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LARGE SOLUTIONS, METASOLUTIONS, AND ASYMPTOTIC BEHAVIOUR OF THE REGULAR POSITIVE SOLUTIONS OF SUBLINEAR PARABOLIC PROBLEMS

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Dedicated to Alan C. Lazer on his 60th birthday

Abstract. In this paper we analyze the existence of regular and large positive solutions for a class of non-linear elliptic boundary value problems of logistic type in the presence of refuges. These solutions describe the asymptotic behaviour of the regular positive solutions of the associated parabolic model. The main tool in our analysis is an extension of the interior estimates found by J. B. Keller in [Ke57] and R. Osserman in [Os57] to cover the case of changing sign nonlinearities combined with the construction of adequate sub and supersolutions. The supersolutions are far from obvious since the nonlinearity vanishes in finitely many regions of the underlying support domain.

1. Introduction

In this paper we analyze the asymptotic behaviour of the positive solutions of

$$\begin{gathered}\frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x) f(x,u) u \quad \text{in } \Omega \times (0,\infty)\,, \\ u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,\infty)\,, \\ u(\cdot,0) = u_0 \ge 0 \quad \text{in } \Omega\,, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \ge 1$, with boundary $\partial\Omega$ of class C^3, $\lambda \in \mathbb{R}$, and $a \ge 0$, $a \ne 0$, is a function of class $C^\mu(\bar\Omega)$, for some $\mu \in (0,1)$, satisfying the following assumptions:

(Ha1) The open set
$$\Omega_+ := \{\, x \in \Omega \,:\, a(x) > 0 \,\}$$
is connected with boundary $\partial\Omega_+$ of class C^3. Moreover, if Γ_+ and Γ are components of $\partial\Omega_+$ and $\partial\Omega$, respectively, such that $\Gamma_+ \cap \Gamma \ne \emptyset$, then $\Gamma_+ = \Gamma$ and $a(x)$ is bounded away from zero on Γ_+.

(Ha2) If Γ_+ is a component of $\partial\Omega_+$ such that $\Gamma_+ \cap \partial\Omega = \emptyset$, then
$$a(x) = o(\operatorname{dist}(x,\Gamma_+)) \quad \text{as } \operatorname{dist}(x,\Gamma_+) \downarrow 0\,, \quad x \in \Omega_+\,. \tag{1.2}$$

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(Ha3) The open set $\Omega_0 := \Omega \setminus \bar{\Omega}_+$ possesses a finite number of components, say $\Omega^i_{0,j}$, $1 \le i \le m$, $1 \le j \le n_i$, such that

$$\bar{\Omega}^i_{0,j} \cap \bar{\Omega}^{\hat{i}}_{0,\hat{j}} = \emptyset \quad \text{if } (i,j) \neq (\hat{i},\hat{j}) \,.$$

In the sequel given a regular subdomain D of Ω and $V \in C^\mu(\bar{D})$, we denote by $\sigma^D[-\Delta+V]$ the principal eigenvalue of $-\Delta+V$ in D under homogeneous Dirichlet boundary conditions, and set $\sigma^D := \sigma^D[-\Delta]$. Without loss of generality we can label the components of Ω_0 so that

$$\sigma^{\Omega^i_{0,j}} = \sigma^{\Omega^i_{0,j+1}} \,, \quad 1 \le i \le m \,, \;\; 1 \le j \le n_i - 1 \,, \tag{1.3a}$$

and

$$\sigma^{\Omega^i_{0,1}} < \sigma^{\Omega^{i+1}_{0,1}} \,, \quad 1 \le i \le m-1 \,. \tag{1.3b}$$

As for the nonlinearity, we suppose the following:

(Hf) The function $f : \bar{\Omega} \times [0,\infty) \to \mathbb{R}$ is continuous and of class $C^{\mu,1+\mu}(\bar{\Omega} \times (0,\infty))$ and it satisfies $f(x,0) = 0$, $f(x,u) > 0$, and $\partial_u f(x,u) > 0$ for all $u > 0$ and $x \in \Omega$. Moreover,

$$\lim_{u \uparrow \infty} f(x,u) = \infty$$

uniformly on compact subsets of $\bar{\Omega}_+$.

Problem (1.1) provides us with the evolution of a single species obeying a generalized logistic growth law [Mu93], [Ok80]. Typically, u is the density of the species, Ω is the inhabiting region, λ is the net growth rate of u, and the coefficient $a(x)$ measures the saturation effect responses to the population stress in Ω_+. In Ω_0 the individuals of the population are free from other effects than diffusion and so each of the components of Ω_0 can be regarded as a *refuge*. The refuges have been ordered accordingly to the size of the principal eigenvalue of $-\Delta$ under homogeneous Dirichlet boundary conditions. The refuges $\Omega^1_{0,j}$, $1 \le j \le n_i$, will be called the lower order refuges. Throughout this paper we use the following notation

$$\sigma_0 := \sigma^\Omega \,, \quad \sigma_i := \sigma^{\Omega^i_{0,1}} \,, \quad 1 \le i \le m \,. \tag{1.4}$$

Thanks to (1.3b),

$$\sigma_0 < \sigma_1 < \sigma_2 < \cdots < \sigma_m \,.$$

In case $m = 1$ one should simply write $\sigma_0 < \sigma_1$.

The classical results for the case when $a(x)$ is bounded away from zero in $\bar{\Omega}$ strongly suggest that the dynamics of the positive solutions of (1.1) should be regulated by its non-negative steady states, which are the non-negative solutions of

$$-\Delta u = \lambda u - a(x) f(x,u) u \quad \text{in } \;\Omega \,, \quad u|_{\partial\Omega} = 0 \,. \tag{1.5}$$

Under the assumptions above, any non-negative solution of (1.5) lies in the Banach space

$$U := \{\, u \in C^{2+\mu}(\bar{\Omega}) \;:\; u|_{\partial\Omega} = 0 \,\} \,.$$

Moreover, if $u \in U \setminus \{0\}$ is a non-negative solution of (1.5), then it follows from the strong maximum principle that $u(x) > 0$ for all $x \in \Omega$ and $\frac{\partial u}{\partial n}(x) < 0$ for all $x \in \partial\Omega$, where n is the outward unit normal to Ω at x, i.e. u lies in the interior of the cone U^+ of non-negative functions of U. Furthermore, for any $p > \frac{N}{2}$ and $u_0 \in U_0 := W_0^{2,p}(\Omega)$, $u_0 \geq 0$, the parabolic problem (1.1) has a unique global regular solution $u_{[\lambda,a,\Omega]}(x,t;u_0)$ (cf. Proposition 24.9 of [DK92]). By a global regular solution we mean that

$$u_{[\lambda,a,\Omega]} \in C(\bar{\Omega} \times [0,\infty)) \cap C^{2+\mu,1+\frac{\mu}{2}}(\bar{\Omega} \times (0,\infty)) \,. \tag{1.6}$$

Indeed, for any non-negative solution u of (1.1) we have

$$\frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x) f(x,u) u \leq \lambda u \,,$$

and hence

$$u(x,t;u_0) \leq T(t) u_0 \,,$$

where $T(t)$ is the L_p-evolution operator associated with $\Delta + \lambda$ under homogeneous Dirichlet boundary conditions. Therefore, the solutions are global in time. If the initial data has less regularity, e.g. $u_0 \in C(\bar{\Omega})$ instead of $u_0 \in U_0$, then (1.6) might fail. Nevertheless, by parabolic regularity, for any $u_0 \in C(\bar{\Omega})$, $u_0 \geq 0$, we have

$$u_{[\lambda,a,\Omega]} \in C^{2+\mu,1+\frac{\mu}{2}}(\bar{\Omega} \times (0,\infty)) \,.$$

Some pioneer results about the dynamics of the positive solutions of (1.1) in the presence of refuges were found in [FKLM96], where assuming $m = 1$ and $n_1 = 1$ it was shown that problem (1.5) possesses a positive solution if, and only if, $\sigma_0 < \lambda < \sigma_1$ and that within this range of values of the parameter λ the unique positive solution of (1.5) is a global attractor for the positive solutions of (1.1), whereas the species is driven to extinction if $\lambda \leq \sigma_0$. Some pioneer results about the existence of positive solutions of (1.5) had already been given in [BO86] and [Ou92].

In [GGLS98] problem (1.5) was analyzed under more restrictive assumptions than those imposed here in. Precisely, it was assumed that $m = n_1 = 1$, that $D := \Omega \setminus \bar{\Omega}_{0,1}^1$ satisfies $\bar{D} \subset \Omega$, and that there exists $p > 0$ such that

$$f(x,u) = |u|^p \,, \quad (x,u) \in \bar{\Omega} \times \mathbb{R} \,. \tag{1.7}$$

One of the many results found in [GGLS98] shows that if u_λ stands for the maximal regular non-negative steady-state of (1.5), then

$$\lim_{\lambda \uparrow \sigma_1} u_\lambda = \infty \quad \text{uniformly in } \bar{\Omega}_{0,1}^1 \,,$$

while in $D := \Omega \setminus \bar{\Omega}_{0,1}^1$ the positive solutions u_λ stabilize as $\lambda \uparrow \sigma_1$ to a large regular positive solution of the problem

$$\begin{aligned} -\Delta u &= \lambda u - a|u|^p u \quad \text{in } D \,, \\ u &= \infty \quad \text{on } \partial D \,, \end{aligned} \tag{1.8}$$

for the value of the parameter $\lambda = \sigma_1$. This result is of great interest by itself because it implies the existence of a large solution in a problem where the weight function $a(x)$ vanishes on a component of the boundary, whereas most of the previous results about the existence of large solutions had been given in the simplest case when the nonlinearity is bounded away from zero (cf. [Ke57], [Os57], [BM91], [Ve92], [LM93], [LM94], [MV97], and the references therein), and in addition it shows how a uniform interior Harnack inequality might fail when the nonlinearity vanishes in some region of the support domain. It should be pointed out that the classical interior estimates of [Ke57] and [Os57] can not be applied straight away to show the existence of a large positive solution of (1.8), since our nonlinearities change of sign, and that this fact provokes the existence of some unimportant gaps in some of the proofs given in references (e.g. the proof of Lemma 1.3 in [MV97]). Nevertheless, the results of [LM93], [LM94] and [GGLS98] can be used to fill in these gaps, since the large solutions in balls provide us with those interior estimates.

Thanks to these results a rather natural question arises. How does behave the population as times grows to infinity when $\lambda \geq \sigma_1$? It can not approach to a regular positive solution, of course. In fact, for any $u_0 > 0$ and $\varepsilon > 0$ we have that

$$u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \geq u_{[\sigma_1-\varepsilon,a,\Omega]}(\cdot,t;u_0)\,,$$

and hence

$$\liminf_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \geq u_{\sigma_1-\varepsilon}\,.$$

On the other hand, thanks to Theorem 3.1 of [GGLS98],

$$\lim_{\varepsilon\downarrow 0} u_{\sigma_1-\varepsilon} = \infty \quad \text{uniformly in } \bar{\Omega}^1_{0,1}\,.$$

Therefore, for each $\lambda \geq \sigma_1$ it follows that

$$\liminf_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0) = \infty \quad \text{uniformly in } \bar{\Omega}^1_{0,1}\,, \tag{1.9}$$

and hence the population grows arbitrarily in the refuge as times passes by. How does behave the population in the complement of the refuge? The answer to this question in the very special case when $m = 1$, $n_1 = 1$, $\bar{\Omega}^1_{0,1} \subset \Omega$ and (1.7) are satisfied was given very recently in [DH99], where it was shown that the limiting population in D as $t \uparrow \infty$ lies in between the minimal and the maximal large solution of the following problem

$$\begin{aligned} -\Delta u &= \lambda u - a(x)u^{p+1} \quad \text{in } D\,,\\ u &= \infty \quad \text{on } \partial D \cap \Omega\,,\\ u &= 0 \quad \text{on } \partial D \cap \partial\Omega\,. \end{aligned} \tag{1.10}$$

For this special location of the refuge one can consider a general boundary operator on $\partial D \cap \partial\Omega$ of mixed tipe (cf. [FKLM96] and [DH99]). Note that in (1.10) $a(x)$ is bounded away from zero in any compact subset of D.

The case when the species possesses an arbitrarily large number of refuges within its inhabiting region and (1.7) is satisfied has been already analyzed in [GL98] and [G99], where the concept of *metasolution* was introduced in order to characterize all the possible limiting profiles of the population as $t \uparrow \infty$. Setting

$$\Omega_k := \Omega \setminus \cup_{i=1}^k \cup_{j=1}^{n_i} \bar{\Omega}^i_{0,j}\,, \quad 1 \leq k \leq m\,,$$

a function

$$\mathcal{M} : \Omega \to [0, \infty]$$

is said to be a metasolution of order k of (1.5) supported in Ω_k if $\mathcal{M}|_{\Omega_k}$ is a large regular solution of

$$\begin{aligned} -\Delta u = \lambda u - a(x) u f(x, u) \quad \text{in } \Omega_k \,, \\ u = \infty \quad \text{on } \partial\Omega_k \setminus \partial\Omega \,, \\ u = 0 \quad \text{on } \partial\Omega_k \cap \partial\Omega \,, \end{aligned} \tag{1.11}$$

and

$$\mathcal{M} = \infty \quad \text{in } (\Omega \setminus \Omega_k) \cup (\partial\Omega_k \setminus \partial\Omega) \,.$$

In other words, a metasolution is the continuous extension by infinity to the totality of Ω of a large solution of (1.11). The main analytical result obtained in [GL98] shows that (1.5) possesses a metasolution of order k supported in Ω_k provided

$$\sigma_k \leq \lambda < \sigma_{k+1}$$

if $k \leq m-1$, and provided $\lambda \geq \sigma_m$ if $k = m$. This result was obtained for the special choice (1.7), where the interior estimates of [GGLS98] work out.

In this paper we obtain some substantial generalizations of all these results to cover the general case when conditions (Ha1-3) and (Hf) are satisfied, and in particular we complete the analysis already begun in [GL98] where it was conjectured that the metasolutions of order $k \leq m-1$ (resp $k = m$) should give the limiting profile of the population as $t \uparrow \infty$ if $\sigma_k \leq \lambda < \sigma_{k+1}$ (resp. $\lambda \geq \sigma_m$).

The distribution of this paper is as follows. In Section 2 we characterize the existence of regular positive solutions for (1.5) and show their global asymptotic stability with respect to the positive solutions of (1.1).

In Section 3 we generalize the interior estimates of J. B. Keller [Ke57] and R. Osserman [Os57] to cover the case when the nonlinearity changes of sign. Some previous results for changing sign nonlinearities were found by A. C. Lazer and P. J. McKenna in [LM94]. In the special case when a, $\lambda \in \mathbb{R}$ and the nonlinearity has the form

$$h(u) := a u f(u) - \lambda u \tag{1.12}$$

it was assumed in [LM94] that $h \in C^1([f^{-1}(\frac{\lambda}{a}), \infty)$ with $h' \geq 0$ and that in addition $h'(u)$ is nondecreasing for u large, and

$$\liminf_{u \uparrow \infty} \frac{h'(u)}{\sqrt{\int_{f^{-1}(\frac{\lambda}{a})}^{u} h(z)\, dz}} > 0 \,. \tag{1.13}$$

Instead of these conditions, for the special choice (1.12) we only need to assume that there exists $u_* > f^{-1}(\frac{\lambda}{a})$ such that

$$\int_{u_*}^{\infty} \left[\int_{u_*}^{u} h(z)\, dz \right]^{-\frac{1}{2}} du < \infty \,. \tag{1.14}$$

Note that for the choice (1.7), (1.14) is satisfied for any $p > 0$, while (1.13) is only satisfied for $p \geq 2$. It should be pointed out that if $f(u)$ satisfies (Hf), then

$$\int_{f^{-1}(\frac{\lambda}{a})}^{\infty} \left[\int_{f^{-1}(\frac{\lambda}{a})}^{u} h(z)\, dz \right]^{-\frac{1}{2}} du = \infty \,,$$

and hence condition (2) of [Ke57] fails. Therefore, the interior estimates of [Ke57] can not be applied straight away to our problem.

Once obtained these interior estimates, we shall use them to get some very general results about the existence of large solutions going to infinity on some of the components of the boundary where $a(x)$ is bounded away from zero. These results are completely new, since $a(x)$ can vanish on a finite number of interior subdomains. This degenerate situation was dealt with by the first time in [GL98]. These results provide us with the first step to obtain the existence of metasolutions of order $1 \leq k \leq m$ by slightly modifying the domain Ω_k along the components of its boundary where $a(x)$ vanishes, but this analysis will be done in Section 5, not in Section 3.

In Section 4 we use the results of Section 3 to characterize the limiting behaviour of the regular positive solution as $\lambda \uparrow \sigma_1$. It will be shown that the regular positive solution is point-wise increasing towards the minimal metasolution of order one of (1.5) supported in Ω_1 (for $\lambda = \sigma_1$).

In Section 5 we use the theory of Section 3 to show that if $k \leq m-1$, then (1.5) exhibits a metasolution of order k supported in Ω_k if, and only if, $\lambda < \sigma_{k+1}$, and that in case $k = m$ (1.5) possesses a metasolution of order m supported in $\Omega_m = \Omega_+$ for each $\lambda \in \mathbb{R}$, as well as to prove that the point-wise limit of the minimal metasolution of order $k \leq m-1$ as $\lambda \uparrow \sigma_{k+1}$ provides us with a metasolution of order $k+1$ supported in Ω_{k+1} for the value of the parameter $\lambda = \sigma_{k+1}$.

In Section 6 we combine the previous results with the parabolic interior estimates obtained in [Re82] and [Re86] to show that for any $1 \leq k \leq m-1$, $\lambda \in [\sigma_k, \sigma_{k+1})$, and $u_0 \in U_0$, $u_0 > 0$, the restriction of the orbit of $u_{[\lambda,a,\Omega]}(\cdot, t; u_0)$, $t \geq 0$, to any compact subset K of Ω_k is relatively compact in $C^2(K)$ with its ω-limit set contained in the closed interval whose ends are the minimal and the maximal regular positive solutions of (1.11), whereas

$$\lim_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot, t; u_0) = \infty \quad \text{uniformly in } \cup_{i=1}^{k} \cup_{j=1}^{n_i} \bar{\Omega}_{0,j}^{i} \,.$$

Therefore, if (1.11) possesses a unique regular solution, say θ, then $u_{[\lambda,a,\Omega]}(\cdot, t; u_0)$ stabilizes to the metasolution defined by θ as $t \uparrow \infty$.

Most of the available results about the uniqueness of large solutions were obtained in domains where $a(x)$ is bounded away from zero and for rather special classes of nonlinearities (cf. [BM91], [LM93], [LM94], [MV97], and the references there in). One of the main results of [DH99] establishes that if $m = 1$, $n_1 = 1$, $\bar{\Omega}_{0,1}^{1} \subset \Omega$, and there exist positive constants α and c such that

$$\lim_{d(x)\downarrow 0} \frac{a(x)}{d^{\alpha}(x)} = c \,, \quad d(x) := \operatorname{dist}(x, \partial\Omega_{0,1}^{1}) \,,$$

then (1.10) possesses a unique regular solution. The idea of the proof consists in combining an extension of Theorem I in [Ke57] with a very well known uniqueness device coming from [BO86]. This seems to be the sole uniqueness result for the case when the coefficient is not bounded away from zero in the totality of Ω. We shall analyze the uniqueness problem in a forthcoming paper.

It should be pointed out that in order to prove most of the results of this paper we need assuming that in any compact subset K of Ω_+ the following estimate holds

$$a(x)uf(x,u) - \lambda u \geq h_K(u)$$

for some function $h_K(u)$ satisfying (1.14). This assumption is far from being of technical nature. In fact, if it fails one would need to impose some additional condition on the size of the domain to get the existence of large solutions even in the most simple situations, but this analysis is out of the scope of this work.

The results contained in this paper were communicated in a talk given by the author in Miami during the celebration of the Conference honoring Alan C Lazer (January/9th/1999). The paper was concluded and sent to the journal on August 1999.

2. Regular positive steady states.

In this section we characterize the existence, prove the uniqueness and analyze the point-wise growth of the regular positive solutions of (1.5), as well as their global attractive character with respect to the positive solutions of (1.1).

In the sequel, given a function $w \in C(\bar{\Omega})$ we say that $w > 0$ if $w \geq 0$ and $w \neq 0$. Given $w \in U$ we say that $w \gg 0$ if w lies in the interior of U^+.

Theorem 2.1. *Assume* (Ha1-3) *and* (Hf). *Then, the following assertions are true:*

(i) *The problem* (1.5) *possesses a regular positive solution if, and only if,*

$$\sigma_0 < \lambda < \sigma_1 \,. \tag{2.1}$$

Moreover, it is unique if it exists. In the sequel it will be denoted by $\theta_{[\lambda,a,\Omega]}$.

(ii) *The map*

$$\begin{array}{ccc} (\sigma_0,\sigma_1) & \longmapsto & U \\ \lambda & \to & \theta_{[\lambda,a,\Omega]} \end{array}$$

is point-wise increasing and differentiable. Moreover, $\partial_\lambda \theta_{[\lambda,a,\Omega]} \in U^+$.

(iii) *Suppose* (2.1). *Then,*

$$\lim_{\lambda\downarrow\sigma_0} \|\theta_{[\lambda,a,\Omega]}\|_{C(\bar{\Omega})} = 0 \,, \tag{2.2}$$

and

$$\lim_{\lambda\uparrow\sigma_1} \theta_{[\lambda,a,\Omega]} = \infty \quad \text{uniformly in} \quad \cup_{j=1}^{n_1} \bar{\Omega}_{0,j}^1 \setminus \partial\Omega \,. \tag{2.3}$$

Proof. (i) Let u be a regular positive solution of (1.5). Then, $u|_{\partial\Omega} = 0$ and

$$(-\Delta + af(\cdot,u))u = \lambda u \,.$$

Hence, u is a positive eigenfunction associated with the eigenvalue λ of the operator $-\Delta + af(\cdot,u_0)$ under homogeneous Dirichlet boundary conditions. Thus, by the uniqueness of the principal eigenvalue (e.g. [Am76]),

$$\lambda = \sigma^\Omega[-\Delta + af(\cdot,u)] \,, \tag{2.4}$$

and hence, by the monotonicity of the principal eigenvalue with respect to the potential

$$\lambda > \sigma^\Omega[-\Delta] = \sigma_0 \,,$$

because $a > 0$, $u \gg 0$ and $f(x, u(x)) > 0$ for all $x \in \Omega$. Moreover, since $a = 0$ in $\Omega^1_{0,1}$, we find from (2.4) that

$$\lambda = \sigma_1^\Omega[-\Delta + af(\cdot, u)] < \sigma^{\Omega^1_{0,1}} = \sigma_1 \,,$$

by the monotonicity of the principal eigenvalue with respect to the domain. Therefore, (2.1) is necessary for the existence of a regular positive solution of (1.5). In order to show that condition (2.1) is sufficient for the existence of a regular positive solution we use the method of sub and supersolutions. Suppose (2.1) and let φ denote the principal eigenfunction associated with σ_0. Then, thanks to (Hf), for any $\varepsilon > 0$ sufficiently small the function $\varepsilon\varphi$ provides us with a positive subsolution of (1.5). To complete the proof of the existence it remains to show the existence of a supersolution $\bar{u}$ of (1.5) such that $\varepsilon\varphi \le \bar{u}$. For any $\delta > 0$ sufficiently small we consider the δ-neighborhood of Ω_0 in Ω

$$\Omega_0^\delta := \{\, x \in \Omega \,:\, \operatorname{dist}(x, \Omega_0) < \delta \,\} \,.$$

Then,

$$\Omega_0^\delta = \cup_{i=1}^m \cup_{j=1}^{n_i} \Omega_{0,j}^{i,\delta} \,, \tag{2.5}$$

where

$$\Omega_{0,j}^{i,\delta} := \{\, x \in \Omega \,:\, \operatorname{dist}(x, \Omega^i_{0,j}) < \delta \,\} \,, \quad 1 \le i \le m \,, \quad 1 \le j \le n_i \,.$$

Thanks to (Ha3), $\bar{\Omega}^i_{0,j} \cap \bar{\Omega}^{\hat{i}}_{0,\hat{j}} = \emptyset$ if $(i,j) \ne (\hat{i}, \hat{j})$, and hence $\delta > 0$ can be chosen sufficiently small so that

$$\bar{\Omega}_{0,j}^{i,\delta} \cap \bar{\Omega}_{0,\hat{j}}^{\hat{i},\delta} = \emptyset \quad \text{if} \;\; (i,j) \ne (\hat{i}, \hat{j}) \,. \tag{2.6}$$

In the sequel we shall assume that $\delta > 0$ has been chosen in this way.

Since $a > 0$, and hence $\Omega_+ \ne \emptyset$, for each $1 \le i \le m$ and $1 \le j \le n_i$ the component $\Omega^i_{0,j}$ is a proper subdomain of Ω. Thus, $\partial\Omega^i_{0,j} \cap \Omega \ne \emptyset$, and so $\Omega^i_{0,j}$ is a proper subdomain of $\Omega^{i,\delta/2}_{0,j}$ and $\Omega^{i,\delta/2}_{0,j}$ is a proper subdomain of $\Omega^{i,\delta}_{0,j}$. Therefore, by the monotonicity of the principal eigenvalue with respect to the domain, we find that

$$\sigma^{\Omega^{i,\delta}_{0,j}} < \sigma^{\Omega^{i,\delta/2}_{0,j}} < \sigma^{\Omega^i_{0,j}} \,, \quad 1 \le i \le m \,, \;\; 1 \le j \le n_i \,. \tag{2.7}$$

On the other hand, thanks to (Ha1), if Γ_0 and Γ are components of $\partial\Omega_0$ and $\partial\Omega$, respectively, such that $\Gamma_0 \cap \Gamma \ne \emptyset$, then $\Gamma_0 = \Gamma$. Thus, it follows from (2.5) and (2.6) that

$$\partial\Omega_0^\delta \setminus \partial\Omega \subset \Omega_+ \,. \tag{2.8}$$

Moreover, for each $1 \le i \le m$ and $1 \le j \le n_i$ we have that

$$\lim_{\delta \downarrow 0} \Omega^{i,\delta}_{0,j} = \Omega^i_{0,j} \,,$$

e.g. in the sense of Definition 4.1 of [Lo96]. Thus, it follows from Theorem 4.2 of [Lo96] that

$$\lim_{\delta \downarrow 0} \sigma^{\Omega^{i,\delta}_{0,j}} = \sigma^{\Omega^i_{0,j}} \,, \quad 1 \le i \le m \,, \quad 1 \le j \le n_i \,. \tag{2.9}$$

Therefore, thanks to (1.3), (2.1), (2.7) and (2.9), we find that for each $\delta > 0$ sufficiently small

$$\lambda < \sigma^{\Omega^{1,\delta}_{0,j}} < \sigma^{\Omega^{i,\delta}_{0,\hat{j}}}, \quad 1 \le j \le n_1 \,, \;\; 2 \le i \le m \,, \;\; 1 \le \hat{j} \le n_i \,. \tag{2.10}$$

Now, for any $1 \le i \le m$ and $1 \le j \le n_i$ let $\varphi^{i,\delta}_{0,j} \gg 0$ denote the principal eigenfunction associated with $\sigma^{\Omega^{i,\delta}_{0,j}}$, and consider the function

$$\Phi(x) := \begin{cases} \varphi^{i,\delta}_{0,j}(x) & \text{if } \; x \in \Omega^{i,\delta/2}_{0,j} \; \text{ for some } \; 1 \le i \le m \,, \;\; 1 \le j \le n_i \,, \\ \psi(x) & \text{if } x \in \bar{\Omega} \setminus \Omega^{\delta/2}_0 \,, \end{cases} \tag{2.11}$$

where $\psi(x)$ is any regular extension of the functions $\varphi^{i,\delta}_{0,j}$ outside $\Omega^{\delta/2}_0$. Thanks to (2.8), we can assume that ψ is positive and bounded away from zero in $\bar{\Omega} \setminus \Omega^{\delta/2}_0$.

We claim that if $\kappa > 1$ is sufficiently large, then the function

$$\bar{u} := \kappa \Phi$$

is a supersolution of (1.5) satisfying $\varepsilon \varphi \le \bar{u}$ for $\varepsilon > 0$ small enough. Indeed, in the set

$$\bar{\Omega} \setminus \Omega^{\delta/2}_0 \subset \bar{\Omega}_+$$

the function ψ is positive and bounded away from zero, as well as $a(x)$, by (Ha1). Thus, for κ large

$$-\Delta \psi \ge \lambda \, \psi - a \psi f(\cdot, \kappa \psi) \quad \text{in } \bar{\Omega} \setminus \Omega^{\delta/2}_0 \,,$$

since for each $x \in \bar{\Omega} \setminus \Omega^{\delta/2}_0$

$$f(x, \kappa \psi(x)) \ge f(x, \kappa \inf_{\bar{\Omega} \setminus \Omega^{\delta/2}_0} \psi)$$

and

$$\lim_{\kappa \uparrow \infty} f(\cdot, \kappa \inf_{\bar{\Omega} \setminus \Omega^{\delta/2}_0} \psi) = \infty$$

uniformly in $\bar{\Omega} \setminus \Omega^{\delta/2}_0$. Moreover, thanks to (2.10), in each of the components of $\Omega^{\delta/2}_0$, $\Omega^{i,\delta/2}_{0,j}$, $1 \le i \le m$, $1 \le j \le n_i$, we have that for any $\kappa > 0$

$$-\Delta(\kappa \Phi) = \kappa \, \sigma^{\Omega^{i,\delta}_{0,j}} \varphi^{i,\delta}_{0,j} > \lambda \, \kappa \, \varphi^{i,\delta}_{0,j} \ge \lambda \, \kappa \, \varphi^{i,\delta}_{0,j} - a \, \kappa \, \varphi^{i,\delta}_{0,j} f(\cdot, \kappa \, \varphi^{i,\delta}_{0,j}) \,.$$

Therefore, $k\Phi$ provides us with a positive supersolution of (1.5) if κ is sufficiently large.

Let Γ be a component of $\partial\Omega$. If Γ is a component of Ω_+, then by the construction itself we have that Φ is positive and bounded away from zero on Γ, while if $\Gamma \cap \partial\Omega_+ = \emptyset$, then $\Phi = 0$ on Γ and $\frac{\partial \Phi}{\partial n}(x) < 0$ for all $x \in \Gamma$, where n is the outward unit normal. Therefore, $\varepsilon \varphi \le \kappa \Phi$ provided $\varepsilon > 0$ is sufficiently small. This shows that (2.1) is sufficient for the existence of a regular positive solution of (1.5).

We now show the uniqueness of the positive solution. Suppose (2.1) and let u, v be two positive solutions of (1.5), $u \ne v$. Then,

$$(-\Delta + ag - \lambda)(u - v) = 0 \quad \text{in } \Omega \,, \quad u - v = 0 \quad \text{on } \partial\Omega \,, \tag{2.12}$$

where

$$g(x) := \begin{cases} \frac{u(x)f(x,u(x))-v(x)f(x,v(x))}{u(x)-v(x)} & \text{if } u(x) \neq v(x)\,, \\ f(x,u(x)) & \text{if } u(x) = v(x)\,, \end{cases} \qquad x \in \bar{\Omega}\,.$$

By the monotonicity of f on its second argument, it is easily seen that

$$g > f(\cdot, u) \quad \text{in } \Omega\,,$$

since $u \neq v$. Thus, it follows from (2.4) and the monotonicity of the principal eigenvalue with respect to the potential that

$$\sigma^\Omega[-\Delta + a\,g - \lambda] \geq \sigma^\Omega[-\Delta + af(\cdot,u) - \lambda] = 0\,.$$

Note that it might happen $g = f(\cdot, u)$ in Ω_+, and hence in the previous inequality we should not substitute $\geq$ by $>$ without some additional work. Assume

$$\sigma^\Omega[-\Delta + a\,g - \lambda] > 0\,.$$

Then, zero can not be an eigenvalue of $-\Delta + a\,g - \lambda$, and hence we find from (2.12) that $u = v$, which is impossible. Thus,

$$\sigma^\Omega[-\Delta + a\,g - \lambda] = 0\,.$$

Moreover, $u \neq v$. Hence, it follows from (2.12) that there exists $\kappa \in \mathbb{R} \setminus \{0\}$ such that

$$u - v = \kappa\varphi\,,$$

where $\varphi \gg 0$ stands for the principal eigenfunction associated with $\sigma^\Omega[-\Delta + a\,g - \lambda] = 0$. Therefore, either $u(x) < v(x)$ for all $x \in \Omega$, or $u(x) > v(x)$ for all $x \in \Omega$. In any of these situations we have that $g > f(\cdot, u)$ in Ω_+ and therefore

$$\sigma^\Omega[-\Delta + a\,g - \lambda] > 0\,,$$

which implies $u = v$. This contradiction shows the uniqueness and concludes the proof of Part (i).

(ii) Consider the operator $\mathcal{F} : \mathbb{R} \times U^+ \to C^\mu(\bar{\Omega})$ defined by

$$\mathcal{F}(\lambda, u) := -\Delta u - \lambda u + auF(\cdot, u)\,, \quad (\lambda, u) \in \mathbb{R} \times U^+\,,$$

where F stands for the substitution operator induced by f, and pick up $\lambda \in (\sigma_0, \sigma_1)$. Then, $\mathcal{F}$ is an operator of class C^1 in the interior of U^+ such that

$$\mathcal{F}(\lambda, \theta_{[\lambda,a,\Omega]}) = 0\,,$$

and

$$D_u\mathcal{F}(\lambda, \theta_{[\lambda,a,\Omega]}) = -\Delta - \lambda + af(\cdot, \theta_{[\lambda,a,\Omega]}) + a\theta_{[\lambda,a,\Omega]}\partial_u f(\cdot, \theta_{[\lambda,a,\Omega]})\,.$$

By (Hf), (2.4) and the monotonicity of the principal eigenvalue with respect to the potential we find that

$$\sigma^{\Omega}[D_u\mathcal{F}(\lambda,\theta_{[\lambda,a,\Omega]})] > \sigma^{\Omega}[-\Delta-\lambda+af(\cdot,\theta_{[\lambda,a,\Omega]})] = 0\,. \tag{2.13}$$

Therefore, $D_u\mathcal{F}(\lambda,\theta_{[\lambda,a,\Omega]})$ is an isomorphism and it follows from the implicit function theorem that there exist $\varepsilon>0$ and a map of class C^1, $u:(\lambda-\varepsilon,\lambda+\varepsilon)\to U^+$, such that $u(\lambda)=\theta_{[\lambda,a,\Omega]}$ and for each $s\in(\lambda-\varepsilon,\lambda+\varepsilon)$

$$\mathcal{F}(s,u(s))=0\,.$$

Moreover, those are the unique zeroes of $\mathcal{F}$ in a neighborhood of $(\lambda,\theta_{[\lambda,a,\Omega]})$ in $\mathbb{R}\times U^+$, and $u(s)\gg 0$, since $u(\lambda)\gg 0$. Thus, thanks to the uniqueness of the positive solution,

$$u(s)=\theta_{[s,a,\Omega]}\,,\quad s\simeq\lambda\,.$$

Furthermore, by implicit differentiation we find that

$$D_u\mathcal{F}(\lambda,\theta_{[\lambda,a,\Omega]})\partial_\lambda\theta_{[\lambda,a,\Omega]}=\theta_{[\lambda,a,\Omega]}\gg 0\,. \tag{2.14}$$

Thanks to (2.13) the differential operator on the left hand side of (2.14) satisfies the strong maximum principle. Therefore, (2.14) implies

$$\partial_\lambda\theta_{[\lambda,a,\Omega]}\gg 0\,.$$

This completes the proof of Part (ii).

(iii) Relation (2.2) follows from the uniqueness of the positive solution taking into account that thanks to the main theorem of [CR71] $\lambda=\sigma_0$ is a bifurcation value to positive solutions of (1.5) from $u=0$. We now show (2.3). Set

$$\Omega_0^1:=\cup_{j=1}^{n_1}\Omega_{0,j}^1\,.$$

Since $a=0$ in Ω_0^1, (2.14) gives

$$(-\Delta-\lambda)\partial_\lambda\theta_{[\lambda,a,\Omega]}=\theta_{[\lambda,a,\Omega]}\quad\text{in }\ \Omega_0^1\,,\quad\lambda\in(\sigma_0,\sigma_1)\,.$$

Now, pick $\hat\lambda\in(\sigma_0,\sigma_1)$ and consider $c>0$ such that for each $1\le j\le n_1$,

$$\theta_{[\hat\lambda,a,\Omega]}>c\varphi_{0,j}^1\quad\text{in }\Omega_{0,j}^1\,.$$

Recall that $\varphi_{0,j}^1$ is the principal eigenfunction associated with $\sigma^{\Omega_{0,j}^1}$. Thanks to Part (ii), for each $\lambda\in(\hat\lambda,\sigma_1)$ we have that

$$\theta_{[\lambda,a,\Omega]}>\theta_{[\hat\lambda,a,\Omega]}>c\varphi_{0,j}^1\quad\text{in }\Omega_{0,j}^1\,,\quad 1\le j\le n_1\,.$$

Moreover, for each $\lambda\in(\hat\lambda,\sigma_1)$ and $1\le j\le n_1$ the operator $-\Delta-\lambda$ satisfies the strong maximum principle in $\Omega_{0,j}^1$. Hence,

$$\partial_\lambda\theta_{[\lambda,a,\Omega]}>c\,(-\Delta-\lambda)^{-1}\varphi_{0,j}^1=\frac{c}{\sigma_1-\lambda}\varphi_{0,j}^1\quad\text{in }\ \Omega_{0,j}^1\,.$$

Note that, thanks to (Ha3), $\sigma_1 = \sigma^{\Omega^1_{0,j}}$ for each $1 \le j \le n_1$. On the other hand, for each $1 \le j \le n_1$, the function $\varphi^1_{0,j}$ is bounded away from zero on any compact subset of $\Omega^1_{0,j}$. Thus,

$$\lim_{\lambda \uparrow \sigma_1} \partial_\lambda \theta_{[\lambda,a,\Omega]} = \infty \quad \text{uniformly in compact subsets of } \Omega^1_0 \,,$$

and therefore

$$\lim_{\lambda \uparrow \sigma_1} \theta_{[\lambda,a,\Omega]} = \infty \quad \text{uniformly in compact subsets of } \Omega^1_0 \,.$$

It remains to show that

$$\lim_{\lambda \uparrow \sigma_1} \theta_{[\lambda,a,\Omega]}(x) = \infty \quad \text{for all } x \in \partial\Omega^1_0 \setminus \partial\Omega \,. \tag{2.15}$$

For this, consider $\delta > 0$ sufficiently small, pick λ satisfying

$$\sigma^{\Omega^{1,\delta}_{0,j}} < \sigma^{\Omega^{1,\delta/2}_{0,j}} < \lambda < \sigma^{\Omega^1_{0,j}} = \sigma_1 \,, \quad 1 \le j \le n_1 \,,$$

and introduce the function $u_\delta \in C(\bar\Omega)$ defined by

$$u_\delta(x) := \begin{cases} C\,\varphi^{1,\delta}_{0,j}(x) & \text{if } \ x \in \bar\Omega^{1,\delta/2}_{0,j} \text{ for some } \ 1 \le j \le n_1 \,, \\ \\ 0 & \text{if } x \in \bar\Omega \setminus \cup_{j=1}^{n_1} \Omega^{1,\delta/2}_{0,j} \,, \end{cases} \tag{2.16}$$

where $C > 0$ is a positive constant. Then, the argument of the proof of Theorem 4.3 in [LS98] can be easily adapted to show that under condition (Ha2) there exists $C = C(\delta) > 0$ such that u_δ is a subsolution of (1.5) satisfying

$$\lim_{\delta \downarrow 0} u_\delta(x) = \infty$$

for each $x \in \partial\Omega^1_0 \setminus \partial\Omega$. By the uniqueness of the positive solution, necessarily $u_\delta \le \theta_{[\lambda,a,\Omega]}$ and hence, (2.15) is satisfied. The uniform divergence in $\partial\Omega^1_0$ follows from the point-wise monotonicity in λ as an immediate consequence from Dini's theorem. This concludes the proof of the theorem. □

We shall say that a non-negative positive steady-state u of (1.1), i.e. a non-negative solution of (1.5), is *globally asymptotically stable* if

$$\lim_{t \uparrow \infty} \| u_{[\lambda,a,\Omega]}(\cdot, t; u_0) - u \|_{C(\bar\Omega)} = 0$$

for each $u_0 \in U_0$, $u_0 > 0$. The main result on the longtime behaviour of the positive solutions of (1.1) reads as follows.

Theorem 2.2. *Under the assumptions of Theorem 2.1, the following assertions are true:*

(i) *If $\lambda \le \sigma_0$, then $u = 0$ is globally asymptotically stable.*

(ii) *If $\sigma_0 < \lambda < \sigma_1$, then $\theta_{[\lambda,a,\Omega]}$ is globally asymptotically stable.*

(iii) *Set*

$$\Omega^i_0 := \cup_{j=1}^{n_i} \Omega^i_{0,j} \,, \quad 1 \le i \le m \,, \tag{2.17}$$

and assume $\sigma_i \le \lambda$ for some $1 \le i \le m$. Then, for any $u_0 \in U_0$ with $u_0 > 0$ we have

$$\lim_{t \uparrow \infty} u_{[\lambda,a,\Omega]}(\cdot, t; u_0) = \infty \quad \text{uniformly in } \cup_{k=1}^{i} \bar\Omega^k_0 \setminus \partial\Omega \,. \tag{2.18}$$

Proof. Recall that in the proof of Theorem 2.1(i) we have shown that (1.5) possesses arbitrarily large supersolutions in the interior of U^+ for any $\lambda < \sigma_1$.

(i) Let $u_0 \in U_0$ such that $u_0 > 0$ and consider the solution $u(x,t;u_0)$ of (1.1). By the parabolic maximum principle for any $t > 0$ the function $u(\cdot,t;u_0)$ lies in the interior of U^+. Fix $t_1 > 0$ and consider a supersolution $\bar{u} \gg 0$ of (1.5) such that

$$u(\cdot,t_1;u_0) \ll \bar{u}\,.$$

Then, for all $t \geq t_1$ we have that

$$u(\cdot,t;u_0) \ll u(\cdot,t-t_1;\bar{u})\,. \tag{2.19}$$

Thanks to the results of [Sa73], the function $t \to u(\cdot,t-t_1;\bar{u})$ is decreasing and converges to a non-negative solution of (1.5). Due to Theorem 2.1(i) it must converge to zero. Therefore, (2.19) completes the proof of Part (i).

(ii) Let $u_0 \in U_0$ be such that $u_0 > 0$ and pick $t_1 > 0$. Since $\sigma_0 < \lambda < \sigma_1$, there exist a subsolution $\underline{u}$ and a supersolution $\bar{u}$ of (1.5), both in the interior of U^+, such that

$$\underline{u} \ll u(\cdot,t_1;u_0) \ll \bar{u}\,.$$

Then, for all $t \geq t_1$ we have that

$$u(\cdot,t-t_1;\underline{u}) \ll u(\cdot,t;u_0) \ll u(\cdot,t-t_1;\bar{u})\,. \tag{2.20}$$

Thanks to the results of [Sa73], the function $t \to u(\cdot,t-t_1;\bar{u})$ is decreasing and converges to a non-negative solution of (1.5), whereas $t \to u(\cdot,t-t_1;\underline{u})$ is increasing and converges to a positive solution of (1.5). By the uniqueness of the positive solution, both functions must converge to $\theta_{[\lambda,a,\Omega]}$. Combining these features with (2.20) concludes the proof of this part.

(iii) Assume that $\lambda \geq \sigma_i$ for some $1 \leq i \leq m$ and consider $u_0 \in U_0$ with $u_0 > 0$. Let $u = u_{[\lambda,a,\Omega]}(x,t;u_0)$ be the unique global regular solution of (1.1). For all $\varepsilon > 0$ we have $\lambda > \sigma_1 - \varepsilon$, and hence

$$\partial_t u - \Delta u = \lambda u - auf(x,u) > (\sigma_1 - \varepsilon)u - auf(x,u)\,.$$

Thus, for each $t > 0$

$$u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \gg u_{[\sigma_1-\varepsilon,a,\Omega]}(\cdot,t;u_0)\,. \tag{2.21}$$

By Part (ii), we already know that

$$\lim_{t\uparrow\infty} \|u_{[\sigma_1-\varepsilon,a,\Omega]}(\cdot,t;u_0) - \theta_{[\lambda,a,\Omega]}\|_{C(\bar{\Omega})} = 0\,.$$

Hence, it follows from (2.21) that for each $\varepsilon > 0$

$$\liminf_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \geq \theta_{[\sigma_1-\varepsilon,a,\Omega]}\,. \tag{2.22}$$

Thanks to Theorem 2.1(iii),

$$\lim_{\varepsilon\downarrow 0} \theta_{[\sigma_1-\varepsilon,a,\Omega]} = \infty \quad \text{uniformly in } \bar{\Omega}_0^1 \setminus \partial\Omega\,.$$

Therefore, we find from (2.22) that

$$\liminf_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0)=\infty \quad \text{uniformly in } \bar{\Omega}_0^1\setminus\partial\Omega\,.$$

This completes the proof if $i=1$.

Assume $i\geq 2$ and

$$\liminf_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0)=\infty \quad \text{uniformly in } \cup_{k=1}^{i-1}\bar{\Omega}_0^k\setminus\partial\Omega\,.$$

Let $\hat{a}\in C^\mu(\bar{\Omega})$ any weight function such that $\hat{a}>a$ and $\hat{a}(x)>0$ if, and only if, $x\in\Omega_+\cup\cup_{k=1}^{i-1}\bar{\Omega}_0^k$. Then, the lower order refuges of $\hat{a}$ are $\Omega_{0,j}^i$, $1\leq j\leq n_i$. Recall that we are assuming

$$\lambda\geq\sigma_i=\sigma^{\Omega_{0,1}^i}\,.$$

Since $\hat{a}>a$, we have

$$u_{[\lambda,a,\Omega]}(\cdot,t;u_0)\gg u_{[\lambda,\hat{a},\Omega]}(\cdot,t;u_0)\,.$$

Moreover, by the corresponding result for the case $i=1$,

$$\liminf_{t\uparrow\infty} u_{[\lambda,\hat{a},\Omega]}(\cdot,t;u_0)=\infty \quad \text{uniformly in } \bar{\Omega}_0^i\setminus\partial\Omega\,.$$

This completes the proof of the theorem. □

3. A priori interior estimates and the existence of large regular solutions.

Beside their own interest, the following results are crucial to show the stabilization in $\Omega\setminus\bar{\Omega}_0^1$ of the regular positive solutions of (1.5) as $\lambda\uparrow\sigma_1$. They also show the existence of large regular solutions of

$$-\Delta u=\lambda u-auf(\cdot,u)$$

in Ω. By a large regular solution we mean a solution of class $C^{2+\mu}$ which grows arbitrarily when the spatial variable approaches to some of the components of $\partial\Omega$. Those solutions will provide us with the limiting behaviour of the population as times grows to infinity for any $\lambda\geq\sigma_1$.

Theorem 3.1. *Suppose* (Ha1-3), (Hf) *and*

$$\partial\Omega\cap\partial\Omega_+\neq\emptyset\,.$$

Let Γ_+^j, $1\leq j\leq q$, be q arbitrary components of $\partial\Omega\cap\partial\Omega_+$, and $\alpha_j>0$, $1\leq j\leq q$, q arbitrary constants, and consider the nonlinear boundary value problem

$$\begin{aligned} -\Delta u&=\lambda u-a(x)uf(x,u) \quad \text{in } \Omega\,,\\ u|_{\Gamma_+^j}&=\alpha_j>0\,,\quad 1\leq j\leq q\,,\\ u&=0 \quad \text{on } \partial\Omega\setminus\cup_{j=1}^q\Gamma_+^j\,. \end{aligned} \tag{3.1}$$

The following assertions are true:

(i) *Problem* (3.1) *possesses a regular positive solution if, and only if,* $\lambda < \sigma_1$. *Moreover, it is unique if it exists. In the sequel we shall denote it by* $\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}$ *where*

$$\Gamma_+ := (\Gamma_+^1, \dots, \Gamma_+^q), \quad \alpha := (\alpha_1, \dots, \alpha_q).$$

(ii) *The map* $(-\infty, \sigma_1) \to C(\bar{\Omega})$, $\lambda \to \Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}$ *is point-wise increasing, as well as the map* $(0,\infty)^q \to C(\bar{\Omega})$, $\alpha = (\alpha_1, \dots, \alpha_q) \to \Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}$. *Moreover,*

$$\lim_{\lambda \uparrow \sigma_1} \Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]} = \infty \quad \text{uniformly in} \quad \cup_{j=1}^{n_1} \bar{\Omega}_{0,j}^1 \setminus \partial\Omega . \tag{3.2}$$

Proof. (i) Suppose (3.1) possesses a regular positive solution, say u. Then,

$$(-\Delta + af(\cdot, u) - \lambda)u = 0 \quad \text{in} \ \ \Omega ,$$

and $u|_{\partial\Omega} > 0$. Thus, u provides us with a positive strict supersolution of $-\Delta + af(\cdot,u) - \lambda$ in Ω under homogeneous Dirichlet boundary conditions. Hence, it follows from the characterization of the strong maximum principle found in [LM94] (cf. [Lo96] and [AL98]) that

$$\sigma^{\Omega}[-\Delta + af(\cdot,u) - \lambda] > 0 . \tag{3.3}$$

Therefore,

$$\lambda < \sigma^{\Omega}[-\Delta + af(\cdot,u)] < \sigma^{\Omega_{0,1}^1} = \sigma_1 ,$$

since $a = 0$ in $\Omega_{0,1}^1$.

Suppose $\lambda < \sigma_1$. To show that (3.1) possesses a regular positive solution we use the method of sub and supersolutions. The function $\underline{u} := 0$ provides us with a subsolution of (3.1). Moreover, we can use the same supersolutions constructed in the proof of Theorem 2.1. Indeed, to show that $\kappa\Phi$ provides us with a positive supersolution of (3.1), where Φ is the function defined in (2.11), it remains to check that for any κ sufficiently large

$$\kappa\Phi|_{\Gamma_+^j} > \alpha_j , \quad 1 \le j \le q . \tag{3.4}$$

By construction,

$$\cup_{j=1}^q \Gamma_+^j \subset \bar{\Omega} \setminus \Omega_0^{\delta/2} ,$$

and hence

$$\kappa\Phi|_{\Gamma_+^j} = \kappa\psi|_{\Gamma_+^j} , \quad 1 \le j \le q .$$

Moreover, ψ is positive and bounded away from zero in $\bar{\Omega} \setminus \Omega_0^{\delta/2}$. Therefore, (3.4) holds provided κ is large enough. This concludes the proof of the existence.

The uniqueness of the regular positive solution follows from (3.3) with the same argument used to show the uniqueness in the proof of Theorem 2.1. This completes the proof of Part (i).

(ii) Let λ_1, $\lambda_2 \in (\sigma_0, \sigma_1)$ such that $\lambda_1 < \lambda_2$. Then, setting

$$\Theta_i := \Theta_{[\lambda_i, a, \Omega, \Gamma_+, \alpha]}\,, \quad i = 1\,,\, 2\,,$$

we find from their definition that

$$(-\Delta + a\,g - \lambda_1)(\Theta_2 - \Theta_1) > 0 \quad \text{in } \Omega\,, \quad (\Theta_2 - \Theta_1)|_{\partial\Omega} = 0\,. \tag{3.5}$$

where

$$g(x) := \begin{cases} \dfrac{\Theta_2(x) f(x, \Theta_2(x)) - \Theta_1(x) f(x, \Theta_1(x))}{\Theta_2(x) - \Theta_1(x)} & \text{if } \Theta_2(x) \neq \Theta_1(x)\,,\ x \in \bar{\Omega}\,, \\ f(x, \Theta_2(x)) & \text{if } \Theta_2(x) = \Theta_1(x)\,,\ x \in \bar{\Omega}\,. \end{cases}$$

By the monotonicity of f on its second argument, it is easily seen that

$$g > f(\cdot, \Theta_1) \quad \text{in } \Omega\,,$$

since $\Theta_2 \neq \Theta_1$. Thus, thanks to (3.3), it follows from the monotonicity of the principal eigenvalue with respect to the potential that

$$\sigma^{\Omega}[-\Delta + a\,g - \lambda_1] \geq \sigma^{\Omega}[-\Delta + a f(\cdot, \Theta_1) - \lambda_1] > 0\,.$$

Henceforth, the operator $-\Delta + a\,g - \lambda_1$ satisfies the strong maximum principle in Ω under homogeneous Dirichlet boundary conditions, and therefore we find from (3.5) that

$$\Theta_2 - \Theta_1 \gg 0\,.$$

This completes the proof of the monotonicity in λ. The same argument can be easily adapted to get the monotonicity in α.

Relation (3.2) follows easily from Theorem 2.1 taking into account that for any $\lambda \in (\sigma_0, \sigma_1)$

$$\Theta_{[\lambda, a, \Omega, \Gamma_+, \alpha]} \gg \theta_{[\lambda, a, \Omega]}\,.$$

The proof is completed. $\square$

Remark 3.2: (a) Theorem 3.1 also holds in the absence of refuges, i.e. in case $\Omega = \Omega_+$. In this special situation its proof is simpler, since large constants provide us with supersolutions for any $\lambda \in \mathbb{R}$. Therefore, in case $\Omega = \Omega_+$ the positive solution exists for each $\lambda \in \mathbb{R}$. This is the case dealt with in most of the references and in particular in [MV97] and [DH99] for the special choice (1.7).

(b) By the uniqueness of the positive solution, any couple $(\underline{u}, \overline{u})$ formed by a nonnegative subsolution $\underline{u}$ and a nonnegative supersolution $\overline{u}$ of (3.1) must satisfy

$$\underline{u} \leq \Theta_{[\lambda, a, \Omega, \Gamma_+, \alpha]} \leq \overline{u}\,.$$

This estimate can be easily obtained with the same comparison argument used to show the uniqueness and the monotonicities, and so we will omit the details of its proof.

In the sequel we shall assume the following:

(Hfb) There exists a continuous function $f_b : [0, \infty) \to \mathbb{R}$ of class $C^{1+\mu}((0, \infty))$ such that $f_b(0) = 0$, $f_b(u) > 0$ and $f_b'(u) > 0$ for all $u > 0$,

$$\lim_{u \uparrow \infty} f_b(u) = \infty\,,$$

and

$$f(\cdot, u) \geq f_b(u)\,, \quad u \geq 0\,.$$

The following result is a substantial generalization of the interior estimates found in [Ke57] and [Os57]. It provides us with uniform interior estimates in Ω_+ for the solutions of (3.1) under assumption (Hfb), and we shall we use it to show the stabilization in Ω_+ of $\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}$ as either $\alpha_j \uparrow \infty$, $1 \leq j \leq q$, or $\lambda \uparrow \sigma_1$, and to obtain very general existence results of large regular solutions.

Theorem 3.3. *Suppose* (Ha1-3), (Hf) *and* (Hfb). *Let $D \subset \Omega_+$ be open and $K \subset \Omega_+$ compact such that $D + B_\delta \subset K$ for some $\delta > 0$, where B_δ stands for the ball of radius δ centered at the origin. Set*

$$A := \inf_{x \in K} a(x) > 0\,,$$

fix $\beta > 0$ and let u_K denote the unique positive zero of the function

$$h(u) := Auf_b(u) - \beta u\,. \tag{3.6}$$

Thanks to (Hfb), *u_K is well defined. Assume in addition that for each $u_* > u_K$*

$$I(u_*) := \int_{u_*}^{\infty} \left[\int_{u_*}^{u} h(z)\,dz \right]^{-1/2} du < \infty\,. \tag{3.7}$$

Then, there exists a universal constant $M = M(\beta)$ such that for any $\lambda \leq \beta$ and any regular positive solution u_λ of

$$-\Delta u = \lambda u - auf(\cdot, u) \quad \text{in } \Omega_+\,, \tag{3.8}$$

the following uniform estimate is satisfied

$$\|u_\lambda\|_{C(\bar{D})} \leq M\,. \tag{3.9}$$

Remarks 3.4: (a) In (3.9) no condition on the growth of u_λ on $\partial\Omega_+$ is imposed. Therefore, (3.9) provides us with a universal interior estimate for the positive solutions of (3.8) which is independent of the behaviour of the solutions on the boundary.

(b) We should point out that

$$\int_{u_K}^{\infty} \left[\int_{u_K}^{u} h(z)\,dz \right]^{-1/2} du = \infty\,. \tag{3.10}$$

Indeed, the auxiliary function

$$g(u) := \int_{u_K}^{u} h(z)\,dz\,, \quad u > u_K\,,$$

satisfies

$$g(u_K) = 0\,, \quad g'(u_K) = h(u_K) = 0\,,$$

and

$$g''(u_K) = h'(u_K) = Af_b(u_K) + Au_K f_b'(u_K) - \beta = Au_K f_b'(u_K) > 0\,.$$

Therefore, the integral diverges.

(c) If there exist $\eta > 0$ and $p > 0$ such that

$$f_b(u) \geq \eta\, u^p\,, \quad u \geq 0\,,$$

then condition (3.7) is satisfied for any $A > 0$ and $\beta > 0$, and therefore the estimate (3.9) is valid in any open set D satisfying $\bar{D} \subset \Omega_+$. Indeed, for any $u_* > u_K$ we have that

$$\begin{aligned} I(u_*) =&\sqrt{2}\int_{u_*}^{\infty}\left[2A\int_{u_*}^{u} z f_b(z)\,dz - \beta(u^2-u_*^2)\right]^{-1/2} du \\ =&\sqrt{2}\int_{1}^{\infty}\left[2A\int_{1}^{\theta} t f_b(u_* t)\,dt - \beta(\theta^2-1)\right]^{-1/2} d\theta\,. \end{aligned}$$

In particular, $I(u_*)$ is decreasing and

$$I(u_*) \leq \sqrt{2}\int_{1}^{\infty}\left[\frac{2A\eta}{p+2}u_*^p(\theta^{p+2}-1)-\beta(\theta^2-1)\right]^{-1/2} d\theta < \infty\,.$$

(d) Condition (3.7) entails the existence of a unique regular positive solution of

$$-u'' = \beta u - A u f_b(u) \quad \text{in } (-R,R)\,, \quad u(-R)=u(R)=\infty\,, \tag{3.11}$$

for each $R > 0$. Indeed, multiplying the differential equation of (3.11) by u' and setting

$$v = u'\,, \quad u(0) = u_*\,,$$

it is easily seen that the solutions of (3.11) are given by the u_*'s such that for each $x \in (0,R)$

$$v^2(x) = 2\int_{u_*}^{u(x)} h(z)\,dz$$

and $\lim_{x\uparrow R} u(x) = \infty$. Necessarily $u_* > u_K$, because otherwise $u \leq u_K$, since u_K is an equilibrium of the equation. Moreover, this occurs if and only if

$$R = \sqrt{2}\int_{u_*}^{\infty}\left[\int_{u_*}^{u} h(z)\,dz\right]^{-1/2} du\,.$$

Therefore, u_* corresponds to a solution of (3.11) if and only if $u_* > u_K$ and

$$R = \sqrt{2} I(u_*)\,.$$

We already know that $\lim_{u_*\downarrow u_K} I(u_*) = \infty$ and that $I(u_*)$ is decreasing. Moreover, since h is nondecreasing, it is easily seen from (3.7) that

$$\lim_{u_*\uparrow\infty} I(u_*) = 0\,.$$

Therefore, there exists a unique $u_* \in (u_K,\infty)$ such that $R = \sqrt{2}I(u_*)$. Actually, this implies the existence of a radially symmetric regular large solution for the problem in a ball (cf. Lemma 6.2 of [GGLS98]). The proof of Theorem 3.3 can be accomplished very easily from these facts, but in this paper we prefer using the techniques introduced in [Ke57], since they are pioneer in the field. If in addition $h''(u) \geq 0$ for u large and

$$\lim_{u\uparrow\infty} \frac{h'(u)}{\sqrt{\int_{u_K}^{u} h(z)\,dz}} > 0\,,$$

then the regular large solution in the ball is unique, [LM94].

(e) Even in the simplest case when there exists $p > 0$ such that $f(x,u) = f_b(u) = u^p$ the condition (2) of [Ke57] fails, because of (3.10). This is because the nonlinearity $h(u)$ of [Ke57] is increasing and bounded away from zero. Therefore, the interior estimates obtained in [Ke57] do not guarantee straight away the existence of interior estimates for the positive solutions of (3.1). Among other things this implies that the proof of Lemma 1.3 of [MV97] contains a gap, since the interior estimates of [Ke57] can not be used. Our Theorem 3.3 completes the proof of Lemma 1.3 of [MV97].

Although the proof of Theorem 3.3 is an easy consequence from Theorem III of [Ke57], as pointed out to us by the referee, we are going to give a complete selfcontained proof of it by means of the technical tools introduced in the proof of Theorem I of [Ke57]. Note that in [Ke57] Theorem III was obtained as a consequence from Theorem I.

Proof of Theorem 3.3. It suffices to show that for each $x_0 \in \bar{D}$ there exists $\eta > 0$ and a constant $M > 0$ such that for any $\lambda \leq \beta$ and any regular positive solution u_λ of (3.8)

$$\|u_\lambda\|_{C(B_\eta(x_0))} \leq M\,,$$

where $B_\eta(x_0)$ stands for the ball of radius η centered at x_0. Let $x_0 \in \bar{D}$ and consider $R > 0$ such that $\bar{B}_R(x_0) \subset K$. Then, for each $\lambda \leq \beta$ the following differential inequality holds

$$\Delta u_\lambda \geq h(u_\lambda) \quad \text{in } B_R(x_0)\,, \tag{3.12}$$

where $h(u)$ is the function defined by (3.6). Set

$$\alpha_\lambda := \max\{u_K + 1\,, \sup_{x\in\partial B_R(x_0)} u_\lambda\}\,, \tag{3.13}$$

and let Θ_λ denote the unique positive solution of

$$\Delta u = h(u) \quad \text{in } B_R(x_0)\,, \quad u|_{\partial B_R(x_0)} = \alpha_\lambda\,, \tag{3.14}$$

whose existence is guaranteed by Theorem 3.1 (cf. Remark 3.2(a)). Thanks to (3.12) and (3.13), for each $\lambda \leq \beta$ any solution u_λ of (3.8) is a positive subsolution of (3.14) in $B_R(x_0)$, and hence it follows from Remark 3.2(b) that

$$u_\lambda \leq \Theta_\lambda \quad \text{in } B_R(x_0)\,.$$

Thus, to complete the proof it suffices to show that there exist $\eta \in (0,R)$ and a constant $M > 0$ such that for any $\alpha_\lambda > 0$ sufficiently large

$$\|\Theta_\lambda\|_{C(B_\eta(x_0))} \leq M\,. \tag{3.15}$$

By definition, $h(u_K)=0$ and $h(u)>0$ for $u>u_K$. Thus, for any $\alpha_\lambda>u_K$ the constant u_K is a strict positive subsolution of (3.14), and hence it follows from Remark 3.2(b) that

$$u_K \le \Theta_\lambda\,.$$

In fact, the strong maximum principle implies $u_K \ll \Theta_\lambda$. On the other hand, by the uniqueness of the positive solution of (3.14), Θ_λ must be radially symmetric, since the problem is invariant by rotations. Thus,

$$\Theta_\lambda(x)=\Psi_\lambda(|x-x_0|)\,,\quad x\in B_R(x_0)\,,$$

where $\Psi_\lambda(r)$ is the unique positive solution of

$$\psi''(r)+\frac{N-1}{r}\psi'(r)=h(\psi(r))\,,\quad 0<r<R\,,\quad \psi'(0)=0\,,\quad \psi(R)=\alpha_\lambda\,. \tag{3.16}$$

Therefore, to prove (3.15) it suffices to show that there exist $\eta>0$ and $M>0$ such that for any $\alpha_\lambda>u_K$

$$\|\Psi_\lambda\|_{C([0,\eta])}\le M\,. \tag{3.17}$$

We already know that $\Psi_\lambda(r)>u_K$ for each $r\in[0,R]$ and $\alpha_\lambda>u_K$. Hence,

$$h(\Psi_\lambda(r))>0\,.$$

Moreover, for each $u>u_K$ we have that

$$h'(u)=Af_b(u)+Auf_b'(u)-\beta>Af_b(u_K)-\beta+Auf_b'(u)=Auf_b'(u)>0\,,$$

and hence h is increasing. We now follow the proof of Theorem I in page 506 of [Ke57]. The functions Ψ_λ satisfy

$$(R^{N-1}\Psi_\lambda'(r))'=r^{N-1}h(\Psi_\lambda(r))\,, \tag{3.18}$$

and hence integrating (3.18) from 0 to r yields

$$\Psi_\lambda'(r)=r^{1-N}\int_0^r s^{N-1}h(\Psi_\lambda(s))\,ds>0\,. \tag{3.19}$$

This shows that $r\to\Psi_\lambda(r)$ is increasing, as well as $r\to h(\Psi_\lambda(r))$. Thus, we find from (3.19) that

$$\Psi_\lambda'(r)\le r^{1-N}h(\Psi_\lambda(r))\int_0^r s^{N-1}\,ds=\frac{r}{N}h(\Psi_\lambda(r))\,. \tag{3.20}$$

Now, substituting (3.20) into (3.16) gives

$$\Psi_\lambda''\ge\frac{h(\Psi_\lambda)}{N}\,.$$

Moreover, since $\Psi_\lambda'\ge 0$, (3.16) gives $\Psi_\lambda''\le h(\Psi_\lambda)$. Hence,

$$h(\Psi_\lambda)\ge\Psi_\lambda''\ge\frac{h(\Psi_\lambda)}{N}\,. \tag{3.21}$$

We now multiply (3.21) by Ψ_λ' and integrate from 0 to r to obtain

$$2\int_{\Psi_\lambda(0)}^{\Psi_\lambda(r)} h(z)\,dz \geq [\Psi_\lambda'(r)]^2 \geq \frac{2}{N}\int_{\Psi_\lambda(0)}^{\Psi_\lambda(r)} h(z)\,dz\,. \tag{3.22}$$

Now, taking the square root of the reciprocal of (3.22) and integrating again gives

$$\frac{1}{\sqrt{2}}\int_{\Psi_\lambda(0)}^{\Psi_\lambda(r)}\left[\int_{\Psi_\lambda(0)}^{u} h(z)\,dz\right]^{-1/2} du \leq r \leq \sqrt{\frac{N}{2}}\int_{\Psi_\lambda(0)}^{\Psi_\lambda(r)}\left[\int_{\Psi_\lambda(0)}^{u} h(z)\,dz\right]^{-1/2} du\,. \tag{3.23}$$

In particular,

$$\frac{1}{\sqrt{2}}\int_{\Psi_\lambda(0)}^{\alpha_\lambda}\left[\int_{\Psi_\lambda(0)}^{u} h(z)\,dz\right]^{-1/2} du \leq R\,.$$

Hence, thanks to (3.10), $\Psi_\lambda(0)$ must be uniformly bounded away from u_K. Moreover,

$$R \leq \sqrt{\frac{N}{2}}\int_{\Psi_\lambda(0)}^{\infty}\left[\int_{\Psi_\lambda(0)}^{u} h(z)\,dz\right]^{-1/2} du\,.$$

So, since h in increasing we find from (3.7) that $\Psi_\lambda(0)$ must be uniformly bounded above. Therefore, using Remark 3.2(b) gives

$$\Psi_\lambda(0) < \limsup_{\alpha_\lambda\uparrow\infty} \Psi_\lambda(0) = \Psi_0 \in (u_K,\infty)\,,$$

for any $\alpha_\lambda > u_K$. Let Ψ_∞ denote the unique positive solution of the Cauchy problem

$$\psi''(r) + \frac{N-1}{r}\psi'(r) = h(\psi(r)) \quad 0<r<R\,, \quad \psi'(0)=0\,, \quad \psi(0)=\Psi_0\,.$$

By continuous dependence, Ψ_∞ is defined in $[0,R)$ and

$$\lim_{r\uparrow R}\Psi_\infty(r) = \infty\,.$$

Moreover, thanks to Remark 3.2(b), $\Psi_\lambda \leq \Psi_\infty$. This completes the proof. $\square$

Note that Ψ_∞ is the minimal large solution of $\Delta u = h(u)$ in the ball B_R. The heart of the proof of Theorem 3.3 consists in showing the existence of such large solution. A different proof of this fact was given in [GGLS98] for the special case when $f(x,u) = f_b(u) = u^p$.

Combining Theorem 3.1 with Theorem 3.3 it follows the existence of large positive solutions for a wide class of semilinear boundary value problems, even in the presence of refuges. In fact the following result provides us with a substantial extension of most of the results available in the literature.

Theorem 3.5. *Suppose* (Ha1-3), (Hf), (Hfb) *and* $\lambda < \sigma_1$. *Assume in addition that* (3.7) *is satisfied for any compact subset* $K \subset \Omega_+$ *and* $\beta = \sigma_1$ *and that*

$$\partial\Omega \cap \partial\Omega_+ \neq \emptyset\,.$$

Let Γ_+^j, $1 \leq j \leq q$, *be* q *arbitrary components of* $\partial\Omega\cap\partial\Omega_+$ *and consider the singular boundary value problem*

$$\begin{aligned} -\Delta u &= \lambda u - a(x) u f(x,u) \quad \text{in } \Omega\,, \\ u &= \infty \quad \text{on } \gamma_\infty := \cup_{j=1}^q \Gamma_+^j\,, \\ u &= 0 \quad \text{on } \gamma_0 := \partial\Omega \setminus \gamma_\infty\,. \end{aligned} \tag{3.24}$$

Then, the following assertions are true:

(i) *The problem* (3.24) *possesses a regular positive solution* u. *By regular solution we mean* $u \in C^{2+\mu}(\Omega)$ *and*

$$\lim_{\operatorname{dist}(x,\gamma_\infty)\downarrow 0} u(x) = \infty\,, \qquad \lim_{\operatorname{dist}(x,\gamma_0)\downarrow 0} u(x) = 0\,.$$

(ii) *The function*

$$\Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]}(x) := \lim_{\alpha\uparrow\infty} \Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}(x)\,, \quad x \in \Omega\,, \tag{3.25}$$

where $\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}$ *is the unique regular positive solution of* (3.1), *is the minimal regular positive solution of* (3.24). *Moreover,*

$$\lim_{\alpha\uparrow\infty} \|\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]} - \Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]}\|_{C^{2+\mu}(\Omega)} = 0\,. \tag{3.26}$$

By $\alpha \uparrow \infty$ *we mean* $\alpha_j \uparrow \infty$ *for each* $1 \leq j \leq q$.

(iii) *For each* $x \in \cup_{j=1}^{n_1} \bar\Omega_{0,j}^1 \setminus \partial\Omega$

$$\lim_{\lambda\uparrow\sigma_1} \Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]}(x) = \infty\,, \tag{3.27}$$

whereas the function

$$\Theta_{[\sigma_1,a,\Omega,\Gamma_+,\infty]}(x) := \lim_{\lambda\uparrow\sigma_1} \Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]}(x)\,, \quad x \in \Omega \setminus \cup_{j=1}^{n_1} \bar\Omega_{0,j}^1\,, \tag{3.28}$$

is well defined and it provides us with the minimal regular positive solution of

$$\begin{aligned} -\Delta u &= \lambda u - a(x) u f(x,u) \quad \text{in } \Omega \setminus \cup_{j=1}^{n_1} \bar\Omega_{0,j}^1\,, \\ u &= \infty \quad \text{on } \gamma_\infty^1 := \gamma_\infty \cup (\cup_{j=1}^{n_1} \partial\Omega_{0,j}^1 \setminus \partial\Omega)\,, \\ u &= 0 \quad \text{on } \partial(\Omega \setminus \cup_{j=1}^{n_1} \bar\Omega_{0,j}^1) \setminus \gamma_\infty^1\,. \end{aligned} \tag{3.29}$$

Proof. Choose $\delta > 0$ sufficiently small so that

$$K_{j,\delta} := \{x \in \Omega \ : \ \operatorname{dist}(x, \Gamma_+^j) \leq \delta\} \subset \Omega_+ \,, \quad 1 \leq j \leq q \,,$$

and $K_{i,\delta} \cap K_{j,\delta} = \emptyset$ if $i \neq j$, and consider the open set

$$D := \Omega \setminus \cup_{j=1}^q K_{j,\delta} \,.$$

By construction

$$D_+ := \{x \in D \ : \ a(x) > 0\} = \Omega_+ \setminus \cup_{j=1}^q K_{j,\delta}$$

and

$$\bar{D}_0 := \{x \in D \ : \ a(x) = 0\} = \bar{\Omega}_0 \,.$$

Moreover,

$$\partial D = \gamma_0 \cup \gamma_\infty^D \,,$$

where

$$\gamma_\infty^D := \cup_{j=1}^q \{x \in \Omega_+ \ : \ \operatorname{dist}(x, \Gamma_+^j) = \delta\} \,.$$

By construction, γ_∞^D is a compact subset of Ω_+, and hence it follows from Theorem 3.3 that there exists a universal constant $M > 0$ such that

$$\|\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}\|_{C(\gamma_\infty^D)} \leq M$$

for all $\alpha = (\alpha_1, \ldots, \alpha_q) \in (0, \infty)^q$. Thus, any solution of (3.1) provides us with a subsolution of

$$\begin{aligned} -\Delta u &= \lambda u - a(x) u f(x,u) \quad \text{in } \ D \,, \\ u &= M \quad \text{on } \gamma_\infty^D \,, \\ u &= 0 \quad \text{on } \gamma_0 \,. \end{aligned} \tag{3.30}$$

Due to Remark 3.2(b) we have that for each $\alpha \in (0, \infty)^q$

$$\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]} \leq \Theta_{[\lambda,a,D,\gamma_\infty^D,M]} \quad \text{in } D \,, \tag{3.31}$$

where $\Theta_{[\lambda,a,D,\gamma_\infty^D,M]}$ is the unique positive solution of (3.30), whose existence is guaranteed by Theorem 3.1.

Since δ can be taken arbitrarily small,

$$\lim_{\delta \downarrow 0} D = \Omega$$

in the sense of [Lo96], and the mapping $\alpha \to \Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}$ is increasing, we find from (3.31) that the point-wise limit (3.25) is well defined.

Let $\mathcal{O}_1 \subset \mathcal{O}$ two open subsets of Ω such that $\bar{\mathcal{O}}_1 \subset \mathcal{O}$, $\bar{\mathcal{O}} \subset \Omega$, and choose $\delta > 0$ sufficiently small so that $\bar{\mathcal{O}} \subset D$. Then, (3.31) implies

$$\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]} \leq \Theta_{[\lambda,a,D,\gamma_\infty^D,M]} \quad \text{in } \mathcal{O} \,,$$

for each $\alpha \in (0, \infty)^q$, and hence, by the L^p-estimates of Agmon, Douglis & Nirenberg, for each $p > 1$ there exists a constant $M_1 = M(p, \mathcal{O}_1)$ such that

$$\|\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}\|_{W^{2,p}(\mathcal{O}_1)} \leq M_1$$

for each $\alpha \in (0,\infty)^q$. Thus, thanks to Morrey's embedding theorem and Schauder's estimates, there exists a constant $M_2 = M(\mathcal{O}_1)$ such that

$$\|\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}\|_{C^{2+\mu}(\bar{\mathcal{O}}_1)} \leq M_2\,, \quad \alpha \in (0,\infty)^q\,.$$

Now, a rather standard compactness argument combined with the uniqueness of the point-wise limit (3.25) shows that $\Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]} \in C^{2+\mu}(\bar{\mathcal{O}}_1)$ is a regular solution of

$$-\Delta u = \lambda u - auf(\cdot,u)$$

in $\mathcal{O}_1$, and that

$$\lim_{\alpha\uparrow\infty} \|\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]} - \Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]}\|_{C^{2+\mu}(\bar{\mathcal{O}}_1)} = 0\,. \tag{3.32}$$

This completes the proof of Parts (i) and (ii).

We now prove Part (iii). By construction, for each $\alpha \in (0,\infty)^q$ we have that

$$\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]} < \Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]}\,. \tag{3.33}$$

Therefore, thanks to (3.2),

$$\lim_{\lambda\uparrow\sigma_1} \Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]} = \infty \quad \text{uniformly in} \quad \cup_{j=1}^{n_1} \bar{\Omega}^1_{0,j} \setminus \partial\Omega\,. \tag{3.34}$$

We now show that the point-wise limit (3.28) is well defined. Reduce $\delta > 0$, if necessary, so that the open neighborhoods

$$\Omega^{1,\delta}_{0,j} := \{x \in \Omega \ : \ \operatorname{dist}(x,\Omega^1_{0,j}) < \delta\}\,, \quad 1 \leq j \leq n_1\,,$$

satisfy

$$\bar{\Omega}^{1,\delta}_{0,j} \cap \bar{\Omega}^{1,\delta}_{0,\ell} = \emptyset$$

for any $j \neq \ell$, and consider the open set

$$D_1 := D \setminus \cup_{j=1}^{n_1} \bar{\Omega}^{1,\delta}_{0,j}\,.$$

Then,

$$\partial D_1 = \partial D \cup \cup_{j=1}^{n_1} (\partial\Omega^{1,\delta}_{0,j} \setminus \partial\Omega) = \gamma_0 \cup \gamma^D_\infty \cup \cup_{j=1}^{n_1} (\partial\Omega^{1,\delta}_{0,j} \setminus \partial\Omega)\,.$$

By construction, the set

$$K_1 := \gamma^D_\infty \cup \cup_{j=1}^{n_1} (\partial\Omega^{1,\delta}_{0,j} \setminus \partial\Omega)$$

is a compact subset of Ω_+. Thus, thanks to Theorem 3.3, there exists a universal constant $M_3 > 0$ such that for any $\lambda < \sigma_1$

$$\|\Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]}\|_{C(K_1)} \leq M_3\,.$$

Hence, for each $\lambda < \sigma_1$ the function $\Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]}$ provides us with a subsolution of the problem

$$\begin{aligned} -\Delta u &= \lambda u - a(x)uf(x,u) \quad \text{in } D_1\,, \\ u &= M_3 \quad \text{on } K_1\,, \\ u &= 0 \quad \text{on } \gamma_0 = \partial D_1 \setminus K_1\,. \end{aligned} \tag{3.35}$$

The lower order refuges of D_1 are $\Omega^2_{0,j}$, $1 \le j \le n_2$. Therefore, thanks to Theorem 3.1 the problem (3.35) possesses a regular positive solution if, and only if, $\lambda < \sigma_2$. Moreover, it is unique if it exists. Let $\Theta_{[\sigma_1,a,D_1,K_1,M_3]}$ denote the unique regular positive solution of (3.35) for $\lambda = \sigma_1$. Thanks to Remark 3.2(b), for each $\lambda < \sigma_1$ we have

$$\Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]} \le \Theta_{[\sigma_1,a,D_1,K_1,M_3]} \quad \text{in } D_1\,. \tag{3.36}$$

Since δ can be taken arbitrarily small and the mapping $\lambda \to \Theta_{[\lambda,a,\Omega,\Gamma_+,\infty]}$ is nondecreasing, it follows from (3.36) that the point-wise limit (3.28) is well defined. The same bootstrapping and compactness arguments as above combined with (3.28) show that $\Theta_{[\sigma_1,a,\Omega,\Gamma_+,\infty]}$ is a regular large solution of (3.29). Its minimal character follows easily from the construction itself. This completes the proof. □

4. Stabilization of the regular positive solutions in $\Omega \setminus \bar\Omega^1_0$ as $\lambda \uparrow \sigma_1$.

Thanks to Theorem 2.1, the problem (1.5) possesses a regular positive solution if, and only if,

$$\sigma_0 < \lambda < \sigma_1\,.$$

Moreover, it is unique if it exists. As in Section 2 we shall denote it by $\theta_{[\lambda,a,\Omega]}$. The following result shows that $\theta_{[\lambda,a,\Omega]}$ converges to a large regular solution in $\Omega \setminus \bar\Omega^1_0$ as $\lambda \uparrow \sigma_1$.

Theorem 4.1. *Suppose* (Ha1-3), (Hf) *and* (Hfb), *and assume in addition that* (3.7) *is satisfied for any compact subset K of Ω_+ and $\beta = \sigma_1$. Set*

$$\Omega_1 := \Omega \setminus \bar\Omega^1_0\,, \quad \Omega^1_0 := \cup_{j=1}^{n_1} \Omega^1_{0,j}\,.$$

Then, the point-wise limit

$$\Xi_{[\sigma_1,a,\Omega_1]}(x) := \lim_{\lambda\uparrow\sigma_1} \theta_{[\lambda,a,\Omega]}(x) \quad x \in \Omega_1\,, \tag{4.1}$$

is well defined. Moreover, $\Xi_{[\sigma_1,a,\Omega_1]} \in C^{2+\mu}(\Omega_1)$,

$$\lim_{\lambda\uparrow\sigma_1} \|\theta_{[\lambda,a,\Omega]} - \Xi_{[\sigma_1,a,\Omega_1]}\|_{C^{2+\mu}(\Omega_1)} = 0\,, \tag{4.2}$$

and $\Xi_{[\sigma_1,a,\Omega_1]}$ is the minimal large regular solution of

$$\begin{aligned} -\Delta u &= \sigma_1 u - a(x)uf(x,u) \quad \text{in } \Omega_1\,, \\ u &= \infty \quad \text{on } \partial\Omega_1 \cap \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega\,, \end{aligned} \tag{4.3}$$

Proof. Basically this result is a particular case of Theorem 3.5(iii), but not exactly, since Theorem 3.5 dealt with the limiting behaviour of large regular solutions of

(3.1) going to infinity on some of the components of $\partial\Omega_+ \cap \partial\Omega$ and now it might occur $\partial\Omega_+ \cap \partial\Omega = \emptyset$. Nevertheless, the proof of Theorem 3.5(iii) carries over almost mutatis mutandis to prove Theorem 4.1. In the sequel the notations introduced in the proof of Theorem 3.5 will be kept out.

Pick up $\delta > 0$ sufficiently small so that the open neighborhoods

$$\Omega^{1,\delta}_{0,j} := \{x \in \Omega \;:\; \mathrm{dist}(x, \Omega^1_{0,j}) < \delta\}\,, \quad 1 \le j \le n_1\,,$$

satisfy

$$\bar{\Omega}^{1,\delta}_{0,j} \cap \bar{\Omega}^{1,\delta}_{0,\ell} = \emptyset$$

for any $j \neq \ell$, and consider the open set

$$D := \Omega \setminus \cup_{j=1}^{n_1} \bar{\Omega}^{1,\delta}_{0,j}\,.$$

By definition, $D \subset \Omega_1$. Moreover, any component of $\partial\Omega^{1,\delta}_{0,j} \setminus \partial\Omega$, $1 \le j \le n_1$, lies within Ω_+. Let Γ^j_+, $1 \le j \le q$, denote all the components of ∂D contained in Ω_+. Note that any component of ∂D either it is entirely contained in Ω_+, or it is a component of $\partial\Omega$. Then,

$$K := \cup_{j=1}^{q} \Gamma^j_+$$

is a compact subset of Ω_+. Thanks to Theorem 3.3, there exists a universal constant $M > 0$ such that

$$\|\theta_{[\lambda,a,\Omega]}\|_{C(K)} \le M$$

for any $\lambda \in (\sigma_0, \sigma_1)$, and hence $\theta_{[\lambda,a,\Omega]}$ provides us with a subsolution of

$$\begin{aligned} -\Delta u &= \lambda u - a(x) u f(x,u) \quad \text{in } \; D\,, \\ u &= M \quad \text{on } K\,, \\ u &= 0 \quad \text{on } \partial D \setminus K\,. \end{aligned} \tag{4.4}$$

The lower order refuges of D are $\Omega^2_{0,j}$, $1 \le j \le n_2$. Therefore, thanks to Theorem 3.1 the problem (4.4) possesses a regular positive solution if, and only if, $\lambda < \sigma_2$. Moreover, it is unique if it exists. Let $\Theta_{[\sigma_1,a,D,K,M]}$ denote the unique regular positive solution of (4.4) for $\lambda = \sigma_1$. Thanks to Remark 3.2(b) we have that

$$\theta_{[\lambda,a,\Omega]} \le \Theta_{[\sigma_1,a,D,K,M]} \quad \text{in } D\,, \tag{4.5}$$

for all $\lambda \in (\sigma_0, \sigma_1)$. Since δ can be taken arbitrarily small,

$$\lim_{\delta \downarrow 0} D = \Omega_1\,,$$

and the mapping $\lambda \to \theta_{[\lambda,a,\Omega]}$ is increasing, it follows from (4.5) that the point-wise limit (4.1) is well defined.

Let $\mathcal{O}_1 \subset \mathcal{O}$ two open subsets of Ω_1 such that $\bar{\mathcal{O}}_1 \subset \mathcal{O}$, $\bar{\mathcal{O}} \subset \Omega_1$, and choose $\delta > 0$ sufficiently small so that $\bar{\mathcal{O}} \subset D$. Then, (4.5) implies

$$\theta_{[\lambda,a,\Omega]} \le \Theta_{[\sigma_1,a,D,K,M]} \quad \text{in } \bar{\mathcal{O}}\,, \quad \lambda \in (\sigma_0, \sigma_1)\,.$$

Thus, by the L^p-estimates of Agmon, Douglis & Nirenberg, for each $p > 1$ there exists a constant $C_1 = C(p, \mathcal{O}_1)$ such that

$$\|\theta_{[\lambda,a,\Omega]}\|_{W^{2,p}(\mathcal{O}_1)} \leq C_1\,, \quad \lambda \in (\sigma_0, \sigma_1)\,.$$

Hence, thanks to Morrey's embedding theorem and Schauder's estimates, there exists a constant $C_2 = C(\mathcal{O}_1)$ such that

$$\|\theta_{[\lambda,a,\Omega]}\|_{C^{2+\mu}(\bar{\mathcal{O}}_1)} \leq C_2\,, \quad \lambda \in (\sigma_0, \sigma_1)\,.$$

Now, a well known compactness argument combined with the uniqueness of the point-wise limit (4.1) shows that $\Xi_{[\sigma_1,a,\Omega_1]} \in C^{2+\mu}(\bar{\mathcal{O}}_1)$ is a regular solution of

$$-\Delta u = \sigma_1 u - auf(\cdot, u)$$

in $\mathcal{O}_1$, and that

$$\lim_{\lambda \uparrow \sigma_1} \|\theta_{[\lambda,a,\Omega]} - \Xi_{[\lambda_1,a,\Omega_1]}\|_{C^{2+\mu}(\bar{\mathcal{O}}_1)} = 0\,.$$

This completes the proof of (4.2). By (2.3) and the construction itself $\Xi_{[\sigma_1,a,\Omega_1]}$ must be the minimal regular positive solution of (4.3). The proof is completed. $\square$

5. The existence of metasolutions for $\lambda \geq \sigma_1$.

The following concept goes back to [GL98] and [Go99], where it was introduced in the special case when $f(x,u) = u^p$ for some $p > 0$.

Definition 5.1. *Suppose* (Ha1-3)*,* (Hf) *and set*

$$\Omega_k := \Omega \setminus \cup_{i=1}^k \cup_{j=1}^{n_i} \bar{\Omega}_{0,j}^i\,, \quad 1 \leq k \leq m\,.$$

A function

$$u : \Omega \to [0, \infty]$$

is said to be a regular metasolution of order k of (1.5) *supported in Ω_k if $u|_{\Omega_k}$ is a large regular solution of*

$$\begin{aligned} -\Delta u &= \lambda u - a(x)uf(x,u) \quad \text{in } \Omega_k\,, \\ u &= \infty \quad \text{on } \partial\Omega_k \cap \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega_k \cap \partial\Omega\,, \end{aligned} \tag{5.1}$$

in the sense of the statement of Theorem 3.5(i), while

$$u = \infty \quad \text{in } (\Omega \setminus \bar{\Omega}_k) \cup (\partial\Omega_k \cap \Omega)\,.$$

Using this concept, the following result is an immediate consequence from Theorem 4.1.

Corollary 5.2. *Suppose* (Ha1-3)*,* (Hf) *and* (Hfb)*, and assume in addition that* (3.7) *is satisfied for any compact subset K of Ω_+ and $\beta = \sigma_1$. Then, the function $\mathcal{M}_{[\sigma_1,a,\Omega]} : \Omega \to [0,\infty]$ defined by*

$$\mathcal{M}_{[\sigma_1,a,\Omega]} := \begin{cases} \Xi_{[\sigma_1,a,\Omega_1]} & \text{in } \Omega_1\,, \\ \infty & \text{in } (\Omega \setminus \bar{\Omega}_1) \cup (\partial\Omega_1 \cap \Omega)\,, \end{cases} \tag{5.2}$$

is a regular metasolution of order one of (1.5)*, with $\lambda = \sigma_1$, supported in Ω_1.*

The following result characterizes the range of values of the parameter λ for which (1.5) admits a regular metasolution of order k supported in Ω_k, $1 \le k \le m$. Thanks to Theorem 2.2 those metasolutions are the candidates to describe the limiting behaviour of the population as time passes by for any $\lambda \ge \sigma_1$.

Theorem 5.3. *Suppose* (Ha1-3)*,* (Hf) *and* (Hfb)*, and assume in addition that* (3.7) *is satisfied for any compact subset K of Ω_+ and $\beta > 0$. Fix $k \in \{1,\dots,m\}$. If $k < m$, then* (1.5) *possesses a regular metasolution of order k supported in Ω_k if, and only if, $\lambda < \sigma_{k+1}$. If $k = m$, then* (1.5) *possesses a regular metasolution of order m supported in Ω_m for each $\lambda \in \mathbb{R}$. Moreover, if the problem possesses a metasolution of order k supported in Ω_k, then it also possesses a minimal metasolution of order k supported in Ω_k, $1 \le k \le m$.*

Proof. Suppose $k < m$ and (5.1) possesses a large solution u. Then,

$$\lambda \le \sigma^{\Omega_k}[-\Delta + af(\cdot,u)] < \sigma^{\Omega_{0,1}^{k+1}} = \sigma_{k+1}\,.$$

Suppose $k < m$ and $\lambda < \sigma_{k+1}$. To prove the existence of a metasolution of order k supported in Ω_k it suffices to show that (5.1) has a large regular solution. As $a(x)$ vanishes on $\partial\Omega_k \setminus \partial\Omega$, Theorem 3.5 can not be applied straight away to prove this, but this trouble may be overcome arguing as follows.

Fix $M > 0$ and consider the problem

$$\begin{aligned} -\Delta u = \lambda u - a(x)uf(x,u) \quad & \text{in } \Omega_k\,, \\ u = M \quad & \text{on } \partial\Omega_k \cap \Omega\,, \\ u = 0 \quad & \text{on } \partial\Omega_k \cap \partial\Omega\,. \end{aligned} \tag{5.3}$$

This problem does not fit into the setting of Theorem 3.1, since $a(x) = 0$ on $\partial\Omega_k \setminus \partial\Omega$, but we can slightly modify Ω_k so that the corresponding problem does it. For each $\delta > 0$ consider the open neighborhoods

$$\Omega_{0,j}^{i,\delta} := \{x \in \Omega \,:\, \mathrm{dist}(x, \Omega_{0,j}^i) < \delta\}\,, \quad 1 \le i \le k\,, \quad 1 \le j \le n_i\,.$$

By (Ha1-3) there exists $\delta_0 > 0$ such that for each $\delta \in (0,\delta_0)$

$$\bar{\Omega}_{0,j}^{i,\delta} \cap \bar{\Omega}_{0,\hat{j}}^{\hat{i},\delta} = \emptyset$$

if $(i,j) \ne (\hat{i},\hat{j})$. Now, consider the open set

$$\Omega_k^\delta := \Omega \setminus \cup_{i=1}^k \cup_{j=1}^{n_i} \bar{\Omega}_{0,j}^{i,\delta}\,.$$

By definition,

$$\Omega_k^\delta \subset \Omega \setminus \cup_{i=1}^k \cup_{j=1}^{n_i} \bar{\Omega}_{0,j}^i = \Omega_k \,,$$

and

$$\lim_{\delta \downarrow 0} \Omega_k^\delta = \Omega_k \,,$$

in the sense of [Lo96]. Moreover, any component of $\partial \Omega_{0,j}^{i,\delta} \setminus \partial \Omega$, $1 \leq i \leq k$, $1 \leq j \leq n_i$, lies within Ω_+. Note that any component of $\partial \Omega_k^\delta$ either it is entirely contained in Ω_+, or it is a component of $\partial \Omega$. Now, consider the modified problem

$$\begin{aligned} -\Delta u = \lambda u - a(x) u f(x,u) \quad & \text{in } \Omega_k^\delta \,, \\ u = M \quad & \text{on } \partial \Omega_k^\delta \cap \Omega \,, \\ u = 0 \quad & \text{on } \partial \Omega_k^\delta \cap \partial \Omega \,. \end{aligned} \tag{5.4}$$

Thanks to Theorem 3.1, (5.4) possesses a unique positive solution, say $\Theta_{[\lambda,a,\delta,M]}$. Let $\mathcal{O}_1 \subset \mathcal{O}$ be two open subdomains of Ω_k such that $\bar{\mathcal{O}}_1 \subset \mathcal{O}$, $\bar{\mathcal{O}} \subset \Omega_k$. By construction, there exists $\delta_1 \in (0, \delta_0)$ such that for any $\delta \in (0, \delta_1)$

$$\bar{\mathcal{O}} \subset \Omega_k^{\delta_1} \subset \Omega_k^\delta \subset \Omega_k \,,$$

Set

$$K := \partial \Omega_k^{\delta_1} \setminus \partial \Omega \,.$$

By construction, for each $\delta \in (0, \delta_1)$ K is a compact subset of

$$\{ x \in \Omega_k^\delta \ : \ a(x) > 0 \} \,.$$

Thus, thanks to Theorem 3.3, there exists a constant $M_1 > 0$ such that for each $\delta \in (0, \delta_1)$

$$\| \Theta_{[\lambda,a,\delta,M]} \|_{C(K)} \leq M_1 \,.$$

Hence, for each $0 < \delta < \delta_1$ the function $\Theta_{[\lambda,a,\delta,M]}$ provides us with a positive subsolution of

$$\begin{aligned} -\Delta u = \lambda u - a(x) u f(x,u) \quad & \text{in } \Omega_k^{\delta_1} \,, \\ u = M_1 \quad & \text{on } \partial \Omega_k^{\delta_1} \cap \Omega \,, \\ u = 0 \quad & \text{on } \partial \Omega_k^{\delta_1} \cap \partial \Omega \,. \end{aligned} \tag{5.5}$$

Let $\Theta_{[\lambda,a,\delta_1,M_1]}$ denote the unique positive solution of (5.5), whose existence is guaranteed by Theorem 3.1. Thanks to Remark 3.2(b), for each $\delta \in (0, \delta_1)$ we have

$$\Theta_{[\lambda,a,\delta,M]} \leq \Theta_{[\lambda,a,\delta_1,M_1]} \quad \text{in } \Omega_k^{\delta_1} \,,$$

and therefore, there exists a constant $M_2 > 0$ such that for each $0 < \delta < \delta_1$

$$\| \Theta_{[\lambda,a,\delta,M]} \|_{C(\bar{\mathcal{O}})} \leq M_2 \,.$$

By the same bootstrapping argument used in the proof of Theorem 3.5 and Theorem 4.1, there exists a constant $M_3 > 0$ such that for each $\delta \in (0, \delta_1)$

$$\| \Theta_{[\lambda,a,\delta,M]} \|_{C^{2+\mu}(\bar{\mathcal{O}}_1)} \leq M_3 \,.$$

Now, by a standard compactness argument combined with a diagonal procedure it is clear that we can substract a subsequence $\delta_n \downarrow 0$ such that

$$\lim_{n\to\infty} \|\Theta_{[\lambda,a,\delta_n,M]} - \Theta_{[\lambda,a,M]}\|_{C^{2+\mu}(\Omega_k)} = 0$$

for some $\Theta_{[\lambda,a,M]} \in C^{2+\mu}(\Omega_k)$. Necessarily,

$$\lim_{x\to[\partial\Omega_k\setminus\partial\Omega]} \Theta_{[\lambda,a,M]}(x) = M\,.$$

Therefore, $\Theta_{[\lambda,a,M]}$ provides us with a solution of (5.3). The same argument of the proof of Theorem 3.1 shows that in fact $\Theta_{[\lambda,a,M]}$ is the unique regular positive solution of (5.3), and that $M \to \Theta_{[\lambda,a,M]}$ is point-wise increasing.

Thanks to Theorem 3.3, there exists a constant $M_4 > 0$ such that for any $M > 0$

$$\|\Theta_{[\lambda,a,M]}\|_{C(K)} \leq M_4\,.$$

Hence, for each $M > 0$ the function $\Theta_{[\lambda,a,M]}$ provides us with a subsolution of

$$\begin{aligned} -\Delta u = \lambda u - a(x)uf(x,u) \quad &\text{in } \Omega_k^{\delta_1}\,,\\ u = M_4 \quad &\text{on } \partial\Omega_k^{\delta_1}\cap\Omega\,,\\ u = 0 \quad &\text{on } \partial\Omega_k^{\delta_1}\cap\partial\Omega\,, \end{aligned} \tag{5.6}$$

and so

$$\Theta_{[\lambda,a,M]} \leq \Theta_{[\lambda,a,\delta_1,M_4]} \quad \text{in } \Omega_k^{\delta_1}\,.$$

Therefore, the point-wise limit

$$\Theta_{[\lambda,a,\infty]}(x) := \lim_{M\uparrow\infty} \Theta_{[\lambda,a,M]}(x)\,, \quad x \in \Omega_k\,,$$

is well defined. Finally, the same regularity and compactness argument given before shows that $\Theta_{[\lambda,a,\infty]}$ is a large solution of (5.1). Its minimality follows easily from the construction itself.

In case $k = m$, by Remark 3.2(a) we do not have any limitation on the size of λ in order to apply Theorem 3.1, and therefore the previous procedure provides us with a minimal large solution of (5.1) for each $\lambda \in \mathbb{R}$. This completes the proof. $\square$

The following result shows the point-wise behavior of the metasolutions of order $k \leq m-1$ supported in Ω_k as $\lambda \uparrow \sigma_{k+1}$. They stabilize in Ω_{k+1}, whereas they grow to infinity in

$$\cup_{j=1}^{n_{k+1}} \bar{\Omega}_{0,j}^{k+1} \setminus \partial\Omega\,.$$

Therefore, they provide us with metasolutions of order $k+1$ supported in Ω_{k+1}.

Theorem 5.4. *Suppose* (Ha1-3), (Hf) *and* (Hfb), *and assume in addition that* (3.7) *is satisfied for any compact subset K of Ω_+ and $\beta > 0$. Fix $k \in \{1,\dots,m-1\}$ and consider a sequence*

$$\lambda_n \in (\sigma_k, \sigma_{k+1})\,, \quad n \geq 1\,,$$

such that

$$\lim_{n\to\infty} \lambda_n = \sigma_{k+1}\,.$$

For each $n \geq 1$, let u_n be a metasolution of order k of (1.5)*, with $\lambda = \lambda_n$, supported in*
$$\Omega_k = \Omega \setminus \cup_{i=1}^{k} \cup_{j=1}^{n_i} \bar{\Omega}_{0,j}^{i} \,,$$
whose existence is guaranteed by Theorem 5.3. Then,

$$\lim_{n\to\infty} u_n = \infty \quad \text{uniformly in } \cup_{j=1}^{n_{k+1}} \bar{\Omega}_{0,j}^{k+1} \setminus \partial\Omega \,. \tag{5.7}$$

Moreover, for any open subdomain $\mathcal{O}$ of Ω_{k+1} with $\bar{\mathcal{O}} \subset \Omega_{k+1}$ there exists a subsequence of (λ_n, u_n), $n \geq 1$, relabeled again by n, and a regular solution of

$$-\Delta u = \sigma_{k+1} u - a u f(\cdot, u)$$

in $\mathcal{O}$, say $u_\omega \in C^{2+\mu}(\bar{\mathcal{O}})$, such that

$$\lim_{n\to\infty} \|u_n - u_\omega\|_{C^{2+\mu}(\bar{\mathcal{O}})} = 0 \,. \tag{5.8}$$

Furthermore, if u_λ stands for the minimal metasolution of order k of (1.5) *supported in Ω_k, then the point-wise limit*

$$u_\omega(x) = \lim_{\lambda \uparrow \sigma_{k+1}} u_\lambda(x) \,, \quad x \in \Omega_{k+1} \,,$$

is well defined and its extension by ∞ to Ω provides us with a metasolution of order $k+1$ of (1.5) *supported in Ω_{k+1} (for the value of the parameter $\lambda = \sigma_{k+1}$).*

Proof. By definition, for each $n \geq 1$ the function $u_n|_{\Omega_k}$ is a large regular solution of
$$\begin{aligned} -\Delta u &= \lambda_n u - a(x) u f(x,u) \quad \text{in } \Omega_k \,, \\ u &= \infty \quad \text{on } \partial\Omega_k \cap \Omega \,, \\ u &= 0 \quad \text{on } \partial\Omega_k \cap \partial\Omega \,. \end{aligned}$$

In particular, $u_n|_{\Omega_k}$ is a positive strict supersolution of

$$\begin{aligned} -\Delta u &= \lambda_n u - a(x) u^{p+1} \quad \text{in } \Omega_k \,, \\ u &= 0 \quad \text{on } \partial\Omega_k \,, \end{aligned} \tag{5.9}$$

and hence,
$$u_n|_{\Omega_k} \geq \theta_{[\lambda_n, a, \Omega_k]} \,, \tag{5.10}$$
where $\theta_{[\lambda_n, a, \Omega_k]} \gg 0$ is the unique positive solution of (5.9), whose existence and uniqueness is guaranteed by Theorem 2.1(i). Note that the lower order refuges of Ω_k are $\Omega_{0,j}^{k+1}$, $1 \leq j \leq n_{k+1}$, and that

$$\lambda_n < \sigma_{k+1} = \sigma^{\Omega_{0,j}^{k+1}} \,.$$

Thanks to Theorem 2.1(iii), we have

$$\lim_{n\to\infty} \theta_{[\lambda_n, a, \Omega_k]} = \infty \quad \text{uniformly in } \cup_{j=1}^{n_{k+1}} \bar{\Omega}_{0,j}^{k+1} \setminus \partial\Omega_k \,,$$

and hence we find from (5.10) that

$$\lim_{n\to\infty} u_n = \infty \quad \text{uniformly in } \cup_{j=1}^{n_{k+1}} \bar{\Omega}_{0,j}^{k+1} \setminus \partial\Omega_k \,. \tag{5.11}$$

Moreover, $u_n = \infty$ in $(\Omega \setminus \bar{\Omega}_k) \cup (\partial\Omega_k \cap \Omega)$, since u_n is a metasolution of order k supported in Ω_k. Thus, (5.11) implies (5.7).

Now, for each $\delta > 0$ sufficiently small consider the open set

$$D := \Omega \setminus \cup_{i=1}^{k+1} \cup_{j=1}^{n_i} \bar{\Omega}_{0,j}^{i,\delta} \,,$$

where

$$\Omega_{0,j}^{1,\delta} := \{x \in \Omega \;:\; \operatorname{dist}(x, \Omega_{0,j}^i) < \delta\} \,, \quad 1 \le i \le k+1 \,, \quad 1 \le j \le n_i \,.$$

By definition,

$$D \subset \Omega \setminus \cup_{i=1}^{k+1} \cup_{j=1}^{n_i} \bar{\Omega}_{0,j}^{i} = \Omega_{k+1} \,,$$

and

$$\lim_{\delta \downarrow 0} D = \Omega_{k+1} \,,$$

in the sense of [Lo96]. Moreover, any component of $\partial\Omega_{0,j}^{i,\delta} \setminus \partial\Omega$, $1 \le i \le k+1$, $1 \le j \le n_i$, lies within Ω_+. Let Γ_+^j, $1 \le j \le q$, denote all the components of ∂D contained in Ω_+. Note that any component of ∂D either it is entirely contained in Ω_+, or it is a component of $\partial\Omega$. Then,

$$K := \cup_{j=1}^{q} \Gamma_+^j$$

is a compact subset of Ω_+. Thanks to Theorem 3.3, there exists a constant $M > 0$ such that for any $n \ge 1$,

$$\|u_n\|_{C(K)} \le M \,,$$

and hence u_n provides us with a positive subsolution of

$$\begin{aligned} -\Delta u = \sigma_{k+1} u - a(x) u f(x,u) \quad &\text{in } \; D \,, \\ u = M \quad &\text{on } K \,, \\ u = 0 \quad &\text{on } \partial D \setminus K \,. \end{aligned} \tag{5.12}$$

The lower order refuges of (5.12) are $\Omega_{0,j}^{k+2}$, $1 \le j \le n_{k+2}$, if $k \le m-2$, and $a(x)$ is bounded away from zero if $k = m-1$. Moreover, $\sigma_{k+1} < \sigma_{k+2}$. Therefore, thanks to Theorem 3.1 the problem (5.12) possesses a unique regular positive solution. Let Θ denote it. By Remark 3.2(b) we find that for each $n \ge 1$

$$u_n \le \Theta \quad \text{in } D \,. \tag{5.13}$$

Let $\mathcal{O}$ be an open subset of Ω_{k+1} with $\bar{\mathcal{O}} \subset \Omega_{k+1}$ and choose $\delta > 0$ sufficiently small so that

$$\mathcal{O} \subset D \,.$$

The same bootstrapping argument of the proof of Theorem 4.1 shows that there exists a constant $M_1 > 0$ such that

$$\|u_n\|_{C^{2+\mu}(\bar{\mathcal{O}})} \leq M_1 \,, \quad n \geq 1 \,.$$

Therefore, we can substract a subsequence of (λ_n, u_n), again labeled by n, such that

$$\lim_{n\to\infty} \|u_n - u_\omega\|_{C^{2+\mu}(\bar{\mathcal{O}})} = 0$$

for some $u_\omega \in C^{2+\mu}(\bar{\mathcal{O}})$. Necessarily, u_ω is a solution of

$$-\Delta u = \sigma_{k+1} u - a u f(\cdot, u)$$

in $\mathcal{O}$.

The last assertion follows easily from the fact that $\lambda \to u_\lambda(x)$ is nondecreasing for each $x \in \bar{\Omega}$. This completes the proof of the theorem. $\square$

6. The asymptotic behaviour of the population for $\lambda \geq \sigma_1$.

In this section we characterize the asymptotic behaviour of the population as $t \uparrow \infty$ for any value of the parameter $\lambda \geq \sigma_1$.

Theorem 6.1. *Suppose* (Ha1-3), (Hf) *and*

$$\partial\Omega \cap \partial\Omega_+ \neq \emptyset \,.$$

Let Γ_+^j, $1 \leq j \leq q$, be q arbitrary components of $\partial\Omega \cap \partial\Omega_+$, and $\alpha_j > 0$, $1 \leq j \leq q$, q arbitrary constants, and consider the evolution problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x) u f(x,u) \quad & \text{in } \Omega \times (0,\infty)\,, \\ u\big|_{\Gamma_+^j} = \alpha_j > 0\,, \quad 1 \leq j \leq q \quad & t > 0 \\ u = 0 \quad \text{on } (\partial\Omega \setminus \cup_{j=1}^q \Gamma_+^j) \times (0,\infty)\,, & \\ u(\cdot, 0) = u_0\,, \quad \text{in } \Omega\,, & \end{aligned} \tag{6.1}$$

The following assertions are true:

(i) *For each $u_0 \in U_0$, $u_0 \geq 0$, the problem* (6.1) *possesses a unique global regular solution $u_{[\lambda,a,\Omega,\Gamma_+,\alpha]}(\cdot,\cdot;u_0) \in C^{2+\mu,1+\frac{\mu}{2}}(\bar{\Omega} \times (0,\infty))$.*

(ii) *For each $u_0 \in U_0$, $u_0 \geq 0$,*

$$\lim_{t\to\infty} \|u_{[\lambda,a,\Omega,\Gamma_+,\alpha]}(\cdot,t;u_0) - \Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}\|_{C(\bar{\Omega})} = 0\,, \tag{6.2}$$

where $\Theta_{[\lambda,a,\Omega,\Gamma_+,\alpha]}$ is the unique regular positive solution of (3.1).

Proof. The existence of a unique regular solution follows from the results of [DK92]. The global existence of these solutions is easily obtained from the estimate

$$\frac{\partial u}{\partial t} - \Delta u = \lambda u - a u f(\cdot, u) \leq \lambda u\,.$$

This completes the proof of Part (i). Adapting the proof of Theorem 2.2 it is easily seen that for any $u_0 \in U_0$, $u_0 \geq 0$, condition (6.2) holds. $\square$

From this result it easily follows the main theorem of this section.

Theorem 6.2. *Suppose* (Ha1-3), (Hf) *and* (Hfb), *and assume in addition that* (3.7) *is satisfied for any compact subset K of Ω_+ and $\beta > 0$ and that either $1 \leq k \leq m-1$ and $\sigma_k \leq \lambda < \sigma_{k+1}$, or $k = m$ and $\lambda \geq \sigma_m$. Then, for any $u_0 \in U_0$ there exists $\nu > 0$ such that for any compact subset K of Ω_k the restriction of the orbit*

$$\Gamma(u_0) := \{u_{[\lambda,a,\Omega]}(\cdot, t; u_0) \ : \ t \geq 0\}$$

to K is relatively compact in $C^{2+\nu}(K)$. Moreover,

$$\Theta^{\min}_{[\lambda,a,\infty]} \leq \liminf_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot, t; u_0) \leq \limsup_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot, t; u_0) \leq \Theta^{\max}_{[\lambda,a,\Omega]} \,,$$

where $\Theta^{\min}_{[\lambda,a,\infty]}$ and $\Theta^{\max}_{[\lambda,a,\infty]}$ stand for the minimal and maximal large solutions of problem (5.1), *respectively. Thus, if* (5.1) *possesses a unique large solution, say $\Theta_{[\lambda,a,\Omega]}$, then*

$$\lim_{t\uparrow\infty} \|u_{[\lambda,a,\Omega]}(\cdot, t; u_0) - \Theta_{[\lambda,a,\Omega]}\|_{C^{2+\nu}(K)} = 0 \tag{6.3}$$

for any compact subset K of Ω_k, and therefore due to Theorem 2.2(iii) the solution $u_{[\lambda,a,\Omega]}(\cdot, t; u_0)$ is point-wise convergent as $t \uparrow \infty$ to the metasolution of order k supported in Ω_k associated with $\Theta_{[\lambda,a,\Omega]}$.

Proof. The notations introduced in the proof of Theorem 5.3 will be kept out. Consider $1 \leq k \leq m$ and λ within its corresponding range of values. Let $K \subset \Omega_k$ compact and consider $\delta > 0$ sufficiently small so that

$$K \subset \Omega^\delta_k \subset \Omega_k \,.$$

Set

$$M_L := \inf_{\partial\Omega^\delta_k \cap \Omega} u_{[\lambda,a,\Omega]}(\cdot, 1; u_0) \,, \quad M_S := \sup_{\partial\Omega^\delta_k \cap \Omega} u_{[\lambda,a,\Omega]}(\cdot, 1; u_0)$$

and consider the auxiliary evolution problems

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u = \lambda u - a(x) u f(x, u) \quad & \text{in } \Omega^\delta_k \times (0, \infty) \,, \\ u = M > 0 \,, \quad & \text{on } (\partial\Omega^\delta_k \cap \Omega) \times (0, \infty) \,, \\ u = 0 \quad & \text{on } (\partial\Omega^\delta_k \cap \partial\Omega) \times (0, \infty) \,, \\ u(\cdot, 0) = u_1 \,, \quad & \text{in } \bar{\Omega}^\delta_k \,, \end{aligned} \tag{6.4}$$

where

$$M \in \{M_L, M_s\} \,, \quad u_1 := u_{[\lambda,a,\Omega]}(\cdot, 1; u_0) \,.$$

Thanks to the parabolic maximum principle, for each $t \geq 1$ we have that

$$u_{[\lambda,a,\delta,M_L]}(\cdot, t-1; u_1) \leq u_{[\lambda,a,\Omega]}(\cdot, t; u_0) \leq u_{[\lambda,a,\delta,M_S]}(\cdot, t-1; u_1) \quad \text{in } \Omega^\delta_k \,,$$

where $u_{[\lambda,a,\delta,M]}(\cdot, t-1; u_1)$ is the unique global regular solution of (6.4), whose existence is guaranteed by Theorem 6.1(i). Moreover, it follows from Theorem 6.1(ii) that

$$\lim_{t\uparrow\infty} \|u_{[\lambda,a,\delta,M]}(\cdot, t-1; u_1) - \Theta_{[\lambda,a,\delta,M]}\|_{C^1(\bar{\Omega}^\delta_k)} = 0 \,, \tag{6.5}$$

where $\Theta_{[\lambda,a,\delta,M]}$ is the unique regular positive steady-state of (6.4), whose existence is guaranteed by Theorem 3.1. In particular, for each $\delta > 0$ we have that

$$\Theta_{[\lambda,a,\delta,M_L]} \leq \liminf_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \leq \limsup_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \leq \Theta_{[\lambda,a,\delta,M_S]} \tag{6.6}$$

in $\bar{\Omega}_k^\delta$. On the other hand, thanks to Theorem 2.2(iii) and the parabolic estimates of [Re82] and [Re86] it is easily seen that

$$\lim_{\delta\downarrow 0} M_L = \infty\,.$$

Thus, thanks to the analysis already done in the proof of Theorem 5.3,

$$\Theta^{\min}_{[\lambda,a,\Omega]} \leq \liminf_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \leq \limsup_{t\uparrow\infty} u_{[\lambda,a,\Omega]}(\cdot,t;u_0) \leq \inf_{\delta>0} \Theta_{[\lambda,a,\delta,\infty]} = \Theta^{\max}_{[\lambda,a,\Omega]}\,.$$

This completes the proof of (6.3). Moreover, it shows that there exists a constant $M > 0$ such that

$$\|u_{[\lambda,a,\Omega]}(\cdot,t;u_0)\|_{C(K)} \leq M\,.$$

Hence, by the results of [Re86], the restriction of $\Gamma(u_0)$ to K, say $\Gamma_K(u_0)$, is relatively compact in $C^{2-}(K)$. Therefore, by the parabolic Schauder estimates, [LSU68], there exists $\nu > 0$ such that $\Gamma_K(u_0)$ is relatively compact in $C^{2+\nu}(K)$. This completes the proof. $\square$

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