t) - a(t + \theta_2))W(Q)\, dt. \tag{2.8}$$

Now

$$\Big|\int_{-\infty}^{t_1}(a(t)-a(t+\theta_2))W(Q)dt\Big|\le 2\frac{\overline{a}}{\underline{a}}\int_{-\infty}^{t_1}L(Q)dt\le 2\frac{\overline{a}}{\underline{a}}\varphi(\rho) \tag{2.9}$$

and similarly

$$\Big|\int_{s_1}^{\infty}(a(t)-a(t+\theta_2))W(Q)dt\Big|\le 2\frac{\overline{a}}{\underline{a}}\int_{s_1}^{\infty}L(Q)dt\le 2\frac{\overline{a}}{\underline{a}}\varphi(\rho) \tag{2.10}$$

while by (a_2)(ii),

$$-\int_{t_1}^{s_1}(a(t)-a(t+\theta_2))W(Q)dt\ge\gamma\int_{t_1}^{s_1}W(Q)\,dt\,. \tag{2.11}$$

In the interval $[t_1,s_1]$, Q goes from $\partial B_\rho(0)$ to $\partial B_\rho(\xi)$. In particular, since Q minimizes (2.1), there is a subinterval $[\sigma,s]$ of $[t_1,s_1]$ in which Q lies in $\mathbb{R}^n\backslash B_r(\mathbb{Z}^n)$ and joins $\partial B_r(0)$ to $\partial B_r(\xi)$. Hence by the definition of M,

$$\begin{aligned}\frac{1}{2} &\le |Q(s)-Q(\sigma)|=\Big|\int_\sigma^s\dot{Q}(t)dt\Big|\\ &\le (s-\sigma)^{1/2}\Big(\int_\sigma^s|\dot{Q}|^2dt\Big)^{1/2}\\ &\le (s-\sigma)^{1/2}(2I(Q))^{1/2}\\ &\le (2M(s-\sigma))^{1/2}\end{aligned}$$

so that $s-\sigma\ge 1/(8M)$, and

$$-\int_{t_1}^{s_1}W(Q)dt\ge-\int_\sigma^s W(Q)dt\ge\frac{1}{8M}\beta(r)\,.$$

Combining (2.9)–(2.11) and the above equation yields

$$0\ge\gamma\frac{1}{8M}\beta(r)-4\frac{\overline{a}}{\underline{a}}\varphi(\rho)$$

contrary to (2.7). Hence it is not possible that $Q(t)\in\partial B_r(0)$ for $t\in(t_1,m_1]$. Similarly using (a_2) (i), $Q(t)\in\partial B_r(\xi)$ for $t\in[m_2,t_2)$ cannot occur. Thus Q is a solution of (1.1) and Theorem 2.7 is proved.

Remark 2.8 (i) Replacing Q_i+j for $j\in\mathbb{Z}^n$ gives a solution of (1.1) in $\Gamma(m_i,m_{i+1})$ heteroclinic from j to $j+\xi$.

(ii) Possibly $Q_i(\infty)\ne Q_{i-1}(\infty)$.

(iii) Although Q_i need not be unique, when $a\in\mathcal{A}^*$ is T-periodic, one choice for $Q_{i-1}(t)$ is $Q_i(t-T)$.

Remark 2.9 Modifying slightly arguments as in [8, 15, 9] gives at least $n+1$ distinct points $\xi_0 \equiv \xi$, $\xi_1, \dots, \xi_n \in \mathbb{Z}^n \backslash \{0\}$ and corresponding heteroclinic solutions $Q_i^0 \equiv Q_i$, $Q_i^1, \dots, Q_i^n \in \Gamma(m_i, m_{i+1})$ provided that (2.7) is strengthened. E.g. once $Q_i^0, \dots, Q_i^{\ell-1}$ have been found, Q_i is the minimizer of the variational problem

$$\inf_{q \in \Gamma_\ell(m_i, m_{i+1})} I(q)$$

where

$$\Gamma_\ell(m_i, m_{i+1}) = \{q \in \Gamma(m_i, m_{i+1}) \mid q(\infty) \notin \mathrm{span}_{\mathbb{N}}\{\xi_0, \dots, \xi_{\ell-1}\}$$

and $\mathrm{span}_{\mathbb{N}} X$ denotes the span with coefficients in $\mathbb{N} \cup \{0\}$ of elements in X. Moreover Q_i^ℓ is a solution of (1.1), $0 \le i \le n$ as in Theorem 2.7 provided that (2.7) is replaced by

$$\varphi(\rho) < \frac{\gamma}{32 M^*} \left(\underline{a}/\overline{a}\right) \beta(r)\,, \tag{2.12}$$

where M^* is defined as follows. Let e.g. $e_1, \dots, e_n$ be the usual basis in $\mathbb{R}^n$, i.e. $e_1 = (1, 0, \dots, 0)$, etc. Set $e_{n+1} = (-1, \dots, -1)$. Replace η in (2.2) by e_i, calling the resulting function R_i. Then at least one of $R_1(1), \dots, R_{\ell+1}(1) \notin \mathrm{span}_{\mathbb{N}}\{\xi_0, \dots, \xi_{\ell-1}\}$. Set

$$M^* = \max_{1 \le i \le n+1} I(R_i)\,..$$

Remark 2.10 As mentioned in the Introduction, the conclusions of Theorem 2.7 hold for a more general class of a's than $\mathcal{A}^*$. Rather than formalizing such a result, we just give an example of this type. Suppose $a = \alpha_1 + \alpha_2$ where $\alpha_1 \in \mathcal{A}^*$ and $\alpha_2 \ge 0$ is continuous and periodic with period $p \le 1$ which for convenience will be taken to be 1. (Some small modifications in the argument that follows are needed if $p < 1$.) It can be assumed that $\omega >> 1$. Let μ denote the greatest integer in θ_2, $\mu = [\theta_2]$, so $0 \le \theta_2 - \mu < 1 \ll \omega$. Now in the proof of Theorem 2.7, choose $\tau = \mu$. Since $\mu \le \theta_2 < m_2 - s_1 - \omega + 1 < m_2 - s_1$, $Q^* \in \Gamma$ as earlier so (2.8) becomes

$$0 \ge -\int_{\mathbb{R}} (\alpha_1(t) - \alpha_1(t+\mu)) W(Q)\, dt - \int_{\mathbb{R}} (\alpha_2(t) - \alpha_2(t+\mu)) W(Q)\, dt\,.$$

By the 1-periodicity of α_2, the second integral on the right vanishes so the earlier argument can be used again to get existence here. The multiplicity results of Remark 2.9 are also valid for this more general class of a's.

3 Solutions of multichain type

Consider a heteroclinic ℓ-chain constructed by gluing together ℓ basic heteroclinics or their translates as obtained in Remarks 2.9 and 2.8(i). Suppose the chain begins at ξ_0 and ends at ξ_ℓ. The goal of this section is to show there are infinitely many heteroclinic solutions of (1.1) that spend as much time as desired near $\xi_1, \ldots, \xi_{\ell-1}$. To be more precise, let r, ρ, and $\mathcal{A}^*$ be as in §2. Let $k \in \mathbb{Z}^{2\ell}$ where $k = (k_1, \ldots, k_{2\ell})$, $k_j < k_{j+1}$, and $k_j = m_{i_j}$ for some i_j where (m_i) is as in (a_1). Define

$$\begin{aligned} \Gamma_k \quad = \quad & \big\{q \in W^{1,2}_{\text{loc}} \mid q(t) \in \overline{B}_r(\xi_0),\ t \le k_1,\ q(t) \in \overline{B}_r(\xi_j),\ t \in [k_{2j}, k_{2j+1}], \\ & 1 \le j \le \ell - 1, \text{ and } q(t) \in \overline{B}_r(\xi_\ell),\ t \ge k_{2\ell}\big\} \end{aligned}$$

Set

$$c_k = \inf_{q \in \Gamma_k} I(q)\,. \tag{3.1}$$

Repeating arguments from §2 gives

Proposition 3.1 *1. There exists $Q = Q_k \in \Gamma_k$ such that $I(Q_k) = c_k$.*

2. There are numbers $t_1 \in [k_1 - \omega, k_1]$, $t_{2j} \in [k_{2j}, k_{2j} + \omega]$, $t_{2j+1} \in [k_{2j+1} - \omega, k_{2j+1}]$, $1 \le j \le \ell - 1$, $t_{2\ell} \in [k_{2\ell}, k_{2\ell} + \omega]$ such that $Q(t_1) \in \overline{B}_\rho(\xi_0)$, $Q(t_{2j})$, $Q(t_{2j+1}) \in \overline{B}_\rho(\xi_j)$, $Q(t_{2\ell}) \in \overline{B}_\rho(\xi_\ell)$.

3. There is a $\varphi(\rho) \to 0$ as $\rho \to 0$ such that

$$\int_{-\infty}^{t_1} L(Q)dt, \quad \int_{t_{2\ell}}^{\infty} L(Q)dt \le \varphi(\rho)$$

and similarly,

$$\int_{t_{2j}}^{t_{2j+1}} L(Q)dt \le \varphi(\rho), \quad 1 \le j \le \ell - 1$$

4. $Q(t) \in B_{r/2}(\xi_j)$, $t \le t_1$, $Q(t) \in B_{r/2}(\xi_j)$, $t \in [t_{2j}, t_{2j+1}]$, $Q(t) \in B_{r/2}(\xi_\ell)$, $t \ge t_{2\ell}$.

5. There is an $s_1 \in [k_1, k_1 + \omega]$, $s_{2j} \in [k_{2j} - \omega, k]$, $s_{2j+1} \in [k_{2j+1}, k_{2j+1} + \omega]$, $1 \le j \le \ell - 1$, $s_{2\ell} \in [k_{2\ell} - \omega, k_{2\ell}]$ such that $Q(s_1)$, $Q(s_2) \in \overline{B}_\rho(\xi_0) \cup B_\rho(\xi_1), \cdots Q(s_{2\ell-1}), Q(s_{2\ell}) \in \overline{B}_\rho(\xi_{\ell-1}) \cup \overline{B}_\rho(\xi_\ell)$.

The characterization of Q_k as a minimum in (3.1) implies there is an $\overline{M} > 0$ and independent of ℓ such that

$$\int_{k_{2j-1}}^{k_{2j}} L(Q_k)dt \le \overline{M}, \quad 1 \le j \le \ell \tag{3.2}$$

Replacing (2.12) by

$$\varphi(\rho) < \frac{\gamma}{80\overline{M}}\beta(r), \tag{3.3}$$

we have

Theorem 3.2 *If (W_1)–(W_2) hold, $\rho \ll r \ll 1$, (3.3) is satisfied, and $a \in \mathcal{A}^*$, then (1.1) has a solution, $Q_k \in \Gamma_k$ with $I(Q_k) = c_k$.*

Proof: As earlier, it suffices to show $Q(t) = Q_k(t) \notin \partial B_r(\xi_0)$, $t \le k_1$; $Q(t) \notin \partial B_r(\xi_j)$, $t \in [k_{2j}, k_{2j+1}]$, $1 \le j \le \ell - 1$; $Q(t) \notin \partial B_r(\xi_\ell)$, $t \ge k_{2\ell}$. The idea is to show if one of these conditions is violated, it is possible to construct an appropriate $Q^* \in \Gamma_k$ and obtain a contradiction as in §2. There are basically two cases to consider.

Suppose first that $Q(t) \in \partial B_r(\xi_0)$ for some $t \in (t_1, k_1]$. Then as in the proof of Theorem 2.7, $Q(s_1) \in \overline{B}_\rho(\xi_i)$, $Q(t) \in B_{r/2}(\xi_1)$ for $t \in [s_1, t_2]$, and

$$\int_{s_1}^{t_2} L(Q)dt \le \varphi(\rho)\,. \tag{3.4}$$

Set

$$Q^*(t) = \begin{cases} Q(t - k_1) \qquad \text{if } t \le s_1 + \theta_{k_1+1}, \\ (s_1 + \theta_{k_1+1} + 1 - t)Q(s_1) + (t - (s_1 + \theta_{k_1+1}))\xi_1, \\ \qquad \text{if } s_1 + \theta_{k_1+1} \le t \le s_1 + \theta_{k_1+1} + 1 \\ (s_1 + \theta_{k_1+1} + 2 - t)\xi_1 + (t - (s_1 + \theta_{k_1+1} + 1))Q(s_1 + \theta_{k_1+1} + 2), \\ \qquad \text{if } s_1 + \theta_{k_1+1} + 1 \le t \le s_1 + \theta_{k_1+1} + 2 \\ Q(t), \qquad \text{if } t \ge s_1 + \theta_{k_1+1} + 2. \end{cases} \tag{3.5}$$

Then $Q^* \in \Gamma_k$ and

$$\begin{aligned} 0 \ge I(Q) - I(Q^*) \quad &= \quad \int_{-\infty}^{s_1} L(Q)dt - \int_{-\infty}^{s_1+\theta_{k_1+1}+1} L(Q^*)\,dt \\ &\quad + \int_{s_1}^{s_1+\theta_{k_1+1}+2} L(Q)dt - \int_{s_1+\theta_{k_1+1}}^{s_1+\theta_{k_1+1}+2} L(Q^*)\,dt\,. \end{aligned}$$

By (3.4), each of the last two terms in this inequality is less than or equal to $\varphi(\rho)$. Therefore,

$$0 \ge -\int_{-\infty}^{s_1} (a(t) - a(t + \theta_{k_1+1}))W(Q)dt - 2\varphi(\rho).$$

As in §2, this leads to

$$0 \ge \gamma \frac{1}{8\overline{M}} \beta(r) - 3\varphi(\rho)$$

contrary to (3.3).

Using (a$_2$)(i), a similar argument holds if $Q(t) \in \partial B_r(\xi_{2\ell})$ for some $t \in [k_{2\ell}, t_{2\ell})$. If $Q(t) \in \partial B_r(\xi_j)$ for some $t \in [k_{2j}, k_{2j+1}]$, then by Proposition 3.1, either $t \in [k_{2j}, t_{2j})$ or $t \in (t_{2j+1}, k_{2j+1}]$. The argument is similar in either event, so suppose $t \in [k_{2j}, t_{2j})$. Then $Q(s_{2j}) \in \overline{B}_\rho(\xi_{j-1})$ and $Q(t) \in B_{r/2}(\xi_{j-1})$ for $t \in [t_{2j-1}, s_{2j}]$. It is now convenient to use two comparison functions. Define

$$\widetilde{Q}(t) = \begin{cases} Q(t) & \text{if } t \le t_{2j-1} \\ \xi_{j-1} & \text{if } t_{2j-1} + 1 \le t \le s_{2j} - 1 \\ Q(t) & \text{if } s_{2j} \le t \le t_{2j} \\ \xi_j & \text{if } t_{2j+1} \le t \le t_{2j+1} - 1 \\ Q(t) & \text{if } t \ge t_{2j} \end{cases}$$

with a linear interpolant, as in (3.5) for the four intermediate intervals. Then $\widetilde{Q} \in \Gamma_k$ and

$$\begin{aligned} 0 &\leq I(\widetilde{Q}) - I(Q) \\ &= \int_{t_{2j-1}}^{t_{2j-1}+1} L(\widetilde{Q})dt + \int_{s_{2j-1}}^{s_{2j}} L(\widetilde{Q})\,dt + \int_{t_{2j}}^{t_{2j+1}} L(\widetilde{Q})\,dt \\ &\quad + \int_{t_{2j+1}-1}^{t_{2j+1}} L(\widetilde{Q})\,dt - \int_{t_{2j-1}}^{s_{2j}} L(Q)\,dt - \int_{t_{2j}}^{t_{2j+1}} L(Q)\,dt\,. \end{aligned}$$

Each of the terms on the right-hand side of this inequality is less than or equal to $\varphi(\rho)$ so

$$0 \leq I(\widetilde{Q}) - I(Q) \leq 6\varphi(\rho)\,. \tag{3.6}$$

Now define

$$Q^*(t) = \begin{cases} \widetilde{Q}(t) & \text{if } t \leq t_{2j} + 1 - \theta_j \\ \widetilde{Q}(t+\theta_j) & \text{if } t_{2j+1} + 1 - \theta_j \leq t \leq t_{2j} + 1 \\ \widetilde{Q}(t) & \text{if } t \geq t_{2j} + 1\,. \end{cases}$$

Again $Q^* \in \Gamma_k$ and

$$0 \leq I(Q^*) - I(Q) = I(Q^*) - I(\widetilde{Q}) + I(\widetilde{Q}) - I(Q)\,. \tag{3.7}$$

Hence by (3.13),

$$I(\widetilde{Q}) - I(Q^*) \leq I(\widetilde{Q}) - I(Q) \leq 6\varphi(\rho)\,. \tag{3.8}$$

But by the definition of Q^* and $\widetilde{Q}$,

$$\begin{aligned} I(\widetilde{Q}) - I(Q^*) &= \int_{t_{2j-1}+1}^{t_{2j}+1} (L(\widetilde{Q}) - L(Q^*))dt \\ &= -\int_{s_{2j-1}}^{t_{2j+1}} (a(t) - a(t-\theta_j))W(Q)dt \\ &\geq -\int_{s_{2j}}^{t_{2j}} (a(t) - a(t-\theta_j))W(Q)dt - 4\varphi(\rho) \\ &\geq \frac{\gamma}{8\overline{M}}\beta(r) - 4\varphi(\rho). \end{aligned} \tag{3.9}$$

Combining (3.7)–(3.9) shows

$$\frac{\gamma\beta(r)}{8\overline{M}} \leq 10\varphi(\rho)$$

contrary to (3.3). The proof is complete.

Remark 3.3 By choosing k appropriately, the solution, Q_k, of (1.1) is near each of the equilibrium points $\xi_1, \ldots, \xi_{\ell-1}$ for as long a time interval as desired.

However Q_k need not be near the original heteroclinic chain joining 0 and ξ_ℓ, i.e. $Q_k\big|_{k_{2j-1}}^{k_{2j}}$ is not necessarily near any basic heteroclinic joining ξ_{j-1} and ξ_j. Nevertheless, a $Q_k\big|_{k_{2j-1}}^{k_{2j}}$ near such P_j can be constructed by taking $k_{2j}-k_{2j-1}$ sufficiently large as in [17]. Indeed (3.2) implies an L^∞ upper bound for $Q_k\big|_{k_{2j-1}}^{k_{2j}}$ independent of $k_{2j}-k_{2j-1}$ and (1.1) then yields such a bound in C^2. As $k_{2j}-k_{2j-1}\to\infty$, by standard arguments as in [17], $Q_k\big|_{k_{2j-1}}^{k_{2j}}$ approaches a chain of heteroclinic $H,\dots,H_s$ joining ξ_{j-1} and ξ_j with

$$\sum_1^s I(H_i) = I(P_j).$$

The construction of P_j as indicated in Remark 2.9 implies $s=1$. Hence for $k_{2j}-k_{2j-1}$ large, $Q_k\big|_{k_{2j-1}}^{k_{2j}}$ will be near a basic heteroclinic P_j joining ξ_{j-1} and ξ_j.

A standard consequence of Theorem 3.2 is the existence of solutions of infinite chain type of (1.1). Consider any formal doubly infinite heteroclinic chain made up of the basic heteroclinics of Remark 2.9. The endpoints of the chain form a sequence $\Xi = (\xi_i)_{i\in\mathbb{Z}}$, $\xi_i\in\mathbb{Z}^n$. Let $k=(k_i)_{i\in\mathbb{Z}}$ with $k_i<k_{i+1}$ and each $k_i = m_{i_j}$ for some j. Now set

$$\Gamma_k = \{q\in W^{1,2}_{\mathrm{loc}} \mid q(t)\in \overline{B}_r(\xi_j),\ t\in[k_{2j},k_{2j+1}],\ j\in\mathbb{Z}\}.$$

Then we have

Theorem 3.4 *Under the hypothesis of Theorem 3.2, for each* Ξ, k *as above, there is a solution,* $Q_k\in\Gamma_k$, *of (1.1).*

Proof: Note that the construction of Theorem 3.2 is independent of ℓ, the number of basic homoclinics. For Ξ and k as above, let $\Xi_\ell = (\xi_{-\ell},\dots,\xi_\ell)$ and $K_\ell = (k_{-2\ell},\dots,k_{2\ell})\in\mathbb{Z}^{4\ell}$. Then by Theorem 3.2, there is a solution Q_ℓ of (1.1) in Γ_{K_ℓ}, heteroclinic from $\xi_{-\ell}$ to ξ_ℓ. Since Q_ℓ is a solution of (1.1), for each $j\in\mathbb{Z}$, the form of Γ_{K_ℓ} yields $C^2([k_{2j},k_{2j+1}],\mathbb{R}^n)$ bounds for Q_ℓ (independent of ℓ). Moreover in the intervals $[k_{2j-1},k_{2j}]$, the bound (3.2) holds with $Q=Q_\ell$. As in Remark 3.3, this gives bounds in $C^2([k_{2j-1},k_{2j}],\mathbb{R}^n)$ for Q_ℓ independent of ℓ. The Arzela-Ascoli Theorem then yields the desired solution Q_k of (1.1).

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