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A condition on the potential for the existence of doubly periodic solutions of a semi-linear fourth-order partial differential equation *

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Abstract

We study the existence of solutions to the fourth order semi-linear equation

$$\Delta^2 u = g(u) + h(x)\,.$$

We show that there is a positive constant C_*, such that if $g(\xi)\xi \geq 0$ for $|\xi| \geq \xi_0$ and $\limsup_{|\xi|\to\infty} 2G(\xi)/\xi^2 < C_*$, then for all $h \in L^2(Q)$ with $\int_Q h dx = 0$, the above equation has a weak solution in $H^2_{2\pi}$.

1 Introduction

This paper is motivated by the study of the differential equation

$$u'' + g(u) = h(t) = h(t + 2\pi)\,, \tag{1.1}$$

where g and h are continuous functions. It is assumed that

$$\int_0^{2\pi} h(t)\,dt = 0\,. \tag{1.2}$$

Indeed, if $\hat{h} = \frac{1}{2\pi}\int_0^{2\pi} h(t)\,dt$, we may replace $g(u)$ by $g(u) - \hat{h}$ and h by $h - \hat{h}$ in (1.1). We write $g \in \Sigma$ if there exists a constant $\xi_0 \geq 0$ such that

$$g(\xi)\xi \geq 0 \quad \text{for} \quad |\xi| \geq |\xi_0\,. \tag{1.3}$$

Given $g \in \Sigma$, let $G'(\xi) = g(\xi)$, $G(0) = 0$.

Recently Fernandes and Zanolin [2] proved the existence of 2π-periodic solutions of (1.1). Their work shows that if $g \in \Sigma$, (1.2) holds and either $\liminf_{\xi\to\infty} 2G(\xi)/\xi^2 < 1/4$ or $\liminf_{\xi\to-\infty} 2G(\xi)/\xi^2 < 1/4$, then there exists a 2π-periodic solution of (1.1). Earlier work of Mawhin and Ward showed that if

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either $\limsup_{\xi\to\infty} g(\xi)/\xi < 1/4$ or $\limsup_{\xi\to-\infty} g(\xi)/\xi < 1/4$, then (1.1) has a solution.

These results led us to consider a more modest question for the partial differential equation

$$\Delta u + g(u) = h(x)\,, \tag{1.4}$$

where $x = (x_1, x_2)$ and $h(x_1 + 2\pi, x_2) = h(x_1, x_2 + 2\pi) = h(x_1, x_2)$. Namely if $Q = [0, 2\pi] \times [0, 2\pi]$, $g \in \Sigma$, $h \in L^2(Q)$, and

$$\int_Q h dx = 0\,; \tag{1.5}$$

does there exist a constant C_* such that the condition

$$\limsup_{|\xi|\to\infty} \frac{2G(\xi)}{\xi^2} < C_* \tag{1.6}$$

implies the existence of a *weak* solution to (1.4) with the "boundary condition" $u(x_1 + 2\pi, x_2) = u(x_1, x_2 + 2\pi)$?

Thus we define a solution to be a member of the function space $H^1_{2\pi}$ such that

$$\int_Q [u_{x_1} v_{x_1} + u_{x_2} v_{x_2} - g(u)v - h(x)v]dx = 0$$

for all $v \in C^\infty_{2\pi}$, the space of C^∞ functions defined on $\mathbb{R}^2$ which are 2π-periodic in each variable. The space $H^1_{2\pi}$ is the completion of this space with respect to the norm

$$\|u\| = \Big[\int_Q (u^2_{x_1} + u^2_{x_2} + u^2)dx\Big]^{1/2}.$$

The difficulty with this problem is that if g is only assumed to be continuous and $u \in H^1_{2\pi}$, it is not generally true that the function $g(u(x))$ *is locally integrable.* Also, unless g satisfies a suitable growth condition, the functional, $f : H^1_{2\pi} \to \mathbb{R}$,

$$f(u) = \int_Q \frac{|\nabla u|^2}{2} - G(u) + h(x)u\,dx$$

is not of class C^1. Thus we abandon this problem and considered the analogous fourth order semi-linear problem

$$\Delta^2 u = g(u) + h(x) \tag{1.7}$$

with $u \in H^2_{2\pi}$, where h is in $L^2(Q)$ and $H^2_{2\pi}$ denotes the completion of $C^\infty_{2\pi}$ with respect to the norm

$$\Big\{\int_Q \Big[\sum_{i=1}^2 \sum_{j=1}^2 u^2_{x_i x_j} + \sum_{i=1}^2 u^2_{x_i} + u^2\Big]dx\Big\}^{1/2}.$$

By a weak solution of (1.7) we mean a $u \in H^2_{2\pi}$ such that $\int_Q [\Delta u \Delta v - g(u)v - h(x)v]dx = 0$ for all v in $C^\infty_{2\pi}$.

Since it can be shown that $H^2_{2\pi} \subset C_{2\pi}$ (this is essentially the Sobolev embedding theorem), $u \in H^2_{2\pi}$ implies that $g(u(x))$ is continuous. Moreover the compactness of $H^2_{2\pi}$ in $C_{2\pi}$ ensures that the functional $f : H^2_{2\pi} \to \mathbb{R}$ defined by

$$f(u) = \int_\Omega \left[\frac{(\Delta u)^2}{2} - G(u) - h(x)u \right] dx$$

is of class C^1. We show that there exists $C_* > 0$ such that if $g \in \Sigma$ and (1.6) holds, then for all h satisfying (1.5), $h \in L^2(Q)$, (1.7) has a weak solution.

We have shown that if

$$C_* = \frac{1}{4\pi^2 a_*^2 + 1}, \quad \text{where} \quad a_*^2 = \frac{1}{\pi^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{(i^2+j^2)^2}$$

then this statement will be true. However, we feel that this is far from the optimal value of C_*.

It is clear that the optimal value must be less than 1, since it can be shown that if $g(\xi) = \xi$, $h(x_1, x_2) = \sin x_1$, then (1.7) does not have a weak solution, because of resonance.

2 Definitions and preliminary lemmas

In this section we state some preliminary lemmas. These results follow more or less from known results (see for example [1]). Full details will be given elsewhere.

Let $Q = \{(x_1, x_2) | 0 \le x_1 \le 2\pi, 0 \le x_2 \le 2\pi\}$. Let $L^2_{2\pi}(\mathbb{R}^2)$ denote the set of real-valued measurable functions defined in $\mathbb{R}^2$ such that if $u \in L^2_{2\pi}(\mathbb{R}^2)$, then $u(x_1 + 2\pi, x_2) = u(x_1, x_2 + 2\pi) = u(x_1, x_2)$ and such that u restricted to Q is in $L^2(Q)$.

We denote $C_{2\pi}$ and $C^\infty_{2\pi}$ the real-valued functions defined on $\mathbb{R}^2$ which are 2π-periodic in each variable, which are continuous and of class C^∞ respectively.

We denote by $H^2_{2\pi}(\mathbb{R}^2)$ the set of $u \in L^2_{2\pi}(\mathbb{R}^2)$ such that for $p = 1, 2$ there exists $v_p \in L^2_{2\pi}(\mathbb{R}^2)$ such that for all $\phi \in C^\infty_{2\pi}$,

$$-\int_Q (D_p \phi) u dx = \int_Q v_p \phi dx$$

and for $1 \le p,\, q \le 2$ there exists $v_{pq} \in L^2_{2\pi}(\mathbb{R}^2)$ such that for all $\phi \in C^\infty_{2\pi}$,

$$\int_Q (D_p D_q \phi) u dx = \int_Q \phi v_{pq} dx.$$

Here $D_p = \partial/\partial x_p$, $p = 1, 2$. It is clear that v_p, $p = 1, 2$, and v_{pq}, $p, q = 1, 2$, are determined uniquely and we write $v_p = D_p u$, $p = 1, 2$, and $v_{pq} = D_p D_q u$, $p, q = 1, 2$.

The space $H^2_{2\pi}(\mathbb{R}^2)$ is a real Hilbert space with inner product given by

$$\langle u, v \rangle = \int_Q \left[uv + \sum_{p=1}^{2} (D_p u)(D_p v) + \sum_{p,q=1}^{2} (D_p D_q u)(D_p D_q v) \right] dx$$

In the following we denote the Hilbert space $H^2_{2\pi}$ by $\mathbb{E}$ and $\|\cdot\|_{\mathbb{E}}$ will denote the norm given by the inner product defined above.

Lemma 2.1 *If $u \in \mathbb{E}$ then u is equal almost everywhere to a unique function in $C_{2\pi}$. If this function is again denoted by u, then there exists a constant a_0 such that for all $u \in \mathbb{E}$, $\|u\|_{C_{2\pi}} = \max_{x\in\mathbb{R}^2}|u(x)| \le a_0\|u\|_{\mathbb{E}}$. (see [1, p 167]).*

We denote by $\hat{\mathbb{E}}$ the set of $u \in \mathbb{E}$ such that $\int_Q u dx = 0$.
The following result can be proved using multiple Fourier series.

Lemma 2.2 *An inner product on $\hat{\mathbb{E}}$ which is equivalent to the $\mathbb{E}$-inner product is given by*

$$\langle u, v\rangle_{\hat{\mathbb{E}}} = \int_Q (\Delta u)(\Delta v)dx$$

where, as usual $\Delta u = D_1^2 u + D_2^2 u$.

Lemma 2.3 *The best possible constant a_* such that for all $u \in \hat{\mathbb{E}}$,*

$$\|u\|_{c_{2\pi}} = \max_{x\in\mathbb{R}^2}|u(x)| \le a_*\|u\|_{\hat{\mathbb{E}}}\,,$$

where $\|u\|_{\hat{\mathbb{E}}} = \|\Delta u\|_{L^2(Q)}$, is

$$a_* = \frac{1}{2\pi}\Big(\sum_{\substack{k\in\mathbf{Z}^2\\ k\neq(0,0)}} \frac{1}{|k|^4}\Big)^{1/2} \tag{2.1}$$

it where $\mathbf{Z}^2 = \mathbf{Z}\times\mathbf{Z}$, $\mathbf{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}$, and if $k = (k_1, k_2) \in \mathbf{Z}^2$, $|k| = \sqrt{k_1^2 + k_2^2}$.

This lemma and the next are proved using multiple Fourier series.

Lemma 2.4 *If $u \in \hat{\mathbb{E}}$, then $\int_Q u^2 dx \le \int_Q (\Delta u)^2 dx$.*

The following result is proved using the idea of the proof given in [5, p. 216] except Fourier series are used instead of Fourier transform.

Lemma 2.5 *Let $0 < \alpha < 1$. There exists $M(\alpha)$ such that if $u \in \mathbb{E}$, then for $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^2$*

$$|u(x) - u(y)| \le M_{(\alpha)}\|u\|_{\mathbb{E}}|x-y|^{\alpha}\,.$$

Here, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$|x-y| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}\,.$$

The final preliminary lemma follows from Lemma 2.1, Lemma 2.5 and Ascoli's Lemma.

Lemma 2.6 *The injection from $\mathbb{E}$ to $C_{2\pi}$ is compact, that is, if $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{E}$, then there exists a subsequence $\{u_{n_i}\}_{i=1}^{\infty}$ such that $\{u_{n_i}\}_{i=1}^{\infty}$ converges uniformly on $\mathbb{R}^2$.*

3 Periodic solutions of a semi-linear elliptic fourth-order partial differential equation

In this section g will always denote a real-valued function defined and continuous on $\mathbb{R}$, and G will denote the function such that $G'(\xi) = g(\xi)$ for $\xi \in \mathbb{R}$ with $G(0) = 0$. $\hat{L}^2_{2\pi}$ will denote the closed subspace of $L^2_{2\pi}(\mathbb{R}^2)$ such that for all $h \in \hat{L}^2_{2\pi}$, $\int_Q h(x)dx = 0$.

We consider the question of existence of *weak* solution of the problem

$$\begin{aligned} \Delta^2 u = g(u) + h(x) \\ u \in H^2_{2\pi}(\mathbb{R}^2) \end{aligned} \tag{3.1}$$

where $h \in \hat{L}^2_{2\pi}$. This is defined to be a function $u \in H^2_{2\pi}(\mathbb{R}^2)$ such that for all $v \in \mathbb{E}$ $(= H^2_{2\pi}(\mathbb{R}^2))$,

$$\int_Q [(\Delta u)(\Delta v) - g(u)v - h(x)v]dx = 0\,. \tag{3.2}$$

If u is a function of class C^4 which is 2π-periodic in each variable, then (3.1) holds if and only if (3.2) holds.

Let $f : \mathbb{E} \to \mathbb{R}$ be the function

$$f(u) = \int_Q \left[\frac{|\Delta u|^2}{2} - G(u) - h(x)u \right] dx\,.$$

Since $\mathbb{E} \subset C_{2\pi}$, standard arguments (see, for example, [4]) show that $f \in C^1$. For $v \in \mathbb{E}$,

$$f'(u)(v) = \int_Q [(\Delta u)(\Delta v) - g(u)v - h(x)v]dx\,.$$

Therefore, weak solutions of (3.1) coincide with critical points of f.

Let Σ denote the set of continuous $g : \mathbb{R} \to \mathbb{R}$ such that there exists some ξ_0, depending on g, such that

$$g(\xi)\xi \geq 0 \quad \text{for} \quad |\xi| \geq \xi_0\,. \tag{3.3}$$

Theorem 3.1 *Let a_* be as in (2.1) and let*

$$C_* = \frac{1}{4\pi^2 a_*^2 + 1} \tag{3.4}$$

If $g \in \Sigma$ and

$$\limsup_{|\xi| \to \infty} \frac{2G(\xi)}{\xi^2} < C^* \tag{3.5}$$

then, for all $h \in \hat{L}^2_{2\pi}$, there exists a weak solution of (3.1).

Sketch of Proof: The proof is an application of Rabinowitz's Saddle-Point Theorem [4]. Assume first that g satisfies the stronger condition: There exist $\delta > 0$ and $\xi_0 \geq 0$ such that

$$|\xi| \geq \xi_0 \text{ implies } \operatorname{sgn}(\xi)g(\xi) \geq \delta. \tag{3.6}$$

Assuming that (3.5) holds there exist constants $C_2 \geq 0$ and C_1 with

$$C_1 < C_* \tag{3.6}$$

such that for all $\xi \in \mathbb{R}$,

$$G(\xi) \leq C_1 \left(\frac{\xi^2}{2}\right) + C_2. \tag{3.7}$$

We claim that the functional f defined above satisfies the Palais-Smale condition. To see this let $\{u_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{E}$ such that $\{f(u_n)\}_{n=1}^{\infty}$ is a bounded sequence in $\mathbb{R}$ and $f'(u_n) \to 0$ in $\mathbb{E}^*$, the topological dual space of $\mathbb{E}$.

We first show that the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^2(Q)$. Assuming the contrary, we may assume, by considering a subsequence, that $\|u_n\|_{L^2} \neq 0$ for all n and that $\|u_n\|_{L^2} \to \infty$ as $n \to \infty$.

By assumption, there exists a constant C_3 such that $f(u_n) \leq C_3$ for all $n \geq 1$ or

$$\int_Q \left[\frac{|\Delta u_n|^2}{2} - G(u_n) - h(x)u_n\right] dx \leq C_3$$

for all n. From (3.7) we have that for $n \geq 1$

$$\int_Q |\Delta u_n|^2 dx \leq C_1 \|u_n\|_{L^2}^2 + 8\pi^2 C_2 + 2\|h\|_{L^2}\|u_n\|_{L^2} + 2C_3$$

Setting $w_n = u_n/\|u_n\|_{L^2}$ for $n = 1, 2, ..$ we obtain

$$\int_Q (\Delta w_n)^2 dx \leq C_1 + \frac{2\|h\|_{L^2}}{\|u_n\|_{L^2}} + \frac{8\pi^2 C_2 + 2C_3}{\|u_n\|_{L^2}^2} \tag{3.8}$$

for all $n \geq 1$.

If $\hat{\mathbb{E}}$ is defined as in the previous section and if we identify the constant functions with the real numbers $\mathbb{R}$, then

$$\mathbb{E} = \hat{\mathbb{E}} \oplus \mathbb{R} \tag{3.9}$$

For $n \geq 1$, let

$$w_n = z_n + \tau_n\,, \tag{3.10}$$

where $z_n \in \hat{\mathbb{E}}$ and $\tau_n \in \mathbb{R}$. Since $\|\Delta z_n\|_{L^2} = \|\Delta w_n\|_{L^2}$, it follows from (3.8) and Lemma 2.2 that the sequence $\{z_n\}_{n=1}^{\infty}$ is bounded in $\hat{\mathbb{E}}$. Therefore, since for all $n \geq 1$, $4\pi^2\tau_n^2 \leq \|w_n\|_{L^2}^2 = 1$, we infer the existence of a constant C_4 such that $\|w_n\|_{\mathbb{E}} < C_4$ for all n.

It follows that there exists a subsequence of $\{w_n\}_{n=1}^{\infty}$ which converges weakly to w in $\mathbb{E}$. By considering a subsequence, we may assume, without loss of generality, that the sequence $\{w_n\}_{n=1}^{\infty}$ itself converges weakly to w.

If $w = z + \tau$ where $z \in \hat{\mathbb{E}}$ and $\tau \in \mathbb{R}$, then z_n converges weakly to z and τ_n converges to τ as $n \to \infty$. From Lemma 2.6, it follows that the sequence $\{w_n\}_{n=1}^{\infty}$ converges uniformly to w on $\mathbb{R}^2$, and since $\lim_{n\to\infty}\tau_n = \tau$, we see that $\{z_n(x)\}_{n=1}^{\infty}$ converges uniformly to $z(x)$ on $\mathbb{R}^2$.

The uniform convergence implies that $\|w\|_{L^2} = \lim_{n\to\infty}\|w_n\|_{L^2} = 1$. From the lower semi-continuity of a norm with respect to weak convergence, it follows from (3.8) that

$$\|\Delta z\|_{L^2}^2 = \|\Delta w\|_{L^2}^2 \leq \liminf_{n\to\infty}\|\Delta w_n\|_{L^2}^2 \leq C_1\,.$$

Therefore, $\|\Delta z\|_{L^2}^2 \leq C_1\|w\|_{L^2}^2 = C_1(\|z\|_{L^2}^2 + 4\pi^2\tau^2)$ and since, according to Lemma 2.4, $\|z\|_{L^2} \leq \|\Delta z\|_{L^2}$, it follows that

$$\|\Delta z\|_{L^2}^2 \leq \frac{C_1 4\pi^2\tau^2}{1 - C_1}\,. \tag{3.11}$$

(That $C_1 < 1$ follows from (3.4) and (3.6)). Since $1 = \|w\|_{L^2}^2 = \|z\|_{L^2}^2 + 4\pi^2\tau^2$, we see that $\tau \neq 0$.

According to lemma 2.3

$$\max_{x\in\mathbb{R}^2}|z(x)|^2 \leq \left(\frac{a_*^2 C_1 4\pi^2}{1 - C_1}\right)\tau^2,$$

and from (3.4) and (3.6)

$$\frac{a_*^2 C_1 4\pi^2}{1 - C_1} < \frac{a_*^2 C_* 4\pi^2}{1 - C_*} = 1.$$

Therefore,

$$\max_{x\in\mathbb{R}^2}|z(x)| < |\tau|.$$

Since $\tau \neq 0$ it follows that either $w(x) = z(x) + \tau > 0$ for all $x \in \mathbb{R}^2$ or $w(x) < 0$ for all $x \in \mathbb{R}^2$. Since $u_n(x) = \|u_n\|_{L^2} w_n$ either $u_n(x) \to \infty$ uniformly with respect to $x \in \mathbb{R}^2$ or $u_n(x) \to -\infty$ uniformly with respect to $x \in \mathbb{R}^2$. From (3.6) it follows that in the first case

$$\int_Q g(u_n(x))dx \geq 4\pi^2\delta$$

for n sufficiently large, and in the second case

$$\int_Q g(u_n(x))dx \leq -4\pi^2\delta$$

for n sufficiently large. But since $h \in \hat{L}_{2\pi}^2$, $f'(u_n)(1) = \int_Q - [g(u_n(x)) + h(x)]dx = -\int_Q g(u_n(x))dx$. Since $f'(u_n)(1) \to 0$ as $n \to \infty$, therefore we have a contradiction. This contradiction proves the sequence $\{u_n\}_{n=1}^{\infty}$ is bounded in $L^2(Q)$.

From the condition $f(u_n) \leq C_3$ for all n and the condition (3.7), it follows from Lemma 2.2, that $\{u_n\}_{n=1}^{\infty}$ is bounded in $\mathbb{E}$. Therefore, from the form of f' and Lemma 2.6, standard arguments (see for example [4]) shows that f' satisfies the Palais-Smale condition.

The existence of a critical point of f follows from Rabinowitz's Saddle Point Theorem [4] corresponding to the direct sum decomposition $\mathbb{E} = \hat{\mathbb{E}} \oplus \mathbb{R}$. Since, according to Lemma 2.4, for all $z \in \hat{\mathbb{E}}$, $\|z\|_{L^2} \leq \|\Delta z\|_{L^2}$, it follows that for all $z \in \hat{\mathbb{E}}$,

$$\begin{aligned} &\int_Q \left[\frac{(\Delta z)^2}{2} - G(z) - h(x)z\right] dx \\ &\geq \int_Q \left[\frac{(\Delta z)^2}{2} - \frac{C_1}{2}z^2 - C_2\right] dx - \|h\|_{L^2}\|z\|_{L^2} \\ &\geq \left(\frac{1-C_1}{2}\right)\int_Q \frac{(\Delta z)^2}{2} dx - C_2 4\pi^2 - \|h\|_{L^2}\|\Delta z\|_{L^2}\,. \end{aligned}$$

Since, as shown above, $C_1 < 1$ it follows that

$$\inf_{z \in \hat{\mathbb{E}}} f(z) > -\infty.$$

The condition $g(\xi) sgn\ \xi \geq \delta$ for $|\xi| \geq \xi_0$ implies that $G(\xi) \to \infty$ as $|\xi| \to \infty$. Therefore, since $h \in \hat{L}^2_{2\pi}$, it follows that for $\xi \in \mathbb{R}$,

$$f(\xi) = \int_Q [-G(\xi) - \xi h(x)] dx \leq -4\pi^2 G(\xi) \to -\infty$$

as $|\xi| \to \infty$. Thus there exists $b > 0$ such that

$$\max\{f(b), f(-b)\} < \inf_{z \in \hat{\mathbb{E}}} f(z).$$

Since f satisfies the Palais-Smale condition, it follows that if Γ denotes the set of all continuous mappings $\gamma : [-b, b] \to \mathbb{E}$ with $\gamma(\pm b) = \pm b$,

$$C_0 = \inf_{\gamma \in \Gamma} \max_{\xi \in [-b,b]} f(\gamma(\xi)),$$

then there exits $u_0 \in \mathbb{E}$ such that $f(u_0) = C_0$ and $f'(u_0) = 0$. This u_0 is a solution of problem (3.1).

To prove that (3.1) has a solution when it is only assumed that $g(\xi)\,\mathrm{sgn}\,\xi \geq 0$ for $|\xi| \geq \xi_0$. We can use a perturbation argument. We define

$$r(\xi) = \begin{cases} -1 & \text{if } \xi \leq -\xi_0\,, \\ -1 + \frac{2(\xi+\xi_0)}{2\xi_0} & \text{if } |\xi| \leq \xi_0\,, \\ 1 & \text{if } \xi \geq \xi_0\,, \end{cases}$$

For $m = 1, 2, 3, \ldots$, set $g_m(\xi) = g(\xi) + \frac{r(\xi)}{m}$. Then $g_m(\xi)\xi \geq \frac{1}{m}$ for $|\xi| \geq \xi_0$ and we still have

$$\limsup_{|\xi|\to\infty} \frac{2G_m(\xi)}{\xi^2} < C^*.$$

By what has been shown, (3.1) has a solution when $g = g_m$. The conditions of the theorem imply that there is a priori bound on this solution (the one characterized by the Saddle Point Theorem) in $\mathbb{E}$, which is independent of m. Using a compactness argument this implies the existence of a solution of (3.1). The computational details of this proof will be published somewhere else.

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