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Infinitely many solutions at a resonance *

Philip Korman & Yi Li

Dedicated to Alan Lazer
on his 60th birthday

Abstract

We use bifurcation theory to show the existence of infinitely many solutions at the first eigenvalue for a class of Dirichlet problems in one dimension.

It has been observed that complexity of the solution curve for the boundary value problem

$$u'' + \lambda f(u) = 0 \quad \text{for } 0 < x < L, \quad u(0) = u(L) = 0 \tag{1}$$

seems to mirror that of the nonlinearity $f(u)$, see e.g. P. Korman, Y. Li and T. Ouyang [6]. Namely, if $f(u)$ is convex ($f = e^u$ is a prominent example), $f(u)$ has at most one critical point, and correspondingly the solution curve of (1) admits only one turn. Similarly for many functions with two inflection points (like cubics, or modified Gelfand's equation) one can show that solution curve admits exactly two turns, see [6] and also P. Korman and Y. Li [5]. It is natural to ask: will solution curve admit infinitely many turns if $f(u)$ changes concavity infinitely many times. It has been known for a while that this indeed might happen, see e.g. R. Schaaf and K. Schmitt [8], D. Costa, H. Jeggle, R. Schaaf and K. Schmitt [2], H. Kielhofer and S. Maier [4]. Recent contributions include Y. Cheng [1] and S.-H. Wang [10]. This note was stimulated by an example in Y. Cheng [1], who has shown that for $f(u) = u + \sin\sqrt{u}$ the problem (1) admits infinitely many solutions at $\lambda = \lambda_1$, where λ_1 denotes as usual the principal eigenvalue of $-u''$ on $(0, L)$, $\lambda_1 = \frac{\pi^2}{L^2}$. The proof in [1] used the quadrature method. In this note we use bifurcation theory to obtain a similar result. The bifurcation approach gives a clear understanding of the solution curve, and opens a way to considering higher dimensions. The papers [8] and [2] also used bifurcation approach, and they considered more general equations, as well as the PDE case in two dimensions. Our approach is different, we work with a smooth curve of solutions (rather than a continuum of solutions bifurcating from

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infinity), and we obtain an extension to more general nonlinearities. Moreover, we can estimate the number of turns the solution curve makes, until it reaches any level $u_{\max} = c$.

We begin with a simple example.

Theorem 1 *The problem*

$$u'' + \lambda(u + \sin u) = 0 \quad \text{for} \quad 0 < x < L, \quad u(0) = u(L) = 0 \tag{2}$$

has a C^1 curve of positive solutions bifurcating (forward) off the trivial solution at $\lambda = \frac{\lambda_1}{2}$. This curve crosses infinitely many times the line $\lambda = \lambda_1$, and tends to infinity at $\lambda = \lambda_1$. This curve exhausts the set of all positive solutions of (2). There are infinitely many positive solutions at $\lambda = \lambda_1$, while at any other λ the number of positive solutions is at most finite.

The proof will use the following lemma, which shows that there are no positive solutions for $\lambda > 0$ small, and for λ large. We did not attempt to get the optimal bounds.

Lemma 1 *If the problem (2) has a positive solution, then*

$$\frac{\lambda_1}{2} < \lambda < \frac{\pi}{\pi - 1}\lambda_1. \tag{3}$$

Proof: Set $f(u) = u + \sin u$. Observe that $0 < f(u) < 2u$ for all $u > 0$. Multiplying the equation (2) by u, integrating by parts, and using the Poincare inequality

$$2\lambda \int_0^L u^2\,dx > \lambda \int_0^L f(u)u\,dx = \int_0^L u'^2\,dx \geq \lambda_1 \int_0^L u^2\,dx,$$

from which the left inequality in (3) follows.

On the other hand, multiplying the equation (3) by $\phi_1 = \sin\frac{\pi}{L}x$, and integrating over $(0, L)$, we obtain (with $\lambda_1 = \frac{\pi^2}{L^2}$)

$$\int_0^L (-\lambda_1 u + \lambda u + \lambda \sin u)\,\phi_1\,dx = 0. \tag{4}$$

Since $\phi_1 > 0$, we obtain a contradiction in (4), provided that

$$g(u) \equiv \frac{(\lambda - \lambda_1)}{\lambda}u + \sin u > 0$$

for all $u > 0$. This will certainly happen if $\lambda > \lambda_1$, and $g(\pi) \geq 1$, which leads to the right inequality in (3).

Proof of the Theorem 1. It is well-known that there is a curve of positive solutions of (2) bifurcating off the trivial one (to the right) at $\frac{\lambda_1}{2}$. Similarly there a curve of positive solutions bifurcating from infinity at $\lambda = \lambda_1$, see e.g. p. 673 in E. Zeidler [11]. We claim that both branches link up, giving a unique solution curve. Indeed, let us start with the branch bifurcating from infinity. This branch extends globally, since at each of its points either the implicit function theorem or a bifurcation theorem of Crandall-Rabinowitz applies, see [6] for details. By Lemma 1 this branch is constrained to a strip $\frac{\lambda_1}{2} < \lambda < \frac{\pi}{\pi-1}\lambda_1$, and so it has to go to zero at λ_1. By uniqueness of the bifurcating solution, this branch has to link up with the lower one.

We show next that the solution curve changes its direction infinitely many times. Define $F(u) = \int_0^u f(t)\,dt$, $G(u) = uf(u) - 2F(u)$. Notice that $G'(u) = uf'(u) - f(u)$. Differentiating the problem (1) with respect to λ, we get

$$u_\lambda'' + \lambda f'(u)u_\lambda + f(u) = 0 \quad \text{for } 0 < x < L, \quad u_\lambda(0) = u_\lambda(L) = 0. \tag{5}$$

Multiplying the equation (1) by u_λ, and (5) by u, then subtracting and integrating, we obtain

$$-\lambda \frac{d}{d\lambda} \int_0^L G(u)\,dx = \int_0^L uf(u)\,dx. \tag{6}$$

Since the integral on the right in (6) is positive, it follows that $\int_0^L G(u(x))\,dx$ is monotone in λ. If we can show that this integral is not monotone as the solution curve goes to infinity, it will follow that the solution curve is not monotone in λ, i.e. it has to change its direction infinitely many times. Let $s = u(\frac{L}{2})$. Clearly, $u(\frac{L}{2}) \to \infty$ as $\lambda \to \lambda_1$, since $u(x)$ achieves its maximum at the midpoint of the interval $(0, L)$. Define $\phi(x)$ by $u(x) = s\phi(x)$, and set

$$I(s) \equiv \int_0^L G(u(x))\,dx = \int_0^L G(s\phi(x))\,dx. \tag{7}$$

To show that $I(s)$ is not monotone in s it suffices to show that $I'(s)$ changes sign infinitely many times. Defining $J(s) = \int_0^L \sin(s\phi(x))\phi(x)\,dx$, we express

$$I'(s) = \int_0^L \left[s\phi(x)\cos(s\phi(x)) - \sin(s\phi(x))\right]\phi(x)\,dx = sJ'(s) - J(s). \tag{8}$$

We claim that the function $J(s)$ changes sign infinitely many times. Indeed, performing a change of variables by letting $u = s\phi(x)$, we have

$$J(s) = 2\int_0^{\frac{L}{2}} \sin(s\phi(x))\phi(x)\,dx = \frac{2}{s}\int_0^{u(\frac{L}{2})} \sin u \frac{dx}{du} u\,du. \tag{9}$$

Here $\frac{dx}{du}$ is the derivative of the inverse function to $u(x)$. Since $\frac{dx}{du}u$ is an increasing function, it is easy to see that

$$J(s) \begin{cases} > 0, & \text{if } u(\frac{L}{2}) = n\pi, \text{ and } n \text{ is odd} \\ < 0 & \text{if } u(\frac{L}{2}) = n\pi, \text{ and } n \text{ is even.} \end{cases}$$

Hence $J(s)$ changes sign infinitely many times. It follows that $J(s)$ has infinitely many points of negative minimum, where by (8) $I'(s)$ is positive, and infinitely many points of positive maximum where $I'(s)$ is negative. It follows that $I'(s)$ changes sign infinitely many times, and so $I(s)$ is not monotone, and hence the solution curve changes its direction infinitely many times.

It remains to show that the solution curve intersects the line $\lambda = \lambda_1$ infinitely many times. From (4) we conclude

$$(\lambda_1 - \lambda) \int_0^L u\phi_1 \, dx = \lambda \int_0^L \sin u \, \phi_1 \, dx = \lambda \int_0^L \sin s\phi \, \phi_1 \, dx. \tag{10}$$

It follows from a standard analysis of bifurcation from infinity that $\phi(x) = \phi_1(x) + o(1)$ for s large. We then have

$$\int_0^L \sin s\phi \, \phi_1 \, dx = J(s) + o(1) \int_0^L \sin(s\phi) \, dx. \tag{11}$$

We now recall an asymptotic formula from sec. 40 in Y.V. Sidorov et al [9] (see also Corollary 1 in sec. 39).

Lemma 2 *Assume that the functions $f(x)$ and $g(x)$ are infinitely differentiable on the interval $[0,1]$, and assume that*

$$g(x) < g(0) \quad \text{for all } x \in (0,1], \quad \text{and} \quad g'(0) = 0, \quad g''(0) < 0.$$

Then as $s \to \infty$

$$\int_0^1 f(x) e^{isg(x)} \, dx = \frac{1}{2} e^{i\left(sg(0) - \frac{\pi}{4}\right)} \sqrt{\frac{2\pi}{s|g''(0)|}} f(0) + O(\frac{1}{s}).$$

Using elliptic regularity, we notice that

$$|\phi(x) - \phi_1(x)|_{C^2[0,1]} = o(1) \quad \text{as } s \to \infty,$$

which implies in particular that $\phi''(\frac{L}{2}) = -\frac{\pi^2}{L^2} + o(1)$. Applying Lemma 2 to $J(s) = 2\Im \int_0^{\frac{L}{2}} e^{is\phi(x)} \phi(x) \, dx = L \Im \int_0^1 e^{is\phi(\frac{L}{2}\xi)} \phi(\frac{L}{2}\xi) \, d\xi$, we conclude that (observe that $\phi(\frac{L}{2}) = 1$, $\phi'(\frac{L}{2}) = 0$)

$$J(s) \sim \frac{c_0}{\sqrt{s}} \sin\left(s - \frac{\pi}{4}\right),$$

for some positive constant c_0. Similarly

$$\int_0^L \sin(s\phi) \, dx \sim \frac{c_0}{\sqrt{s}} \sin\left(s - \frac{\pi}{4}\right).$$

It follows from (11) that the integral on the left changes sign infinitely many times, and then from (10) we conclude that the solution curve crosses the line $\lambda = \lambda_1$ infinitely many times, completing the proof.

We now consider more general equations. Recall that we denote $F(u) = \int_0^u f(t)\,dt$.

We say that a function $f(u)$ satisfies Schaaf-Schmitt condition if there exist two monotone sequences $\{x_n\}$ and $\{y_n\}$ with $x_n \to \infty$, $y_n \to \infty$ such that

$$\begin{aligned} &F(x_n) - F(x) \geq 0 \quad \text{for all } 0 \leq x < x_n \\ &F(y_n) - F(x) \leq 0 \quad \text{for all } 0 \leq x < y_n, \end{aligned} \tag{12}$$

both inequalities being strict on a set of positive measure. Setting $x = 0$, we observe that (12) implies in particular

$$F(x_n) \geq 0, \quad F(y_n) \leq 0 \quad \text{for all } n. \tag{13}$$

Lemma 3 *Let $f(u) \in C^1[0,\infty)$ satisfy Schaaf-Schmitt condition, and $g(u) \in C^1[0,\infty)$ be a positive function with $g'(u) > 0$ for all $u > 0$. Then*

$$\int_0^{x_n} f(u)g(u)\,du > 0, \quad \int_0^{y_n} f(u)g(u)\,du < 0. \tag{14}$$

Proof: Using (12) and (13) we have

$$\begin{aligned} &\textstyle\int_0^{x_n} f(u)g(u)\,du = \int_0^{x_n} g(u)d\,[F(u) - F(x_n)] \\ &= g(0)F(x_n) - \textstyle\int_0^{x_n} g'(u)\,[F(u) - F(x_n)]\,du > 0, \end{aligned}$$

and the second inequality in (14) is proved similarly. ♢

Theorem 2 *Consider the problem*

$$u'' + \lambda\,(u + f(u)) = 0 \quad \textit{for } 0 < x < L, \quad u(0) = u(L) = 0. \tag{15}$$

Assume that $f(u) \in C^2[0,\infty)$ satisfies

$$f(0) = 0, \quad f'(0) > -1, \tag{16}$$

$$u + f(u) > 0 \quad \textit{for all } u > 0, \tag{17}$$

$$\frac{f(u)}{u} \to 0 \quad \textit{as } u \to \infty. \tag{18}$$

Assume finally that either $f(u)$ or $uf(u)$ satisfies Schaaf-Schmitt condition. Then the problem (15) has a C^1 curve of positive solutions bifurcating off the trivial solution at $\lambda = \frac{\lambda_1}{1+f'(0)}$. This curve makes infinitely many turns, and tends to infinity at $\lambda = \lambda_1$. This curve exhausts the set of positive solutions of (15).

Proof: Similar to Theorem 1. Examining our derivation of (9), one sees that this time

$$J(s) = \frac{2}{s}\int_0^{u(\frac{L}{2})} f(u)\frac{dx}{du}u\,du = \frac{2}{s}\int_0^s f(u)\frac{dx}{du}u\,du.$$

Since by (17) the function $\frac{dx}{du}$ is increasing, applying Lemma 3 we see that $J(s)$ changes sign infinitely many times, and hence the solution curve changes direction infinitely many times. Clearly, an analog of Lemma 1 holds, so that the curves bifurcating from zero and infinity have to link up. ♢

Remarks.

1. One can have several variations of the Theorem 2. For example, we could drop the condition (16), and have a curve bifurcating from infinity, and making infinitely many turns. Alternatively, we could drop the condition (18) and obtain a curve bifurcating from zero, and making infinitely many turns.

2. It follows from [8] that the problem (15) has infinitely many solutions at $\lambda = \lambda_1$ if $f(u)$ satisfies Schaaf-Schmitt condition. Clearly, it may happen that $uf(u)$ satisfies Schaaf-Schmitt condition, while $f(u)$ does not.

3. Another novelty of our approach is that we can estimate the number of turns the solution curve makes until it reaches any level $u(0) = c$.

4. Arguing as in Lemma 1, we see that either the solution curve crosses the line $\lambda = \lambda_1$ infinitely many times, or else it makes infinitely many turns to the left of this line. (Assuming that $\lambda > \lambda_1$ for sufficiently large s, we multiply the equation (15) by u, integrate by parts, and obtain a contradiction.)

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Philip Korman
Institute for Dynamics and
Department of Mathematical Sciences
University of Cincinnati
Cincinnati Ohio 45221-0025
e-mail: kormanp@math.uc.edu

Yi Li
Department of Mathematics
University of Iowa
Iowa City Iowa 52242
e-mail: yi-li@uiowa.edu