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A Second look at the first result of Landesman-Lazer type *

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Dedicated to Paul Frederickson

Abstract

We discuss some results concerning periodic and almot periodic solutions of ordinary differential equations which are precursors of a result on weak solutions of a semilinear elliptic boundary due to E. M. Landesman and the author. It is observed that in the earliest of these, if one looks for periodic solutions instead of almost peridic solutions, then the conditions can be relaxed.

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that the limits

$$g(\pm\infty) = \lim_{s\to\pm\infty} g(s)$$

exist and are finite, and that $\forall \xi \in \mathbb{R}$, $g(-\infty) < g(\xi) < g(\infty)$. In [3], E.M. Landesman and the author considered the boundary-value problem

$$\begin{aligned} \Delta u + \lambda_k u + g(u) = h(x) \quad \text{in } D \\ u\big|_{\partial D} = 0\,, \end{aligned} \tag{1.1}$$

where D is a bounded domain in $\mathbb{R}^n$, $h \in L^2(D)$, and λ_k is a simple eigenvalue of $-\Delta$. It was shown that if φ_k is an eigenfunction corresponding to λ_k, $D^+ = \{x \in D : \varphi_k(x) > 0\}$, and $D^- = \{x \in D : \varphi_k(x) < 0\}$, then the condition

$$\begin{aligned} g(-\infty)\int_{D^+}\varphi_k dx + g(\infty)\int_{D^-}\varphi_k dx &< \int_D h\varphi_k dx \\ &< g(\infty)\int_{D^+}\varphi_k dx + g(-\infty)\int_{D^-}\varphi_k dx \end{aligned}$$

is both necessary and sufficient for the existence of a weak solution of (1.1).

Only a short time before D.E. Leach and the author [5] considered the ordinary differential equation

$$u'' + n^2 u + g(u) = e(t) = e(t + 2\pi)\,, \tag{1.2}$$

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where $n > 0$ is an integer and g satisfies the same conditions as before, it was shown that if

$$A = \int_0^{2\pi} e(t)\cos(nt)\,dt \quad \text{and} \quad B = \int_0^{2\pi} e(t)\sin(nt)\,dt\,,$$

then the inequality $2(g(\infty) - g(-\infty)) > \sqrt{A^2 + B^2}$ is both necessary and sufficient for the existence of a 2π-periodic solution of (1.2).

It is natural to ask what happens if g satisfies the same conditions and $n = 0$ in (1.2). If we consider more generally

$$u'' + cu' + g(u) = e(t) = e(t + 2\pi)\,, \tag{1.3}$$

where $c \in \mathbb{R}$, and we assume that there is a 2π-periodic solution $\hat{u}(t)$, then integration from 0 to 2π shows that

$$\int_0^{2\pi} (e(t) - g(\hat{u}(t)))\,dt = 0\,.$$

Therefore, if

$$e_0 \equiv \frac{1}{2\pi}\int_0^{2\pi} e(t)\,dt\,,$$

then a necessary condition for the existence of a 2π-periodic solution of (1.3) is

$$g(-\infty) < e_0 < g(\infty)\,. \tag{1.4}$$

That this condition is also sufficient can be derived from work of the author [4] which predates [5]. Let c be constant and consider the differential equation

$$u'' + cu' + h(u) = p(t) = p(t + 2\pi)\,. \tag{1.5}$$

It was shown in [4], that if p and h are continuous, there exists ξ_0 such that $h(\xi)\xi \geq 0$ for $|\xi| \geq \xi_0$, $h(\xi)/\xi \to 0$ as $|\xi| \to \infty$, and

$$\frac{1}{2\pi}\int_0^{2\pi} p(t)\,dt = 0\,,$$

then (1.5) has a 2π-periodic solution. If g is continuous, $g(\infty)$ and $g(-\infty)$ are finite, and (1.4) holds, then we can write (1.3) in the form (1.5) where $h(\xi) = g(\xi) - e_0$, $p(t) = e(t) - e_0$, and the assumptions of [4] will hold so (1.3) will have a 2π-periodic solution.

It was sometime after the publication of [4] and [5] that the author realized that the condition (1.4) was both necessary and sufficient for the existence of a 2π-periodic solution of (1.3) for a restricted type of functions g. The first to see a Landesman-Lazer type condition was P.O. Frederickson. This condition appeared in [2], written jointly by Frederickson and the author, which dealt with almost periodic solutions of the two differential equations

$$x'' + F(x') + x = E(t) \tag{1.6}$$

and

$$x'' + f(x)x' + x = e(t) \tag{1.7}$$

The two differential equations are related in that if F and E are C^1, $F' = f$, and $E' = e$, then the derivative of a solution of (1.6) is a solution of (1.7). We shall limit our discussion mainly to (1.7) and let F denote an antiderivative of f, which will be assumed to be continuous, when referring to this differential equation.

Under the assumption that f is strictly positive except at isolated points and that

$$F(\infty) - F(-\infty) = \infty$$

Levinson [6] showed that for any continuous periodic $e(t)$ with least period $T > 0$ there exists a unique T-periodic solution of (1.7) which is globally asymptotically stable.

In [7] Reissig considered (1.6) under the assumptions that F is a continuous, strictly increasing function, $E(t)$ is a continuous periodic function with least period $T > 0$, and that for arbitrary x_0, y_0, and t_0, there is a unique solution of (1.6) with $x(t_0) = x_0$, $x'(t_0) = y_0$. Reissig showed that the condition

$$F(\infty) - F(-\infty) > \max E(t) - \min E(t)$$

implies the existence and uniqueness of a T-periodic solution of (1.6) and that this solution is globally asymptotically stable.

In [2] the assumption that $e(t)$ is almost periodic implies the existence of

$$M[e(t)\exp(-it)] = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} e(t)\exp(-it)\,dt$$

uniformly with respect to $t_0 \in \mathbb{R}$. It was shown in [2] that if f is continuous and strictly positive except at isolated points, so that its antiderivative F is strictly increasing, then

$$F(\infty) - F(-\infty) > \pi\big|M[e(t)\exp(-it)]\big| \tag{1.8}$$

is both necessary and sufficient for the existence of an almost periodic solution of (1.7). Moreover, if this condition holds, there is a unique almost periodic solution which is globally asymptotically stable.

In [2] the assumption that F is strictly increasing was used in an essential way. The proof was based on a result of Amerio [1]. To use this result it was necessary to show that if whenever $\{h_m\}_1^\infty$ is a sequence of real numbers such that

$$e^*(t) = \lim_{m\to\infty} e(t + h_m)$$

exists uniformly with respect to $t \in \mathbb{R}$, then the differential equation

$$x'' + f(x)x' + x = e^*(t)$$

has a unique solution bounded on $\mathbb{R}$. Since $e^*(t)$ is almost periodic and

$$\big|M[e^*(t)\exp(-it)]\big| = \big|M[e(t)\exp(-it)]\big|,$$

verification of Amerio's condition for (1.7) amounts to showing that condition (1.8) and the other assumptions on f imply that (1.7) itself has a unique solution bounded on $\mathbb{R}$. Examination of the arguments given in [2] shows that it is enough to assume the existence of the limits $F(\infty)$ and $F(-\infty)$ and the condition (1.8) in order to ensure the existence of at least one solution of (1.7) which is bounded on $\mathbb{R}$. It is the proof of uniqueness which depends on the strict monotonicity of F. This was accomplished by noting that if $x_1(t)$ and $x_2(t)$ are two solutions of (1.7) and we set

$$\begin{aligned} x_1'(t) &= y_1(t) - F(x_1(t)),\\ x_2'(t) &= y_2(t) - F(x_2(t)),\\ d(t) &\equiv \sqrt{(x_1(t)-x_2(t))^2 + (y_1(t)-y_2(t))^2}, \end{aligned}$$

then

$$d(t)d'(t) \equiv -(x_1(t)-x_2(t))(F(x_1(t)) - F(x_2(t))) \le 0\,.$$

Actually, for reasons of exposition, we have simplified what was done in [2]. A more complicated system which contains (1.6) and (1.7) as special cases was considered but the same type of reasoning described above was used.

What we would like to point out is in the case that $e(t)$ is a 2π-periodic function, the assumptions that F is strictly increasing can be replaced by the assumption that the limits $F(\infty)$ and $F(-\infty)$ exist and that $\forall\xi\in\mathbb{R}$,

$$F(-\infty) < F(\xi) < F(\infty).$$

With these assumptions (1.8) is still a necessary and sufficient condition for the existence of a 2π-periodic solution. This has probably been observed before by someone familiar with both [2] and [5].

A simple computation shows that if $e(t)$ is a 2π-periodic function, then

$$\big|M[e(t)\exp(-it)]\big| = \frac{1}{2\pi}\sqrt{A^2+B^2}\,,$$

where

$$A = \int_0^{2\pi} e(t)\cos t\,dt, \quad \text{and} \quad B = \int_0^{2\pi} e(t)\sin t\,dt.$$

As before, we assume that f is continuous.

Addendum to the Frederickson-Lazer Theorem: If $e(t)$ is 2π-periodic, $F(\infty)$ and $F(-\infty)$ are finite and $\forall\xi\in\mathbb{R}$, $F(-\infty) < F(\xi) < F(\infty)$, then the condition

$$2(F(\infty) - F(-\infty)) > \sqrt{A^2+B^2}$$

is necessary and sufficient for the existence of a 2π-periodic solution of (1.7).

We sketch the proof of the sufficiency using the same reasoning as in [2] and the Brouwer fixed-point theorem.

If $x(t)$ and $y(t)$ satisfy

$$x' = y - F(x), \quad y' = -x + e(t) \tag{1.9}$$

then $x(t)$ will be a solution of (1.7) so we look for a 2π-periodic solution of this system.

If $r(t) = \sqrt{x(t)^2 + y(t)^2}$ and $r(t) \neq 0$, then

$$r'(t) = -x(t)F(x(t))/r(t) + y(t)e(t)/r(t),$$

so, by the boundedness of F, there exists a constant $M > 0$ such that $r(t) \neq 0$ implies $|r'(t)| \leq M$. From this we infer the existence of $r_0 > 0$ such that $x(0)^2 + y(0)^2 \geq r_0^2$, implies $x(t)^2 + y(t)^2 > 0$ for $0 \leq t \leq 2\pi$.

If $x(0)^2 + y(0)^2 \geq r_0^2$ and $t \in [0, 2\pi]$ we can set $x(t) = r(t)\sin\theta(t)$, $y(t) = r(t)\cos\theta(t)$, where

$$r'(t) = -F(r(t)\sin\theta(t))\sin\theta(t) + e(t)\cos\theta(t) \tag{1.10}$$

$$\theta'(t) = 1 - \frac{F(r(t)\sin\theta(\theta)\cos\theta(t)}{r(t)} - \frac{e(t)\sin\theta(t)}{r(t)}\,. \tag{1.11}$$

If for $c \geq r_0$ and $\varphi \in \mathbb{R}$, $r(t, c, \varphi)$ and $\theta(t, c, \varphi)$ denote the components of the solution of the system (1.10)-(1.11) such that $r(0, c, \varphi) = c$, $\theta(0, c, \varphi) = \varphi$, then

$$r(t, c, \varphi) = c + O(1) \quad \text{as } c \to \infty$$

uniformly with respect to $t \in [0, 2\pi]$ and $\varphi \in \mathbb{R}$. Therefore, integration of (1.11) yields $\theta(t, c, \varphi) = t + \varphi + 0(1/c)$ as $c \to \infty$ uniformly with respect to $t \in [0, 2\pi]$ and $\varphi \in \mathbb{R}$.

Since

$$\begin{aligned} &r(2\pi, c, \varphi) - c \\ &= \int_0^{2\pi} -F(r(t, c, \varphi)\sin\theta(t, c, \varphi))\sin\theta(t, c, \varphi) + e(t)\cos\theta(t, c, \varphi)\, dt \end{aligned}$$

the asymptotic estimates for $r(t, c, \varphi)$ and $\theta(t, c, \varphi)$ together with the assumptions on F imply that

$$r(2\pi, c, \varphi) - c \to \int_0^{2\pi} e(t)\cos(t + \varphi)dt - 2[F(\infty) - F(-\infty)]$$

as $c \to \infty$ uniformly with respect to $\varphi \in \mathbb{R}$. Since

$$\int_0^{2\pi} e(t)\cos(t + \varphi)dt = A\cos\varphi - B\sin\varphi \leq \sqrt{A^2 + B^2},$$

our basic assumption implies the existence of $c_* \geq r_0$ such that if $c \geq c_*$, then $r(2\pi, c, \varphi) < c = r(0, c, \varphi)$ for all $\varphi \in \mathbb{R}$.

Let $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ denote the components of the solution of the system which satisfy $x(0,x_0,y_0)=x_0$, $y(0,x_0,y_0)=y_0$. Since F is C^1, the mapping $\mathbb{R}^2\to\mathbb{R}^2$ given by $(x_0,y_0)\to(x(2\pi,x_0,y_0),y(2\pi,x_0,y_0))$ is continuous. Therefore, there exists $\hat{c}\geq c_*$ such that $x_0^2+y_0^2\leq c_*^2$ implies that $x(2\pi,x_0,y_0)^2+y(2\pi,x_0,y_0)^2\leq\hat{c}^2$. Since, as shown above, $x_0^2+y_0^2\geq c_*^2$ implies $x(2\pi,x_0,y_0)^2+y(2\pi,x_0,y_0)^2<x_0^2+y_0^2$, the closed disk $D=\{(x_0,y_0)|x_0^2+y_0^2\leq\hat{c}^2\}$ is mapped into itself. Letting $(\hat{x}_0,\hat{y}_0)$ denote a fixed point of the mapping, it follows that $\mathrm{col}(x(t,\hat{x}_0,\hat{y}_0),y(t,\hat{x}_0,\hat{y}_0))$ is a 2π-periodic solution of (1.9) so $x(t,\hat{x}_0,\hat{y}_0)$ is a 2π-periodic solution of (1.7).

Necessity can be proved as in [2] and [5]: If φ is chosen so that

$$\cos\varphi=B/\sqrt{A^2+B^2}\,,\quad \sin\varphi=A/\sqrt{A^2+B^2}\,,$$

then if $u(t)$ is a 2π-periodic solution of (1.7) and $v(t)=\sin(t+\varphi)$, several integrations by parts give

$$\begin{aligned}\int_0^{2\pi} e(t)v(t)+F(u(t))v'(t)\,dt &= \int_0^{2\pi} e(t)v(t)-f(u(t)u'(t)v(t)\,dt\\ &= \int_0^{2\pi}[u''(t)+u(t)]v(t)\,dt\\ &= \int_0^{2\pi}[v''(t)+v(t)]u(t)\,dt=0\,.\end{aligned}$$

Since $\forall\xi\in\mathbb{R}$, $F(-\infty)<F(\xi)<F(\infty)$, if P and N denote the subintervals of $[0,2\pi]$ on which $v'>0$ and $v'<0$ respectively, then

$$\begin{aligned}\sqrt{A^2+B^2} &= \int_0^{2\pi} e(t)v(t)\,dt\\ &= -\int_0^{2\pi}F(u(t))v'(t)\,dt\\ &< -F(\infty)\int_N v'(t)\,dt-F(-\infty)\int_P v'(t)\,dt\\ &= 2(F(\infty)-F(-\infty)).\end{aligned}$$

If $E(t)$ is 2π-periodic and continuous and F is a locally Lipschitzian function of the type considered above, then a necessary and sufficient condition for (1.6) to have a 2π-periodic solution is

$$2(F(\infty)-F(-\infty))>\sqrt{C^2+D^2}$$

where

$$C=\int_0^{2\pi}E(t)\cos t\,dt,\quad D=\int_0^{2\pi}E(t)\sin t\,dt$$

The proof follows from considering the system

$$x'=y,\quad y'=-F(y)-x+e(t)\,.$$

After introducing polar coordinates, one can obtain asymptotic estimates and show that the period map maps a closed disk into itself.

That Frederickson was the major contributor to [2] was acknowledged in [5]. After a third of a century his contribution to the development to what are called Landesman-Lazer type results needs to be acknowledged again.

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