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Existence of a principal eigenvalue for the Tricomi problem *

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Dedicated to Alan Lazer
on his 60th birthday

Abstract

The existence of a principal eigenvalue is established for the Tricomi problem in normal domains; that is, the existence of a positive eigenvalue of minimum modulus with an associated positive eigenfunction. The argument here uses prior results of the authors on the generalized solvability in weighted Sobolev spaces and associated maximum/minimum principles [17] coupled with known results of Krein-Rutman type.

1 Introduction

In this note, we are interested in establishing the existence of a principal eigenvalue associated to generalized solutions of the classical linear Tricomi problem. That is; we seek to find an eigenvalue-eigenfunction pair (λ, u) with $\lambda > 0$ and u a positive generalized solution to the problem

$$\begin{aligned} Tu &= \lambda u \quad \text{in } \Omega \\ u &= 0 \quad \text{on } AC \cup \sigma. \end{aligned} \tag{LTE}$$

The results will apply as well to the conjugate problem (LTE)* in which the boundary conditions $u = 0$ are placed on $BC \cup \sigma$ instead. Here $T \equiv -y\partial_x^2 - \partial_y^2$ is the Tricomi operator on $\mathbb{R}^2$ and Ω is a bounded region in $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$ of the classical Tricomi form. That is, $\partial\Omega$ consists of a smooth arc σ in the elliptic region $y > 0$, with endpoints on the x-axis at $A = (-x_0, 0)$ and $B = (x_0, 0)$, and two characteristic arcs AC and BC for the Tricomi operator in the hyperbolic region $y < 0$ issuing from A and B and

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meeting at the point C on the y-axis (we assume without loss of generality that A and B are symmetric with respect to the y-axis). One knows that

$$AC: \ (x+x_0) - \frac{2}{3}(-y)^{3/2} = 0 \quad \text{and} \quad BC: \ (x-x_0) + \frac{2}{3}(-y)^{3/2} = 0\,.$$

We will call such a domain a *Tricomi domain*. Moreover, we will assume that the domain is *normal* in the sense that σ is perpendicular to the x-axis in the points A and B.

The underlying Tricomi problem

$$\begin{aligned} Tu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } AC \cup \sigma \end{aligned} \tag{LT}$$

has a notable physical importance. It describes, in the hodograph plane, the problem of transonic flow through a nozzle; a connection first established by Frankl' [6]. The placement of the boundary condition on only a portion $AC \cup \sigma$ of the boundary can yield a well posed problem for classical solutions as first established by Tricomi himself [19] in special cases, whereas placement of data on larger portions of the boundary will overdetermine the problem for classical solutions due to the presence of hyperbolicity; energy integral methods based on the work of Friedrichs [7] yield suitable uniqueness theorems. In addition, there are a wealth of results on existence and uniqueness of strong solutions in Hilbert spaces well adapted to the boundary condition. However, despite its physical importance and despite some 70 years of study, results on the linear Tricomi problems (LT) and (LT)* are not complete. In particular, there is an almost complete absence of spectral theory which is glaring in its own right and impedes substantially progress on associated nonlinear problems. Recent works such as [16] and [15] have attempted to make progress in this direction. One major difficulty lies in the fact that the problem (LT) is **not** self-adjoint.

In part as preparation for the current study, we have established results on the generalized solvability of (LT) in weighted Sobolev spaces and associated maximum/minimum principles for normal domains [17] (which will be recalled in section 2). It is well known that the presence of extremum principles if interpretable as the invariance of a positive cone in a suitable Banach space can be a first step in developing a spectral theory even in cases where the operator is not self adjoint. Such ideas were pioneered by Krein and Rutman [12] and have played an important role in nonlinear analysis (cf. Krasnoselskii [11]). As noted in [17], the solution operators $\widetilde{T}_{AC}^{-1}$ and $\widetilde{T}_{BC}^{-1}$ to the Tricomi problems (LT) and (LT)* are compact as operators on $L^2(\Omega)$ and thanks to the maximum/minimum principles preserve the positive cone in $L^2(\Omega)$. Combining these facts with an argument of Krein-Rutman type using the Hopf maximum principle in the elliptic part of the domain, we establish the following result on the existence of a principal eigenvalue. The relevant definitions and notations will be recalled in section 2.

Theorem 1.1 *Let Ω be a normal admissible Tricomi domain. Then there exists a positive eigenvalue λ_0 of minimum modulus in the sense that $|\lambda_0| \leq |\lambda|$ for* *every $\lambda \in \sigma(\widetilde{T}_{AC})$. Moreover, associated to λ_0 there are corresponding eigenfunctions $u_0 \in \widetilde{W}^1_{AC\cup\sigma} \subset L^2(\Omega)$ and $v_0 \in \widetilde{W}^1_{BC\cup\sigma} \subset L^2(\Omega)$ for the problems (LTE) and $(LTE)^*$ which are positive; that is, $u_0, v_0 \geq 0$ a.e. in Ω.*

We point out that λ_0 is called a principal eigenvalue due to the positivity of the associated eigenfunction and its being of minimum modulus, as done by Lazer in [14, 4]. However, we cannot say at present that the associated eigenspace is simple, nor that other eigenspaces do not contain eigenfunctions that are non-negative almost everywhere, as happens in the purely elliptic case (cf. Manes - Micheletti [18] as well as Hess - Kato [10]).

2 Generalized solvability and maximum/minimum principles

In this section, we will recall the main definitions, notation, and results on generalized solutions that will be used in the proof of the main theorem. We begin by recalling the spaces of functions in which we will work. In all that follows, Γ will be a connected subset of $\partial\Omega$ which is assumed to be piecewise C^1 (in order to apply the divergence theorem). We consider the following spaces of smooth functions

$$C^\infty_{0,\Gamma}(\overline{\Omega}) = \left\{\psi \in C^\infty(\overline{\Omega}) : \psi \equiv 0 \text{ on } N_\epsilon\Gamma \ \text{ for some } \epsilon > 0\right\}, \tag{2.1}$$

where $N_\epsilon\Gamma$ is an ϵ neighborhood of Γ. We denote by $\widetilde{W}^1_\Gamma(\Omega)$ the Sobolev space obtained as closure of the spaces in (2.1) with respect to the norm

$$||\psi||^2_{\widetilde{W}^1_\Gamma(\Omega)} = ||\psi||^2_{\widetilde{W}^{1,2}(\Omega)} = \int_\Omega \left(|y|\psi_x^2 + \psi_y^2 + \psi^2\right)\, dx\, dy,$$

and denote by $\widetilde{W}^{-1}_\Gamma$ the dual space to $\widetilde{W}^1_\Gamma$ equipped with its negative norm in the sense of Lax (cf. [13]). We note that using the definition of the $\widetilde{W}^{-1}_\Gamma(\Omega)$ norm, one has the following estimates: for each Ω with $\partial\Omega$ piecewise C^1, there exist constants $C_1, C_2 > 0$ such that

$$||Tu||_{\widetilde{W}^{-1}_{BC\cup\sigma}(\Omega)} \leq C_1||u||_{\widetilde{W}^1_{AC\cup\sigma}(\Omega)}, \quad u \in C^\infty_{0,AC\cup\sigma}(\overline{\Omega}) \tag{2.2}$$

and

$$||Tv||_{\widetilde{W}^{-1}_{AC\cup\sigma}(\Omega)} \leq C_2||v||_{\widetilde{W}^1_{BC\cup\sigma}(\Omega)}, \quad v \in C^\infty_{0,BC\cup\sigma}(\overline{\Omega}) \tag{2.3}$$

which are just continuity estimates. They give rise to continuous extensions of the Tricomi operator T (defined on dense subspaces of smooth functions) such as

$$\widetilde{T}_{AC} : \widetilde{W}^1_{AC\cup\sigma}(\Omega) \to \widetilde{W}^{-1}_{BC\cup\sigma}(\Omega) \ \text{ and } \ \widetilde{T}_{BC} : \widetilde{W}^1_{BC\cup\sigma}(\Omega) \to \widetilde{W}^{-1}_{AC\cup\sigma}(\Omega). \tag{2.4}$$

We recall that the placement of the boundary conditions on only a portion of the boundary implies that the problems (LT) and (LT)* are not self adjoint. In fact, one has that the operators in (2.4) satisfy $\widetilde{T}_{BC} = \widetilde{T}^*_{AC}$.

As shown by Didenko, a necessary and sufficient condition to have the generalized solvability for the problem (LT) and (LT)* for each $f \in L^2(\Omega)$ is to have the continuity estimates (2.2) and (2.3) as well as the *a priori* estimates of *admissibility* as encoded in the following definition.

Definition 2.1. *A Tricomi domain Ω will be said to be admissible if there exist positive constants C_3 and C_4 such that*

$$\|u\|_{L^2(\Omega)} \leq C_3 \|Tu\|_{\widetilde{W}^{-1}_{BC\cup\sigma}}, \qquad u \in C^\infty_{0,AC\cup\sigma}(\overline{\Omega}) \tag{2.5}$$

and

$$\|v\|_{L^2(\Omega)} \leq C_4 \|Tv\|_{\widetilde{W}^{-1}_{AC\cup\sigma}(\Omega)}, \qquad v \in C^\infty_{0,BC\cup\sigma}(\overline{\Omega}). \tag{2.6}$$

The class of admissible normal domains includes convex domains as well as those that contain a convex subdomain; we record the following result of [17] as an example.

Example 2.2. *Let Ω be a normal Tricomi domain, with boundary $AC \cup BC \cup \sigma$ such that*

i) *Ω contains Ω_0 as a subdomain where Ω_0 has boundary $AC \cup BC \cup \sigma_0$ such that the elliptic boundary arc σ_0 is given as a graph $\{(x,y) : y = g(x),\ -x_0 \leq x \leq x_0\}$ which satisfies the following hypotheses: $g \in C^2((-x_0, x_0))$, $g(\pm x_0) = 0$, $g'(\mp x_0^{\pm}) = \pm\infty$ and, for every $x \in (-x_0, x_0)$, $g(x) > 0$ and $g''(x) \leq -k < 0$.*

ii) *There exists an $\epsilon > 0$ such that the elliptic boundaries σ and σ_0 of Ω and Ω_0 coincide in a strip $\{(x,y) \ : \ 0 \leq y \leq \epsilon\}$.*

Then Ω is admissible in the sense of Definition 2.1.

As noted above, the *a priori* estimates of admissibility together with some functional analysis yield the following kind of solvability result (cf. section 2 of [17]).

Theorem 2.3. *Let Ω be normal admissible Tricomi domain in the sense of Definition 2.1. Then for every $f \in L^2(\Omega)$ there exists a unique generalized solution $u \in \widetilde{W}^1_{AC\cup\sigma}(\Omega)$ to the Tricomi problem (LT) in the sense that there exists a sequence $\{u_j\} \subset C^\infty_{0,AC\cup\sigma}(\overline{\Omega})$ such that*

$$\lim_{j\to\infty} \|u_j - u\|_{\widetilde{W}^1_{AC\cup\sigma}(\Omega)} = 0 \quad \text{and} \quad \lim_{j\to\infty} \|Tu_j - f\|_{\widetilde{W}^{-1}_{BC\cup\sigma}(\Omega)} = 0.$$

As a final preliminary, we recall the following maximum/minimum principle for generalized solutions which is proven by regularizing the problem, applying a variant of the maximum principle of Agmon-Nirenberg-Protter for regular solutions [2], and using the continuity of the solution operator $(\widetilde{T}_{AC})^{-1}$ (cf. Theorem 3.1 of [17]).

Theorem 2.4. *Let Ω be an admissible normal Tricomi domain, $f \in L^2(\Omega)$ and $u \in \widetilde{W}^1_{AC\cup\sigma}(\Omega)$ the unique generalized solution to the problem (LT). Then $f \geq 0 (\leq 0)$ a.e. in Ω implies $u \geq 0 (\leq 0)$ a.e. in Ω. A similar statement holds for the conjugate problem $(LT)^*$*

3 Proof of Theorem 1.1

Having recalled the machinery of generalized solvability and extremum principles, we will show how they give rise to the existence of a principal eigenvalue. The proof proceeds in three steps. The first makes use of the solvability theory.

Step 1. (Passage to the inverse operator)

An immediate consequence of the solvabilty result Theorem 2.3 is the existence of a continuous right inverse K defined on all of $L^2(\Omega)$ whose image $\widetilde{W}_A$ is a dense proper subspace of $\widetilde{W}^1_{AC\cup\sigma}(\Omega)$. The operator K gives rise to a compact operator $K = \widetilde{T}^{-1}_{AC} : L^2(\Omega) \to L^2(\Omega)$. This follows from standard functional analysis (cf. section 2 of [16]). It is therefore clear by a classical result (cf. [3] or [5]) that this injective, non surjective compact operator K has spectrum $\sigma(K) = \{\mu_j\}_{j\in\mathbb{Z}}$ consisting of eigenvalues of finite multiplicity with 0 as the only possible accumulation point. Therefore it suffices to find an eigenvalue $\mu_0 \in \mathbb{R}^+$ and an associated eigenfunction $v_0 \in L^2(\Omega)$ for K satisfying $v_0 \geq 0$ a.e. in Ω, as then

$$\lambda_0 = 1/\mu_0 \quad \text{is an eigenvalue for } \widetilde{T}_{AC} \tag{3.1}$$

and

$$u_0 = K v_0 \quad \text{is an eigenfunction for } \widetilde{T}_{AC}. \tag{3.2}$$

Indeed, if $Kv_0 = \mu_0 v_0$ with $v_0 \in L^2(\Omega)$ then one has $u_0 = \mu_0(\widetilde{T}_{AC} u_0)$ from which it follows that $\widetilde{T}_{AC} u_0 = \lambda_0 u_0$. Moreover, since $u_0 = Kv_0$ is the unique generalized solution to $\widetilde{T}_{AC} u_0 = v_0$, Theorem 2.4 shows that this solution must obey $u_0 \geq 0$ a.e. in Ω if $v \geq 0$ a.e. in Ω.

Step 2. (A result of Krein and Rutman)

By Step 1, it suffices to find a positive eigenvalue μ_0 for $K = \widetilde{T}^{-1}_{AC}$ with an associated positive eigenfunction $v_0 \in L^2(\Omega)$ satisfying $v_0 \geq 0$. If we denote by $L(\Omega)^+ = \{f \in L^2(\Omega) : f \geq 0 \ \text{ a.e. in } \Omega\}$, the positive cone in $L^2(\Omega)$, Theorem 2.4 shows that the compact operator K leaves this cone invariant in the sense that $K\left(L^2(\Omega)^+\right) \subset L^2(\Omega)^+$. We recall the following special case of the result of Krein and Rutman (cf. Theorem 6.2 of [12]) which suits our needs.

Lemma 3.1. *Let K be a compact linear operator on $L^2(\Omega)$ that leaves the positive cone $L^2(\Omega)^+$ invariant. If there exist an $\alpha > 0$ and an $f \in L^2(\Omega)^+$ with $||f||_{L^2(\Omega)} = 1$ such that*

$$Kf \geq \alpha f \ \text{ a.e. in } \ \Omega, \tag{3.3}$$

then

i) *K has nonzero eigenvalues*

ii) *Among the eigenvalues of maximum modulus there is a positive one $\mu_0 \geq \alpha$ to which there correspond eigenvectors $v_0, w_0 \in L^2(\Omega)^+$ of the operators K and K^* respectively.*

The work that remains is to construct a suitable function $f \in L^2(\Omega)^+$ and parameter α so that (3.3) holds. We claim that it is enough to select any nontrivial f such that

$$f \in C^0(\overline{\Omega}), \tag{3.4}$$

$$f \geq 0 \quad \text{in } \Omega, \tag{3.5}$$

$$\operatorname{supp} f \subset \overline{B_r} \subset \Omega^+ = \Omega \cap \{y > 0\}, \tag{3.6}$$

where B_r is any ball with center in the elliptic region Ω^+ and radius r small enough so that $\overline{B_r} \subset \Omega^+$ holds and to pick

$$\alpha = 1/n \tag{3.7}$$

for n sufficiently large. Indeed, selecting $f \not\equiv 0$ satisfying (3.4) – (3.6) and then normalizing f to satisfy $||f||_{L^2(\Omega)} = 1$, it remains to verify condition (3.3) by choosing n large enough in (3.7).

Step 3. (Verification of condition (3.3))

Since $f \in C^0(\overline{\Omega}) \subset L^2(\Omega)$ with Ω normal and admissible, one can apply both Theorem 2.3 as well as the classical solvability result of Agmon [1] to show that there exists a unique generalized solution $u \in \widetilde{W}^1_{AC\cup\sigma}(\Omega) \bigcap C^0(\overline{\Omega})$ to the Tricomi problem (LT).

If one restricts the solution u to the set $\Omega_\delta^+ = \Omega^+ \cap \{y > \delta\}$, where $\delta > 0$ is chosen so that $\overline{B_r} \subset \Omega_\delta^+$, one has that this restriction belongs to $C^2(\Omega_\delta^+) \bigcap C^0(\overline{\Omega_\delta^+})$ and satisfies

$$Tu = f \quad \text{in } \Omega_\delta^+$$

$$u_{|\sigma\cap\{y\geq\delta\}} = 0\,.$$

Moreover, the restriction of u to $\overline{A'B'}$ is an element $g(x) \in C^0(\overline{A'B'})$ with $u(A') = u(B') = 0$ where A' and B' are the unique points of intersection of σ with the line $y = \delta$. In light of the needed admissibility of Ω, one might as well assume that $\Omega \cap \{y < \delta\}$ is a strip for δ small enough (cf. the hypotheses in Example 2.2).

One now may apply the Hopf maximum principle (cf. Theorem 3.5 of [9]) to the restriction of u to $\overline{\Omega_\delta^+}$ where we note that the Tricomi operator is uniformly elliptic on the subdomain Ω_δ^+. This yields

$$u > 0 \quad \text{on } \Omega_\delta^+\,,$$

where we note that u is not constant since u vanishes on the upper boundary which is a subset of σ but $u \not\equiv 0$ since $f \not\equiv 0$ in Ω_δ^+.

We are now ready to show that

$$u(x,y) \geq \alpha f(x,y) \quad \text{for each} \quad (x,y) \in \Omega \tag{3.8}$$

by choosing α small enough. We have $u \geq 0$ using (3.5) and the maximum principle of Theorem 2.4 and hence outside the support of $f = \overline{B_r}$ one has (3.8) for each $\alpha > 0$. Within the support of f, since $u, f \in C^0(\overline{B_r})$, there exist minimum and maximum values m, M such that

$$0 < m = \min_{\overline{B_r}} u \quad \text{and} \quad M = \max_{\overline{B_r}} f\,. \tag{3.9}$$

Using the bounds (3.9) it suffices to pick $\alpha = 1/n$ with n large enough to give $n \geq M/m$. This completes the proof.

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