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Bifurcation of reaction-diffusion systems related to epidemics *

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Dedicated to Alan Lazer on his 60th birthday

Abstract

The article considers the reaction-diffusion equations modeling the infection of several interacting kinds of species by many types of bacteria. When the infected species compete significantly among themselves, it is shown by bifurcation method that the infected species will coexist with bacterial populations. The time stability of the postitive steady-states are also considered by semigroup method. If the infected species do not interact, it is shown that positive coexistence states with bacterial populations are still possible.

1 Introduction, Epidemic Models

This article considers reaction-diffusion equations modeling the infection of several interacting kinds of species by many types of virus or bacteria. Rigorous mathematical treatments are presented in the series of papers [1], [2] and [3] for analyzing the infection of one species by one type of bacteria. The situation can become even more involved in reality when there are more than one kind of bacteria and species, which may also interact among themselves. We consider the system:

$$\begin{gathered} -\Delta u_i + a_i(x)u_i = \textstyle\sum_{j=1}^m b_{ij}v_j \quad \text{for } x \in \Omega,\ i = 1,\dots,n, \\ -\Delta v_k + \tilde{a}_k(x)v_k = \textstyle\sum_{j=1}^n f_{kj}(u_j) + v_k \sum_{j=1}^m c_{kj}v_j \quad \text{for } x \in \Omega,\ k = 1,\dots,m \\ u_i = v_k = 0 \quad \text{for } x \in \partial\Omega,\ i = 1,\dots,n,\ k = 1,\dots,m \end{gathered} \tag{1.1}$$

where $b_{ij} > 0$ and $c_{kj} \geq 0$ are constants, $f_{kj} \in C^1(R)$, and Ω is a bounded domain in R^N, with $\partial\Omega$ of class $C^{2+\alpha}, 0 < \alpha < 1$. Here, Δ denotes the Laplacian operator; and the corresponding parabolic system, with $\partial u_i/\partial t$ and $\partial v_k/\partial t$

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added to the first n and the second m equations respectively on the left of (1.1), will also be considered. The functions u_i represent n different kinds of bacterial population densities and v_k represent m different types of infected species population densities. The populations are assumed to diffuse in space Ω. The functions $a_i(x)$ are assumed to be positive, because the bacterial populations tend to die in the absence of other factors; and the terms $b_{ij}v_j$ represent the growth of the number bacteria due to infected species. The functions $\tilde{a}_k(x)$ are assumed to be positive, because a certain proportion of the infected species recover per unit time; the terms $f_{kj}(u_j)$ describe the rate the k-th species becomes infected by u_j, and the terms $v_k c_{kj} v_j$ describe interaction between the k-th and j-th infected species. The model can be more readily interpreted in the form of the corresponding parabolic system, with (1.1) considered as the steady state solution. The prototype form with $m = n = 1$, $c_{11} = 0$, is introduced and explained in e.g. [1] and [2]. We will consider the case when all c_{kj} are zero as well as other cases.

For convenience, we will adopt the following conventions. Let B and K_0 be respectively $n \times m$ and $m \times n$ constant matrices as follows:

$$B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \qquad K_0 = \begin{bmatrix} f'_{11}(0) & \cdots & f'_{1n}(0) \\ \vdots & & \vdots \\ f'_{m1}(0) & \cdots & f'_{mn}(0) \end{bmatrix}. \tag{1.2}$$

Let $\mathfrak{E} = \{w = \text{col}(w_1, \ldots, w_{n+m}) | w_i \in C^1(\bar{\Omega}), w_i = 0 \text{ on } \partial\Omega, i = 1, \ldots, m+n\}$, with norm $\|w\|_{\mathfrak{E}} = \max\{\|w_i\|_{C^1(\Omega)} | i = 1, \ldots, m+n\}$; and $\mathfrak{P}$ denotes the cone $\mathfrak{P} = \{col.(w_1, \ldots, w_{n+m}) \in \mathfrak{E} | w_i \geq 0 \text{ in } \bar{\Omega},\ i = 1, \ldots, n+m\}$. Also, let $\mathfrak{F} = \{w = col.(w_1, \ldots, w_{n+m}) | w_i \in C^{2+\alpha}(\bar{\Omega}), w_i = 0 \text{ on } \partial\Omega\}$ with its norm denoted as $\|w\|_{\mathfrak{F}} = \max\{\|w_i\|_{C^{2+\alpha}(\bar{\Omega})} | i = 1, \ldots, m+n\}$. As operators from $C^{2+\alpha}(\Omega)$ into $C^{\alpha}(\Omega)$, we write $L_i = -\Delta + a_i$ for $i = 1, \ldots, n$, and $L_{n+k} = -\Delta + \tilde{a}_k$, for $k = 1, \ldots, m$. As an operator from $\mathfrak{F}$ into $[C^{\alpha}(\bar{\Omega})]^{n+m}$, we write $L = \text{col}(L_1, \ldots, L_{n+m})$. For abbreviation, we write $\hat{F} = \text{col}(F_1, \ldots, F_{n+m})$, where F_j are operators from $[C^1(\bar{\Omega})]^{n+m}$ or $\mathfrak{F}$ into $C^1(\bar{\Omega})$ defined by:

$$\begin{aligned} F_i[\text{col}(u_1, \ldots, u_n, v_1, \ldots, v_m)] &= \sum_{j=1}^{m} b_{ij} v_j \quad \text{for} \quad i = 1, \ldots, n \\ F_{n+k}[\text{col}(u_1, \ldots, u_n, v_1, \ldots, v_m)] &= \sum_{j=1}^{n} f_{kj}(u_j) + v_k \sum_{j=1}^{m} c_{kj} v_j \\ &\quad \text{for } k = 1, \ldots, m \end{aligned} \tag{1.3}$$

We now label a few key assumptions, some or all of which will be used in various theorems in this article.

[H1] The functions a_i and $\tilde{a}_k$ are members of $C^{\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, and satisfy $a_i(x) > 0$, $\tilde{a}_k(x) > 0$ for all $x \in \bar{\Omega}$, $i = 1, \ldots, n$, $k = 1, \ldots, m$.

[H2] The functions $f_{kj} \in C^1(R)$ satisfy $f'_{kj}(0) > 0$ and $f_{kj}(0) = 0$, $f_{kj}(s) \geq 0$ for all $s \geq 0$, $k = 1, \ldots, m$, $j = 1, \ldots, n$. For each k, there exists at least one j such that $0 < f_{kj}(s)$ for all $s > 0$.

[H2*] In addition to all the properties in [H2], the functions of f_{kj} satisfy $f_{kj}(s) \leq K_1 s$ for all $s > 0$, where K_1 is some positive constant.

[H3] There exists a constant vector $\vec{d} = \mathrm{col}(d_1, \dots, d_n)$, with $d_i > 0$, $i = 1, \dots, n$ such that: $BK_0\vec{d} > (\lambda_1 + a^*)^2\vec{d}$.

In [H3] and the rest of the paper, the (strict) inequality between the two vectors is interpreted to be satisfied for each component. The quantity λ_1 denotes the first eigenvalue of the problem: $-\Delta\phi = \lambda\phi$ in Ω, $\phi = 0$ on $\partial\Omega$; and ϕ will denote the corresponding positive eigenfunction with $\|\phi\|_\infty = 1$. The symbols a^* and a^{**} are defined as $a^* = \sup\{a_i(x), \tilde{a}_k(x) | x \in \bar{\Omega},\ i = 1, \dots, n,\ k = 1, \dots, m\}$ and $a^{**} = \inf\{a_i(x), \tilde{a}_k(x) | x \in \bar{\Omega},\ i = 1, \dots, n,\ k = 1, \dots, m\}$. Another assumption which will sometimes be used concerning the interaction of the species v_k is as follows:

[H4] $c_{kk} < 0$ and $|c_{kk}| > \sum_{j=1, j\neq k}^{m} |c_{kj}|$, for each $k = 1, \dots, m$.

In Section 2, we will show that problem (1.1) has a positive solution under appropriate conditions. Essentially, we assume that intraspecies interaction among infected species is large compared with interspecies interactions (cf. hypothesis [H4]). In Section 3 we consider the time stability of the positive solution as a steady-state of the corresponding parabolic system, when the infected species v_k compete among themselves (i.e. $c_{kj} < 0$). In Section 4 we consider the situation when the infected species v_k do not interact (i.e. $c_{kj} = 0$). We show that a positive solution still exists, if we assume appropriate infection rates when large numbers of bacteria are present (cf. [H5]). We will use bifurcation methods to show the existence of positive steady-state solutions, and semigroup method to analyze stability of steady-states. In many cases, we only outline the ideas of the proofs, the complete details will be shown elsewhere.

2 Bifurcation of Infected Species

In this section, we will show that under hypotheses [H1], [H2*], [H3] and [H4], the problem (1.1) has a positive solution. The main result of this section is Theorem 2.2. Let $w = \mathrm{col}(u_1, \dots, u_n, v_1, \dots, v_m)$, the system (1.1) can be abbreviated as

$$L[w] = F[w], \quad \text{where} \quad w \in \mathfrak{F}, \tag{2.1}$$

with the operators L and F as introduced before. To study this nonlinear problem, we consider the auxiliary problem:

$$L[w] = \lambda F[w], \quad w \in \mathfrak{F}, \tag{2.2}$$

where λ is a parameter, and investigate the bifurcation from the trivial solution $w = 0$ as the parameter λ passes through a certain value λ_0. Under conditions [H1] and [H2], we will see that this bifurcation actually occurs in Theorem 2.1. Moreover, in Theorem 2.2, we will see that hypotheses [H2*] and [H3] insure that $\lambda_0 < 1$; and hypothesis [H4] insures that the bifurcation curve of nontrivial solutions connects to $\lambda = +\infty$. Thus (2.2) has a nontrivial solution when $\lambda = 1$, i.e. (2.1) has a nontrivial solution under appropriate conditions. For convenience, we let M_0 denotes the $(n+m) \times (n+m)$ square constant matrix:

$$M_0 = \begin{bmatrix} 0 & B \\ K_0 & 0 \end{bmatrix}$$

where the 0's along the diagonal are zero matrices with appropriate dimensions. Applying L^{-1} to both sides of (2.2), using zero Dirichlet boundary condition, we obtain: $w = \lambda L^{-1}F[w]$. Thus, (2.2) can be written as:

$$Q(\lambda, w) = 0, \quad (\lambda, w) \in R \times \mathfrak{E}, \tag{2.3}$$

where $Q : R \times \mathfrak{E} \to \mathfrak{E}$ is an operator given by

$$Q(\lambda, w) := w - \lambda L^{-1}F[w]$$

(for the entire paper the inverse operators L^{-1} or L_i^{-1} will always mean finding the solution using zero Dirichlet boundary condition).

Theorem 2.1 *Under hypotheses [H1] and [H2], the point $(\lambda_0, 0)$ is a bifurcation point for problem (2.3). Here $\lambda = \lambda_0$ is the unique positive number so that the problem:*

$$L[w] = \lambda M_0 w \quad \textit{in}\ \Omega, \quad w = 0 \quad \textit{on} \quad \partial\Omega, \tag{2.4}$$

has a nonnegative eigenfunction in $\mathfrak{E}$. (The eigenvalue λ_0 is simple.) Moreover, the component of $\bar{\mathfrak{S}}$ containing the point $(\lambda_0, 0)$ is unbounded, where

$$\mathfrak{S} := \{(\lambda, w) \in R^+ \times \mathfrak{P} | Q(\lambda, w) = 0, \lambda > 0 \quad \textit{and} \quad w \in \mathfrak{P}\backslash\{0\}\};$$

and it also has the property that $\bar{\mathfrak{S}} \cap (R \times \partial\mathfrak{P}) = (\lambda_0, 0)$.

We first state a sequence of Lemmas which will lead to the proof of Theorem 2.1.

Lemma 2.1 *(Comparison). Let $w, \hat{w} \in [C^2(\Omega) \cap C^1(\bar{\Omega})]^{n+m}, w \not\equiv 0, \hat{w}_i \geq 0, w_i \not\equiv 0$ in Ω, for $i = 1, \ldots, n+m$, satisfy*

$$L_i[w_i(x)] = \sum_{j=1}^{n+m} p_{ij}(x) w_j(x), \quad \textit{for} \quad x \in \Omega, \quad i = 1, \ldots, n+m$$

$$w|_{\partial\Omega} = 0, \quad w = \text{col}(w_1, \ldots, w_{n+m}),$$

$$L_i[\hat{w}_i(x)] \geq \sum_{j=1}^{n+m} q_{ij}(x) \hat{w}_j(x), \quad \textit{for} \quad x \in \Omega, i = 1, \ldots, n+m$$

$$\hat{w} = \text{col}(\hat{w}_1, \ldots, \hat{w}_{n+m})$$

where p_{ij} and q_{ij} are bounded functions in Ω. Suppose that

$$q_{ij} \geq p_{ij} \quad in \quad \bar{\Omega} \quad for \quad i,j = 1,\ldots,n+m, \\ and \quad q_{ij}, p_{ij} \geq 0 \quad in \quad \bar{\Omega} \quad for\ all \quad i \neq j,$$

then there exists an integer k, $1 \leq k \leq n+m$ and a real number δ such that

$$\hat{w}_k \equiv \delta w_k, \quad p_{kj} \equiv q_{kj} \quad in \quad \bar{\Omega} \quad for\ all \quad j = 1,\ldots,n+m \\ and \quad \hat{w}_j - \delta w_j \geq 0 \quad for\ all \quad j = 1,\ldots,n+m.$$

This lemma is exactly the same as Lemma 2.1 in [10], the proof will thus be omitted.

Lemma 2.2 *Under hypotheses [H1] and [H2], there exists $(\lambda_0, w^0) \in R \times \mathfrak{F}$, $\lambda_0 > 0$, such that*

$$L[w^0] = \lambda_0 M_0 w^0 \quad in \quad \Omega, \quad w^0 = 0 \quad on \quad \partial\Omega \tag{2.5}$$

with each component $w_i^0 > 0$ in Ω, $\partial w_i^0/\partial\nu < 0$ on $\partial\Omega$ for $i = 1,\ldots,n+m$. Furthermore, $1/\lambda_0$ is a simple eigenvalue of the operator $L^{-1}M_0 : [C^1(\bar{\Omega})]^{n+m} \to [C^1(\bar{\Omega})]^{n+m}$ (that is the eigenfunction corresponding to this eigenvalue is unique up to a multiple). The number $\lambda = \lambda_0$ is the unique positive number so that the problem $w = \lambda L^{-1}M_0 w$ has a nontrivial nonnegative solution for $w \in \mathfrak{P}$.

Lemma 2.3 *Let $G : \mathfrak{E} \to \mathfrak{E}$ be the operator defined by:*

$$G[w] = L^{-1}[F(w) - M_0 w],$$

then $\|G[w]\|_{\mathfrak{E}}/\|w\|_{\mathfrak{E}} \to 0$ as $\|w\|_{\mathfrak{E}} \to 0$.

The details for proving these lemmas will be shown in a forthcoming article.

Outline of Proof of Theorem 2.1: The operator G described above is completely continuous; and the operator $L^{-1}M_0$ described in Lemma 2.2 is compact and positive with respect to $\mathfrak{P}$. Equation (2.3) can be written as:

$$w - \lambda L^{-1}[M_0 w] - \lambda G[w] = 0, \quad \text{for} \quad (\lambda, w) \in R^+ \times \mathfrak{E}.$$

By means of Lemma 2.3 and the existence and uniqueness part of Lemma 2.2 we can apply Theorem 29.2 in [5] to conclude that $(\lambda_0, 0)$ is a bifurcation point for problem (2.3), and the component of $\bar{\mathfrak{S}}$ containing the point $(\lambda_0, 0)$ as described above is unbounded.

Let $(\lambda_i, w_i) \in \mathfrak{S}$, $i = 1, 2, \ldots$ be a sequence tending to a limit point $(\bar{\lambda}, \bar{w})$ in $R \times \partial\mathfrak{P}$. Using the limiting equation for $(\bar{\lambda}, \bar{w})$, we can show by means of the maximum principle and the uniqueness part of Lemma 2.2 that $(\bar{\lambda}, \bar{w}) = (\lambda_0, 0)$. The details will be shown in a later article.

Using additional hypotheses, we can show by means of Theorem 2.1 that the original problem (2.1) or (1.1) has a nonnegative solution.

Theorem 2.2 *Under hypotheses [H1], [H2*], [H3] and [H4], the problem (2.1) has a solution* $w = \mathrm{col}(w_1, \ldots, w_{n+m}) \in \mathfrak{F}$*, such that* $w_i \geq 0$ *in* Ω *for each* i *and* $w \not\equiv 0$ *(i.e.* $w \in \mathfrak{P}\backslash\{0\}$*).*

In order to prove Theorem 2.2, we first show the following two lemmas.

Lemma 2.4 *Under hypotheses [H1], [H2] and [H3], the positive number where bifurcation occurs described in Theorem 2.1 satisfies* $\lambda_0 < 1$*.*

To prove this lemma, we first use [H3] to find a constant vector $\vec{g}$ such that $L^{-1}[M_0\vec{g}\phi] \geq r\phi\vec{g}$ in Ω for some $r > 1$. Then we use Theorem 2.5 in [7] to obtain $\lambda_0 \leq \frac{1}{r} < 1$.

Lemma 2.5 *Under the hypotheses [H1] and [H2], let* $(\bar{\lambda}, \bar{w}) \in \mathfrak{S}$*, where* $\mathfrak{S}$ *is described in Theorem 2.1. Suppose* R_k *are positive constants such that* $0 \leq \bar{w}_{n+k}(x) \leq R_k$ *for all* $x \in \bar{\Omega}$*,* $k = 1, \ldots, m$*. Then for each* $i = 1, \ldots, n$

$$0 \leq w_i(x) \leq \bar{\lambda}(\inf a_i)^{-1} \sum_{k=1}^{m} b_{ik} R_k \quad \textit{for all} \quad x \in \bar{\Omega}.$$

This lemma is proved by the method of upper-lower solution.

Outline of Proof of Theorem 2.2: Let the component of $\bar{\mathfrak{S}}$ containing the point $(\lambda_0, 0)$, described in Theorem 2.1, be denoted by $\mathfrak{S}^+$. Since $\lambda_0 < 1$, by Lemma 2.4, it suffices to show that the set $I := \{\lambda \in R^+ | (\lambda, w) \in \mathfrak{S}^+$ for some $w\}$ is unbounded.

We can show by means of [H4] and [H2*] that if there exists λ such that $(\lambda, w) \in \mathfrak{S}^+$, then $w = \mathrm{col}(w_1, \ldots, w_{n+m})$ must satisfy:

$$0 \leq w_i(x) \leq \hat{N} \quad \text{for all} \quad x \in \bar{\Omega},\ i = n+1, \ldots, n+m. \tag{2.6}$$

for some positive constant $\hat{N}$.

Finally, inequality (2.6), Lemma 2.5 and gradient estimates by means of equation (2.2) imply that $\mathfrak{S}^+$ cannot be unbounded if I is bounded. Consequently, I must be unbounded, and this completes the proof of the Theorem 2.2.

3 Stability of Infected Competing Species

In this section, we will consider the stability of the steady-state solutions found in the last section as a solution of the corresponding parabolic system. It will be seen in Theorem 3.2 that if [H2] and [H4] are strengthened, then the bifurcating steady-states near the bifurcation point are asymptotically stable in time. Before obtaining further results with additional hypotheses, we first deduce a few more consequences of hypotheses [H1] and [H2].

Lemma 3.1 *Under hypotheses [H1] and [H2], the problem*

$$L[w] = \lambda M_0^T w \quad \text{in } \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega, \tag{3.1}$$

has a solution $(\lambda, w) = (\lambda_0, \hat{w}^0), \hat{w}^0 \in \mathfrak{F}$, with each component $\hat{w}_i^0 > 0$ in Ω, $\partial\hat{w}_i^0/\partial\nu < 0$ on $\partial\Omega$ for $i = 1, \dots, n+m$. (Here, λ_0 is exactly the same number as in Lemma 2.2.) Moreover, any solution of (3.1) with $\lambda = \lambda_0$ is a multiple of $\hat{w}^0$.

Proof. The existence of a positive solution and the simplicity of the corresponding eigenvalue is proved in exactly the same way as Lemma 2.2 with the role of B and K_0 interchanged. The fact that λ_0 is exactly the same as in Lemma 2.2 follows exactly the same procedure as in the proof of Lemma 2.4 in [10], and will thus be omitted.

For convenience, we will define two operators L_0 and $L_1 : \mathfrak{E} \to \mathfrak{E}$ as follows:

$$L_0 := I - \lambda_0 L^{-1} M_0 \tag{3.2}$$

$$L_1 := -L^{-1} M_0 \tag{3.3}$$

Lemma 3.2 *Under hypotheses [H1] and [H2], the null space and range of L_0, denoted respectively by $N(L_0)$ and $R(L_0)$ satisfy:*

(i) $N(L_0)$ is one-dimensional, spanned by w^0;

(ii) $\dim[\mathfrak{E}/R(L_0)] = 1$;

(iii) $L_1 w \notin R(L_0)$.

Proof. Part (i) was proved in Lemma 2.2. The remaining parts are proved in the same way as in Lemma 2.5 in [10]. For the proof of part (iii), the positivity property of w^0 and $\hat{w}^0$ is used.

Theorem 3.1 *Assume hypotheses [H1], [H2] and the additional condition that $f_{kj} \in C^2(R)$ for $k = 1, \dots, n$, $j = 1, \dots, m$. Then there exists $\delta > 0$ and a C^1-curve $(\hat{\lambda}(s), \hat{\phi}(s)) : (-\delta, \delta) \to R \times \mathfrak{E}$ with $\hat{\lambda}(0) = \lambda_0$, $\hat{\phi}(0) = 0$ such that in a neighborhood of $(\lambda_0, 0)$, any solution of (2.3) is either of the form $(\lambda, 0)$ or on the curve $(\hat{\lambda}(s), s[w^0 + \hat{\phi}(s)])$ for $|s| < \delta$. Furthermore, the set $\mathfrak{S}^+ \backslash \{(\lambda_0, 0)\}$ is contained in $R^+ \times (Int\ \mathfrak{P})$, where $\mathfrak{S}^+$ is the component of the closure of $\mathfrak{S}$ (described in Theorem 2.1) containing the point $(\lambda_0, 0)$ in $R^+ \times \mathfrak{E}$.*

Outline of Proof. Equation (2.3) can be written as

$$Q(\lambda, w) := L_0[w] + (\lambda - \lambda_0) L_1[w] - \lambda G[w] = 0 \tag{3.4}$$

where the operator $G : \mathfrak{E} \to \mathfrak{E}$ is defined in Lemma 2.3. Under the additional smoothness condition on f_{jk}, we can show that $Q \in C^2(R^+ \times \mathfrak{E}, \mathfrak{E})$ and the Frechet derivative of G is continuous on $\mathfrak{E}$. Moreover, we can readily deduce as in Lemma 2.3 that $L_0 = D_2Q(\lambda_0, 0)$, $L_1 = D_{12}Q(\lambda_0, 0)$, and $G[0] = D_2G[0] = 0$. Hence we can apply the local bifurcation theorem in [4] to obtain the C^1-curve $(\hat{\lambda}(s), \hat{\phi}(s))$ describing the nontrivial solution of (2.3) near $(\lambda_0, 0)$ as stated above. The remaining details will be shown elsewhere.

In the remaining part of this section, we will consider the linearized and asymptotic stability of the positive bifurcating solution described in Theorem 3.1, near $(\lambda_0, 0)$. Applying the bifurcation theory in [4], and the fact that

$$\int_\Omega \hat{w}^0 \cdot \Delta^{-1} w^0 dx \neq 0 \tag{3.5}$$

(note that each component of both $\hat{w}^0$ and w^0 is strictly positive in Ω), we can assert that there exist $\delta_1 \in (0, \delta)$ and two functions:

$$\begin{aligned} (\gamma(\cdot), z(\cdot)) \quad &: \quad (\lambda_0 - \delta_1, \lambda_0 + \delta_1) \to R \times \mathfrak{E}, \\ (\eta(\cdot), h(\cdot)) \quad &: \quad [0, \delta_1) \to R \times \mathfrak{E}, \end{aligned}$$

with $(\gamma(\lambda_0), z(\lambda_0)) = (\eta(0), h(0)) = (0, w^0)$, such that

$$D_2Q(\lambda, 0)z(\lambda) = \gamma(\lambda)(-L)^{-1}(z(\lambda)), \text{ and} \tag{3.6}$$

$$D_2Q(\hat{\lambda}(s), s(w^0 + \hat{\phi}(s))h(s) = \eta(s)(-L)^{-1}(h(s)) \tag{3.7}$$

Here (3.5) and the theory in [4] imply that $\gamma(\lambda)$ and $\eta(s)$ are respectively L^{-1}-simple eigenvalues of $D_2Q(\lambda, 0)$ and $D_2Q(\hat{\lambda}(s),\, s(w^0 + \hat{\phi}(s))$, with eigenfunctions $z(\lambda)$ and $h(s)$. Moreover, the theory in [4] further leads to the following lemmas.

Lemma 3.3 *Assume all the hypotheses in Theorem 3.1. There exists $\rho > 0$ such that for each $s \in [0, \delta_1)$, there is a unique (real) eigenvalue $\eta(s)$ for the linear operator*

$$Q_s^* := -LD_2Q(\hat{\lambda}(s), s(w^0 + \hat{\phi}(s)) : \mathfrak{F} \to [C^\alpha(\bar{\Omega})]^{m+n} \tag{3.8}$$

satisfying $|\eta(s)| < \rho$ with eigenfunction $h(s) \in \mathfrak{F}$. That is,

$$Q_s^* h(s) \equiv -L[h(s)] + \hat{\lambda}(s)F_w[s(w^0 + \hat{\phi}(s))]h(s) = \eta(s)h(s) \tag{3.9}$$

The next few lemmas study the behavior of the eigenvalues $\hat{\lambda}(s)$, $\eta(s)$ for small $s \geq 0$, and $\gamma(\lambda)$ near $\lambda = \lambda_0$.

Lemma 3.4 *Assume all the hypotheses of Theorem 3.1. Suppose further*

*[H2**] $f''_{kj}(0) < 0$ for $k = 1, \ldots, m$, $j = 1, \ldots, n$;*

[H4] $c_{kj} < 0$, for all $k, j = 1, \ldots, m$.*

Then the function $\hat{\lambda}(s)$ satisfies $\hat{\lambda}'(0) > 0$.

Lemma 3.5 *Under all the hypotheses in Theorem 3.1, the function $\gamma(\lambda)$ satisfies $\gamma'(\lambda_0) > 0$.*

Lemma 3.6 *Under the hypotheses of Theorem 3.1, [H2**] and [H4*], there exists* $\delta_2 \in (0, \delta_1)$ *such that* $\eta(s) < 0$ *for all* $s \in (0, \delta_2)$*.*

The proof of the last three lemmas will be shown elsewhere.

Equation (2.2) or (2.3) can be written as

$$-L[w] + \lambda F[w] = 0 \tag{3.10}$$

The linearized eigenvalue problem for (3.10) at the bifurcating solution $w = s(w^0 + \hat{\phi}(s))$ is precisely (3.9). When $\lambda = \hat{\lambda}(0) = \lambda_0$, the eigenvalue problem corresponding to (3.9) becomes

$$-L[h] + \lambda_0 M_0 h = \eta h \qquad\qquad h \in \mathfrak{E} \tag{3.11}$$

where η is the eigenvalue. Under hypotheses [H1] and [H2], Theorem 2.1 asserts that $\eta = 0$ is an eigenvalue for (3.11), with positive eigenfunction. Using this property and the fact that the off diagonal terms of M_0 are all nonnegative, we can proceed to show the following.

Lemma 3.7 *Under hypotheses [H1] and [H2], all eigenvalues in equation (3.11) except* $\eta = 0$ *satisfies* $Re(\eta) < -r$ *for some positive number* r*.*

Lemma 3.4 to 3.6 above essentially shows the eigenvalue $\eta = 0$ corresponding to (3.9) at $s = 0$ moves to the left as s increases. As to the other eigenvalues with $Re(\eta) < -r$ described in Lemma 3.7, one can show by perturbation arguments that they should still be in the left open half plane for $s > 0$ sufficiently small. More precisely, we have the following.

Lemma 3.8 *Under the hypotheses of Theorem 3.1, there exists a number* $\delta^* \in (0, \delta)$ *and a positive function* $\eta(s)$ *for* $s \in (0, \delta^*)$ *such that the real parts of all the numbers in the point spectrum of the linear operator* Q_s^* *are contained in the interval* $(-\infty, -\eta(s))$*, for* $s \in (0, \delta^*)$*. (Here,* δ *is described in Theorem 3.1 and* Q_s^* *is described in (3.8) in Lemma 3.3).*

For each $s \in (0, \delta^*)$, the function $\bar{w}_s := s[w^0 + \hat{\phi}(s)]$ described in Theorem 3.1 can be considered as a steady-state solution of the problem:

$$\begin{aligned} \frac{\partial w}{\partial t}(x,t) + L[w(x,t)] &= \lambda F[w(x,t)] \quad \text{in } \Omega \times (0, \infty) \\ w(x,t) &= 0 \quad x \in \partial\Omega,\ t \geq 0\,. \end{aligned} \tag{3.12}$$

We now consider the time asymptotic stability of this steady-state as a solution of the parabolic system (3.12). In order to obtain a precise statement, we let B_1 and B_2 be Banach spaces as follow:

$$\begin{aligned} B_1 &= \{u : u \in [C(\bar{\Omega})]^{n+m}, u = 0 \text{ on } \partial\Omega\} \text{ and} \\ B_2 &= \{u : u \in [L_p(\Omega)]^{n+m}\} \text{ for } p \text{ large enough such that } N/(2p) < 1. \end{aligned}$$

Let A_1 be the operator L on B_1 with domain $D(A_1) = \{u : u \in [W^{2,p}(\Omega)]^{n+m}$ for all p, $\Delta u \in [C(\bar{\Omega})]^{n+m}$, $u = 0$ and $\Delta u = 0$ on $\partial\Omega\}$; and A_2 be the operator

L on B_2 with domain $D(A_2) = \{u \in B_2 : u \in [W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)]^{n+m}\}$. For $w = \mathrm{col}(w_1, \ldots, w_{n+m})$, we consider the following nonlinear initial-boundary value problem for each $i = 1, 2$ corresponding to (3.12):

$$\frac{dw}{dt} + A_i w(t) = \lambda F(w(t)), \quad w(0) = w_0 \quad \text{for} \quad t \in (0, T] \tag{3.13}$$

A solution of (3.13) in B_i is a function $w \in C([0,T], B_i) \cap C^1((0,T], B_i)$, with $w(0) = w_0$, $w(t) \in D(A_i)$ for all $t \in (0,T]$; and $w(t)$ satisfies (3.13) for $t \in (0,T]$. The operator A_2 is an infinitesimal generator of an analytic semigroup, say $M(t)$, on B_2 for $t \geq 0$. It is well known that for $\alpha > 0$

$$(-A_2)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \tau^{\alpha-1} M(\tau) d\tau$$

defines a bounded linear operator on B_2. Moreover, $[(-A_2)^{-\alpha}]^{-1} = (-A_2)^{\alpha}$ is a closed operator on B_2 with dense domain $D((-A_2)^{\alpha}) = (-A_2)^{-\alpha}(B_2)$. We denote by X_α the Banach space $(D(-A_2)^{\alpha}, \| \quad \|_\alpha)$ where $\|w\|_\alpha = \|(-A_2)^{\alpha} w\|_{L^p}$ for all $w \in D((-A_2)^{\alpha})$.

Using Lemma 3.8, we can use semigroup theory and apply the stability Theorem 5.1.1 in [6] for sectorial operators to obtain the following asymptotic stability theorem for the steady-state solution $\bar{w}_s$. The details of the proof are omitted here.

Theorem 3.2 *Assume all the hypotheses of Theorem 3.1, [H2**] and [H4*]. For each fixed* $s \in (0, \delta^*)$, *let* $\lambda = \hat{\lambda}(s)$, $\bar{w} = s[w^0 + \hat{\phi}(s)]$. *Then for each* $i = 1, 2$, *there exists* $\rho > 0$, $\beta > 0$ *and* $M > 1$ *such that equation (3.13) has a unique solution in* B_i *for all* $t > 0$ *if* $w_0 \in B_1$ *and* $\|w_0 - \bar{w}\|_\infty \leq \rho/(2M)$ *for* $i = 1$, *(or* $w_0 \in X_\alpha$ *and* $\|w_0 - \bar{w}\|_\alpha \leq \rho/(2M)$ *for* $i = 2$.*) Moreover, the solution satisfies*

$$\|w(t) - \bar{w}\|_\infty \leq 2Me^{-\beta t}\|w_0 - \bar{w}\|_\infty \quad \textit{for all } t \geq 0,\ i = 1, \textit{ or} \tag{3.14}$$

$$\|w(t) - \bar{w}\|_\alpha \leq 2Me^{-\beta t}\|w_0 - \bar{w}\|_\alpha \quad \textit{for all } t \geq 0,\ i = 2. \tag{3.15}$$

(For solutions in B_2, *we assume* $\alpha \in (N/(2p), 1)$ *for the space* X_α.*)*

4 Bifurcation of Infected Species with No Interactions

In this section, we consider system (1.1), under the special situation when all $c_{kj} = 0$, $k, j = 1, \ldots, m$; that is, we consider the problem:

$$\begin{gathered} -\Delta u_i + a_i(x) u_i = \textstyle\sum_{j=1}^m b_{ij} v_j \quad \text{for } x \in \Omega,\ i = 1, \ldots, n \\ -\Delta v_k + \tilde{a}_k(x) v_k = \textstyle\sum_{j=1}^n f_{kj}(u_j) \quad \text{for } x \in \Omega,\ k = 1, \ldots, m \\ u_i = v_k = 0 \quad \text{for } x \in \partial\Omega,\ i = 1, \ldots, n,\ k = 1, \ldots, m\,. \end{gathered} \tag{4.1}$$

In other words, the infected species v_k will not interact among themselves. This situation is a direct generalization of the theory in [1]. Under additional assumptions on $f_{kj}(u_j)$ for large u_j (see [H5] below), Theorem 4.1 shows that problem (4.1) has a positive solution. Letting $w = \mathrm{col}(w_1, \ldots, w_{n+m}) = \mathrm{col}(u_1, \ldots, u_n, v_1, \ldots, v_m)$, system (4.1) can be written as:

$$L[w] = \tilde{F}[w], \quad w \in \mathfrak{E}, \tag{4.2}$$

where $\tilde{F}$ is the same as F described in (1.3) with the special restriction $c_{kj} = 0$, for all $k, j = 1, \ldots, m$. For convenience, we define

$$\hat{f}_{ij}(\eta) := \begin{cases} f_{ij}(\eta)/\eta & if \quad \eta \neq 0 \\ \\ f'_{ij}(0) & if \quad \eta = 0 \end{cases} \qquad \text{for } i = 1, \ldots, m, \ j = 1, \ldots, n. \tag{4.3}$$

Also define the $m \times n$ matrix

$$\left[\hat{f}_{ij}(\eta_{ij})\right]_{ij=1}^{m,n} := \begin{bmatrix} \hat{f}_{11}(\eta_{11}) & \cdots & \hat{f}_{1n}(\eta_{1n}) \\ \vdots & & \vdots \\ \hat{f}_{m1}(\eta_{m1}) & \cdots & \hat{f}_{mn}(\eta_{mn}) \end{bmatrix} \tag{4.4}$$

where η_{ij} are real numbers for $i = 1, \ldots, m, \ j = 1, \ldots, n$. We will use the following hypothesis:

[H5] There exist a real number $\eta_0 > 0$, and a constant vector $\vec{q} = \mathrm{col}(q_1, \ldots, q_n)$, with $q_i > 0$, $i = 1, \ldots, n$, such that:

$$\vec{q}^T B \left[\hat{f}_{ij}(\eta_{ij})\right]_{i,j=1}^{m,n} < (\lambda_1 + a^{**})^2 \vec{q}^T \quad \text{for all} \quad \eta_{ij} \geq \eta_0$$

Under hypothesis [H5], one can always choose a number ρ_1 with $0 < \rho_1 < (\lambda_1 + a^{**})$ such that:

$$\vec{q}^T B \left[\hat{f}_{ij}(\eta_{ij})\right]_{i,j=1}^{m,n} < \rho_1^2 \vec{q}^T < (\lambda_1 + a^{**})^2 \vec{q}^T \quad \text{for all} \quad \eta_{ij} \geq \eta_0 \tag{4.5}$$

The following theorem is the main result of this section.

Theorem 4.1 *Under hypotheses [H1], [H2*], [H3] and [H5], the problem (4.2) (alternatively, problem (4.1) with $w = \mathrm{col}(u_1, \ldots, u_n, v_1, \ldots, v_m)$) has a solution $w = \mathrm{col}(w_1, \ldots, w_{n+m}) \in \mathfrak{F}$, such that $w_i \geq 0$ in $\bar{\Omega}$ for each i and $w \not\equiv 0$ (i.e. $w \in \mathfrak{P} \backslash \{0\}$).*

In this entire section, F is considered with the special restriction $c_{kj} = 0$, and [H4] is not assumed. We are thus led to the problem:

$$L[w] = \lambda \tilde{F}[w] \quad w \in \mathfrak{E} \tag{4.6}$$

with $\tilde{F}$ as described above. Since $\tilde{F}$ is a special case of F, Therem 2.1 applies. Under the assumptions of Theorem 4.1, let $\mathfrak{S}$ be as defined in Theorem 2.1, and $\mathfrak{S}^+$ be the component of $\bar{\mathfrak{S}}$ containing the point $(\lambda_0, 0)$. Recall that $\mathfrak{S}^+$ is proved to be unbounded in Theorem 2.1. The following Lemma will be needed in the proof of Theorem 4.1.

Lemma 4.1 *Assume all the hypotheses of Theorem 4.1. Suppose* $\{(\tilde{\lambda}_r, \tilde{w}^r)\}$, $r = 1, 2, \ldots$ *is a sequence in* $\mathfrak{S}^+$ *with the property:* $\tilde{\lambda}_r \to \hat{\lambda}$, $0 < \hat{\lambda} < \infty$, *and* $\|\tilde{w}^r\|_{\mathfrak{E}} \to \infty$, *as* $r \to \infty$. *Then there exists a subsequence* $\{(\tilde{\lambda}_{r(j)}, \tilde{w}^{r(j)})\}$ *such that the first* n *components of* $\tilde{w}^{r(j)}$ *tend to* $+\infty$ *uniformly in compact subsets of* Ω, *as* $r(j) \to \infty$.

Outline of Proof of Theorem 4.1: Recall that Lemma 2.4 implies that $\lambda_0 < 1$. To prove this Theorem, it suffices to show the fact that if there exists a sequence in $\mathfrak{S}^+$ with property as described in Lemma 4.1, then we must have $\hat{\lambda} > 1$. To show this fact, we use hypothesis [H5] concerning the functions $f_{kj}(u_j)$ for large u_j. It will lead to $\hat{\lambda} \geq (\lambda_1 + a^{**})/\rho_1 > 1$. This leads to the existence of $(\tilde{\lambda}, \tilde{w})$ in $\mathfrak{S}^+$ with $\tilde{\lambda} = 1$; that is, we obtain a nontrivial, nonnegative solution of Problem (4.1).

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