Nonlinear eigenvalue problems with semipositone structure *

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*Dedicated to Alan Lazer
on his 60th birthday*

Abstract

In this paper we summarize the developments of semipositone problems to date, including very recent results on semipositone systems. We also discuss applications and open problems.

1 Introduction

Many problems in physics, chemistry, engineering, and biology lead to the study of reaction diffusion processes. A simple example of a diffusion process is the heat conduction in a solid. Let $u(x, t)$ be the temperature at position x and time t, $k(x)$ the "heat conductivity" and let $H(t)$ be the amount of heat contained in a region Ω in the solid. If c, ρ are constants, and if there is an external source (or sink) $f(x, u, t)$, then the general inhomogeneous diffusion equation takes the form

$$c\rho u_t = \nabla.(k\nabla u) + f(x, u, t).$$

Now, if we assume that the external source (or sink) is independent of t, in particular $f(x, u, t) = f(x, u)$, then the "steady state" (time independent state) of the diffusion equation is

$$\nabla.(k\nabla u) + f(x, u) = 0 \, .$$

For the past forty years or so, the study of such steady states of diffusion problems subject to Dirichlet (u is specified on $\partial\Omega$) boundary conditions has been of considerable interest. In particular, when k is a constant and $f(x, u) = f(u)$, leads to nonlinear eigenvalue problems of the form

$$-\Delta u = \lambda f(u); \quad \Omega \tag{1.1}$$

$$u = 0; \quad \partial\Omega \, , \tag{1.2}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^n$; $n \geq 1$ with smooth boundary $\partial\Omega$ and Δ is the Laplacian operator. Cohen and Keller in [37] initiated the study of such problems when f was positive and monotone, and also introduced the terminology "positone" problems. Their motivation to study such reaction terms f is the fact that the resistance increases with temperature. For an excellent review on positone problems see [53]. For important results on positone problems see [5],[18], [38], [41], [43], [50],[51], [52], [61] and [62].

We next describe briefly a population dynamics model which leads to the study of steady states different from positone problems. Let $N(x,t)$ denote the population of a species which is harvested at a constant rate. Assuming that the logistic growth model fits the normal growth of the population (without harvesting) and supposing that the quantity harvested per unit time is independent of time and is denoted by $H(x)$, the resulting population model is of the form:

$$\frac{\partial N}{\partial t} = c\Delta N + (B - SN)N - H; \quad \Omega \times (0, \infty)$$

$$N(x, 0) = A; \quad \Omega$$

$$N(x, t) = 0; \quad \partial\Omega \times [0, \infty),$$

where Ω is a bounded domain in $\mathbb{R}^3$, c, B, S are positive constants, and A denotes the initial population. The natural question is whether the population exists after a long time. This question is equivalent to establishing the existence of a positive solution to the steady state problem

$$c\Delta N + (B - SN)N - H = 0; \quad \Omega$$

$$N = 0; \quad \partial\Omega.$$

Here one needs to find not only nonnegative solutions but also solutions that are pointwise larger than H, the amount being harvested. It is worthwhile mentioning that from a practical point of view, constant effort harvesting is favored over density dependent ($f(x, u) = \tilde{f}(u) - H(x)u$) harvesting (see [63]). These observations lead to the study of *positive* solutions to the problems of the form $(1.1) - (1.2)$ with $f(0) < 0$. Such problems are referred in the literature as "semipositone" problems and are the main focus of this review paper.

Semipositone problems arise in buckling of mechanical systems, design of suspension bridges, chemical reactions, astrophysics (thermal equilibrium of plasmas), combustion and management of natural resources (constant effort harvesting as derived above).

As pointed out by P.L. Lions in [53], semipositone problems are mathematically very challenging, and from the point of view of many important natural applications, interesting, particularly for positive solutions. In fact, during the last ten years finding positive solutions to problems of the form (1.1)-(1.2) with $f(0) < 0$ has been actively pursued. The difficulty of studying positive solutions to such problems was first encountered by Brown and Shivaji in [19] when they studied the perturbed bifurcation problem

$$-\Delta u = \lambda(u - u^3) - \epsilon; \quad \Omega$$

$$u = 0; \quad \partial\Omega$$

with $\epsilon > 0$. However, the study of semipositone problems was formally introduced by Castro and Shivaji in [30]. Also semipositone problems lead to symmetry breaking phenomena (see [65]). In [65], the authors proved that $f(0) < 0$ is a necessary condition for symmetry breaking of positive solutions in a ball in $\mathbb{R}^n$. Significant progress on semipositone problems has taken place in the last ten years; in particular, due to the pioneering efforts by Castro and Shivaji.

In general, studying positive solutions for semipositone problems is more difficult compared to that of positone problems. The difficulty is due to the fact that in the semipositone case, solutions have to live in regions where the reaction term is negative as well as positive. We will briefly introduce the method of sub-super solutions (see [62]), which has been a very successful method in handling positone problems, to demonstrate the difficulty of studying positive solutions for semipositone problems over positone problems.

A super solution to (1.1)-(1.2) is defined as a function $\psi \in C^2(\overline{\Omega})$ such that

$$-\Delta\psi \geq \lambda f(\psi); \quad \Omega$$
$$\psi \geq 0; \quad \partial\Omega.$$

Sub solutions are defined similarly with the inequalities reversed and it is well known that if there exists a sub solution ϕ and a super solution ψ to (1.1)-(1.2) such that $\phi(x) \leq \psi(x)$ for $x \in \overline{\Omega}$, then (1.1)-(1.2) has a solution u such that $\phi(x) \leq u(x) \leq \psi(x)$ for $x \in \overline{\Omega}$. Further note that if $\phi(x) \geq 0$ for $x \in \Omega$ then $u \geq 0$ for $x \in \Omega$.

In the positone case, $\phi \equiv 0$ is a sub solution to (1.1)-(1.2). Thus to ensure the existence of a positive solution it is enough to find a positive super solution. On the other hand, in the semipositone case $\psi \equiv 0$ is a super solution. Suppose we try to find a nonnegative sub solution ϕ such that

$$-\Delta\phi = h; \quad \Omega$$
$$\phi = 0; \quad \partial\Omega,$$

then we have to choose h to be negative near the boundary of Ω since $f(0) < 0$ while to ensure that ϕ is nonnegative it is necessary that h is sufficiently positive in the interior of Ω. Further h must be such that $h \leq f(\phi)$ for each x in Ω. In short, finding a nonnegative solution is not an easy task in the case of semipositone problems. We encounter similar problems when we try to use other methods that have been successful in the case of positone problems.

However, mathematicians have found their way via sub-super solution methods, variational methods, degree theory, fixed point theory, shooting methods, reflection arguments, Maximum principles, bifurcation theory etc. to establish a rich collection of results for the single equations case(see Section 2).

The study of semipositone systems for positive solutions is even more challenging since not only one has to deal with the systems, but also needs to establish the positivity of the solution componentwise. In the single equations

case a popular plan was to find a solution with "big" enough supremum norm and use this to establish that the solution is positive. In systems, knowing that the supremum norm of $u = (u_1, \ldots, u_m)$ (say) is large does not necessarily mean that the supremum norm of each u_i is large. Thus establishing that each component u_i of the solution is positive is an additional challenge.

Semipositone systems occur naturally in important applications; for example, predator - prey systems with constant effort harvesting. In particular, with diffusion included, such systems in steady states will be of the form:

$$\begin{aligned}
-\Delta u &= \lambda[f(u,v) - K]; \quad \Omega \\
-\Delta v &= \lambda[g(u,v) - H]; \quad \Omega \\
u &= v = 0; \quad \partial\Omega \,,
\end{aligned}$$

where Ω is a smooth bounded region in $\mathbb{R}^n$, and K and H represent harvesting (or stocking if they are negative) densities of the predator u and prey v respectively. See [14], [15], [16], [39], [58] and [63] for examples of f's and g's where the authors study various predator-prey systems with constant effort harvesting but without diffusion. Thus the study when diffusion is included (i.e., the study of semipositone systems) will greatly enhance the understanding of these problems in population dynamics.

In summary, while strengthening the result for single equations it would be challenging and important to extend the theory for single equations to systems in the following two directions:
[A] To study systems arising in applications such as predator-prey, cooperative and competitive models with constant effort harvesting.
[B] To identify and study systems that exhibit qualitative properties similar to that of the single equations case.
To date, no results are known in the direction [A], while some developments have occured recently in the direction [B] which we will outline in Section 3.

We now conclude this introduction by outlining the rest of the paper. Namely, in section 2 we provide the known literature on semipositone single equations to date and open problems. In section 3, we discuss the very recent developments on semipositone systems.

2 Survey of semipositone problems for single equations case

Semipositone problems were formally introduced by Castro and Shivaji in [30] where the authors studied two point boundary value problems of the form

$$\begin{aligned}
-u'' &= \lambda f(u); \quad 0 < x < 1 \\
u(0) &= u(1) = 0
\end{aligned}$$

and obtained detailed results via a quadrature method for various classes of reaction terms f. In particular they considered classes of superlinear nonlinearities

(eg: $f(u) = u^p - \epsilon$; $\epsilon > 0$, $p > 1$, $\quad f(u) = e^u - 2$ etc.) for which they proved that there exists $\lambda_1 > 0$ such that for $0 < \lambda \le \lambda_1$ there is a unique positive solution while for $\lambda > \lambda_1$ there are no positive solution. For classes of sublinear nonlinearities (eg: $f(u) = (1+u)^{1/3} - 3$, $\quad f(u) = au - bu^2 - c$; $a > 0, b > 0, c > 0$ etc.) they proved that there exists $0 < \mu < \lambda_1$ such that for $\lambda < \mu$ there are no positive solutions, for $\mu < \lambda \le \lambda_1$ there are at least two positive solutions, and for $\lambda = \mu$ and $\lambda > \lambda_1$ there is exactly one positive solution. They also studied classes of superlinear nonlinearities which were initially concave and later convex (eg: $f(u) = u^3! - au^2 + bu - c$; $a > 0, b > 0, c > 0$ and $b > (32/81)a^2, a^3 > 54c$) and established that there are ranges of λ for which there are at least three positive solutions. Further, in all cases, they established a sequence $\{\lambda_n\}; n = 1, 2, \dots$ such that for $\lambda = \lambda_n$, the problem had a unique nonnegative solution with $(n-1)$ interior zeros, which satisfy both the Dirichlet as well as Neumann boundary conditions. Note that such solutions are possible only if $f(0) < 0$.

Many mathematicians during the past ten years or so have successfully extended these results to higher dimensions. The first major breakthrough came in [31] when the authors proved that all nonnegative solutions for

$$-\Delta u = \lambda f(u); \quad \Omega$$
$$u = 0; \quad \partial \Omega$$

with $\lambda > 0$, $f(0) < 0$ and $\Omega = B_n$ a ball in $\mathbb{R}^n$; $n > 1$ are in fact positive. Since positivity implies radial symmetry (see [43]), various results appeared in the literature following this result for positive (radial) solutions for semipositone problems.

This positivity result when Ω is a ball in $\mathbb{R}^n$; $n > 1$ is unlike the case when $n = 1$, and authors in [31] used the fact that the boundary is connected. In [43], due to the result for $n = 1$, it was conjectured that the problem may have nonnegative solutions with interior zeros in higher dimensions as well, which we see is false when at least Ω is a ball. However, the conjecture remains to be proven/disproven in general bounded regions.

2.1   Superlinear nonlinearities

In this section we discuss results on superlinear nonlinearities. See [32] and [66] where the authors establish the existence of positive solutions for λ small for classes of superlinear nonlinearities when Ω is a ball. In [66] authors use shooting methods to prove this existence result for nonlinearities of the form $f(u) = u^p - \epsilon$; $\epsilon > 0$, $1 < p < \frac{n}{n-2}$. In [32] authors do better by combining shooting methods with Pohozaev's identity. In fact, their theorem gives this existence result for $f(u) = u^p - \epsilon$; $\epsilon > 0$, $1 < p < \frac{n+2}{n-2}$ which is an optimal result since it can be proven that positive solution do not exist if $p \ge \frac{n+2}{n-2}$ (the critical exponent). This existence result has been extended to the general bounded regions (see [3], [6] and [69]). In [3] the authors use degree theory, in [69] the authors use variational methods while in [6] the result is obtained via

bifurcation from infinity. The often successful technique has been to obtain a solution with big enough supremum norm and use this fact to show that the solution is positive. See also [9] and [13] for extensions of this existence result for λ small.

For classes of superlinear problems non-existence result for λ large has been proven in [3], [13] and [17]. Here the authors prove the result by analyzing the qualitative behavior of solutions (if they exist) near the boundary and obtaining a contradiction using an energy analysis or on the positivity of the solution.

The instability of the solution for convex monotone nonlinearities was first established in [19] where the authors use Green's identities to prove that the first eigenvalue of the linearized equation about the solution has the appropriate sign. See [67] and [55] for an extension of this result for non-monotone functions.

The uniqueness result for superlinear nonlinearities for λ small for the case when Ω is a ball was established in [1] and [23] using the Implicit function theorem, variations with respect to parameters, the Pohozaev's identity and comparison arguments. In [23], it was further established in the ball that if $f(u) = u^p - \epsilon;\ \epsilon > 0,\ 1 < p < \frac{n+2}{n-2}$ then the problem has at most one positive solution for any λ. However, this uniqueness result remains an open question in general bounded regions even in convex regions other than a ball. Also the case when the nonlinearity is concave first and then convex needs to be studied beyond the $n = 1$ case discussed in [30].

2.2   Sublinear nonlinearities

In this section we discuss results on sublinear semipositone problems. For classes of sublinear concave nonlinearities when Ω is a general bounded region in $\mathbb{R}^n$; $n > 1$, existence results were established in [6], [8], [26] and [36]. In [36] for nonlinearities with falling zeros, variational methods were used to prove an existence result for λ large. In [8] and [26], the method of sub-super solutions was employed to establish existence results, one for λ large, and the other near the first eigenvalue of the Laplacian operator with Dirichlet boundary conditions. For this latter existence result, the Anti-maximum principle (see [35]) was used to create a nonnegative sub solution. Further in [8] nonexistence of positive solutio! ns for λ small was established via Green's identities using the fact that $f(0) < 0$ and the sublinearity assumption. In [6], an existence result for λ large was established via bifurcation from infinity. See also [56] where existence result from semipositone problems is used to establish existence and multiplicity results for classes of sublinear nonlinearities which vanish at the origin but negative near the origin.

When Ω is a ball, for classes of sublinear nonlinearities complete details are known. In [22] and [33] they established the exact bifurcation diagram. In particular, there exists $0 < \mu < \lambda_1$ such that for $\lambda < \mu$ no solution, for $\mu < \lambda \leq \lambda_1$ exactly two solution, and for $\lambda \geq \lambda_1$ as well as for $\lambda = \mu$ exactly one solution. Further, the upper branch of the solution is stable while the lower branch including at $\lambda = \mu$ is unstable. In [22], the authors study increasing nonlinearities (eg: $f(u) = (u+1)^{1/3} - 2$) while in [33] nonlinearities with falling

zeros (eg: $f(u) = au^2 - bu - c$; $a, b, c > 0$) are discussed. The Implicit Function Theorem, variation with respect to parameters, the above mentioned existence and nonexistence results, and the uniqueness and stability results established in [1]! were useful in proving this exact bifurcation diagram. In [1], the authors use again the Implicit function theorem, variations with respect to parameters and Sturm comparison theorem to establish the uniqueness and stability results. See also [24] and [25] where evolution of bifurcation curves for positive solution are studied with concave nonlinearities.

However to date, the uniqueness result for λ large, in general bounded regions, has been proven only for classes of increasing nonlinearities and when the outer boundary of the region is convex (see [27]). Here the authors prove their results by first establishing qualitative properties of the solution near the boundary, namely they establish that for large λ the solution $u(x) \geq k\,dist(x, \partial\Omega)$ where $k > 0$ is a constant. Uniqueness result for λ large when the outer boundary of the region is non-convex is open. For the case when the nonlinearity has a falling zero, this uniqueness result has been proven only in the case when Ω is a ball (see [33]). Finally, the multiplicity result is also open in regions other than a ball.

2.3   Quasilinear equations

In this section we discus existence results for radial solutions to quasilinear equations of the form

$$-\operatorname{div}(\alpha(|\nabla u|^2)\nabla u) = \lambda f(|x|, u); \quad x \in \Omega$$
$$u = 0; \quad x \in \partial\Omega,$$

where $\lambda > 0$ and Ω is an annulus. Here $\alpha(s^2)s$ is an odd increasing homeomorphism on the real line. The special case when $\alpha(|\nabla u|^2) = |\nabla u|^{p-2}$, $p > 1$ corresponds to the p-Laplacian case. Some existence results has been established in [47] via fixed point theory in a cone for such systems. For classes of superlinear functions they prove the existence result for λ small and for classes of sublinear functions they prove the existence result for λ large. The existence result for λ small for classes of superlinear functions has been extended to the case when Ω is a ball in $\mathbb{R}^n$ in [44] via degree theory. To our knowledge these are the only two papers in this direction and thus many questions on uniqueness, non-existence, multiplicity all remain open even for radial solutions when Ω is a ball/annulus. Further, the study of such quasilinear equations with semipositone structure is open in the case of general bounded regions.

2.4   Remarks

Here we summarize the known results to date for semipositone problems with Neumann/Robin boundary conditions and in unbounded regions. For results on positive solutions for semipositone problems with Neumann boundary conditions see [2] and [57]. In [57] the authors study two point boundary value problems via

a quadrature method. In [2] the authors study classes of superlinear problems via degree theory.

See also [10] and [11] where classes of two point boundary value problems with Robin boundary conditions are discussed via quadrature methods. Further note that the results in [2], [3], [8], [19], [55] and [67] holds for Robin boundary conditions as well.

For results on positive solutions in unbounded regions see [4]. Here the authors discuss existence result for λ small for classes of superlinear nonlinearities via variational methods.

Finally, for the study of sign-changing solutions see [12] and [23]. In [12], a two point boundary value problem is discussed via a quadrature method while in [23], a thorough study is carried out for all branches of solutions when Ω is a ball in $\mathbb{R}^n$; $n > 1$ via the Implicit function theorem and variations with respect to parameters.

3   Recent developments on semipositone systems

Here we describe recent developments on semipositone systems, namely results in [28], [29], [44], [45], [48] and [49]. These results give the complete picture to date in this direction.

Result 1: In[28], the authors study cooperative semipositone systems in a ball. In particular, they consider a classical nonnegative solution $u := (u_1 \geq 0, u_2 \geq 0, \ldots, u_m \geq 0)$ for the system

$$-\Delta u_i = f_i(u_1, u_2, \ldots, u_m); \quad \Omega \quad 1 \leq i \leq m,$$
$$u_i = 0 \quad \partial\Omega,$$

where Ω is a ball in $\mathbb{R}^n$; $n > 1$ and $f_i : \underbrace{[0, \infty) \times [0, \infty) \ldots, [0, \infty)}_{m \text{ times}} \to \mathbb{R}$ are C^1

functions satisfying

$$f_i(0, 0, \ldots, 0) < 0, \qquad i = 1, 2, \ldots, m \quad \text{(semipositone system)} \qquad \text{and}$$
$$\frac{\partial f_i}{\partial u_j} \geq 0, \qquad i \neq j \quad \text{(cooperative system)} .$$

Then they prove that $u_i > 0$ for each $i = 1, 2, \ldots, m$ i.e., nonnegative solutions are componentwise positive. This result is of great importance since positivity implies that the solutions are radially symmetric and radially decreasing (see [68]). They prove the result by combining Lemma 4.2 of [68], Maximum principle/reflection arguments and analysis of solutions near the boundary. This result holds even if Ω is a region between two balls or the union of balls. However, the question of positivity of nonnegative solutions in general bounded region remains open including in the single equation case. On the other hand, in unbounded regions there are nonnegative solutions with interior zeros. Indeed,

consider the two point boundary value problem

$$-u'' = \lambda f(u); \quad 0 < x < 1) \tag{3.1}$$
$$u(0) = u(1) = 0, \tag{3.2}$$

where $f(0) < 0$, $f'(u) > 0$ and $\lim_{u \to \infty} f(u) = \infty$. Then from [30] it follows that there exists an increasing sequence of positive numbers λ_n such that $(3.3) - (3.4)$ has a nonnegative solution $u_n(x)$ with n interior zeros x_n in $(0, 1)$. Now consider

$$-\Delta w = \lambda f(w); \quad \Omega := \{(x, y) : 0 < x < 1, \, y \in \mathbb{R}\} \tag{3.3}$$
$$w = 0; \quad \partial\Omega := \{(z, y) : z \in \{0, 1\}, y \in \mathbb{R}\}. \tag{3.4}$$

Clearly for $\lambda = \lambda_n$, $w_n(x, y) = u_n(x)$ is a nonnegative solution of $(3.5) - (3.6)$ which vanishes on

$$\tilde{\Omega} := \{(x_n, y) : y \in \mathbb{R}, \, n = 1, 2, \ldots, \} \subset \Omega.$$

Result 2: In [48], the authors discuss existence results for radial solutions in an annulus for classes of semilinear semipositone systems. In particular, they consider the existence of positive solutions for the system

$$-(r^{n-1}u')' = \lambda r^{n-1} f(u, v); \quad a < r < b$$
$$-(r^{n-1}v')' = \lambda r^{n-1} g(u, v); \quad a < r < b$$
$$u(a) = u(b) = 0; \quad v(a) = v(b) = 0,$$

where $\lambda > 0$ is a parameter, $f, g : [0, \infty) \times [0, \infty) \to \mathbb{R}$ are continuous and there exists $M > 0$ such that $f(u, v) \geq -\frac{M}{2}$, $g(u, v) \geq -\frac{M}{2}$ for every $(u, v) \in [0, \infty) \times [0, \infty)$. They first consider the case when f, g further satisfy:

(A1) $\lim_{v \to \infty} f(u, v) = \infty$, $\lim_{u \to \infty} g(u, v) = \infty$, where each limit is uniform with respect to the other variable and $\lim_{z \to \infty} \frac{h^*(z)}{z} = \infty$, where
$$h^*(z) := \inf_{u, v \geq z} \{\min(f(u, v), g(u, v))\}.$$

and prove that the system has a positive solution (u_λ, v_λ) for λ small with $|(u_\lambda(t), v_\lambda(t))|_\infty \to \infty$ as $\lambda \to 0$ uniformly for t in compact intervals of (a, b). Next, they consider the case when f, g satisfy:

(A2) $\lim_{v \to \infty} f(u, v) = \infty$, $\lim_{u \to \infty} g(u, v) = \infty$, where each limit is uniform with respect to the other variable and $\lim_{z \to \infty} \frac{\tilde{h}(z)}{z} = 0$, where
$$\tilde{h}(z) := \sup_{0 \leq u, v \leq z} \{\max(f(u, v), g(u, v))\}$$

and prove that the system has a positive solution (u_λ, v_λ) for λ large with $\lambda^{-1} \max(u_\lambda(t), v_\lambda(t)) \to \infty$ as $\lambda \to \infty$ uniformly for t in compact intervals of (a, b).
Finally, they consider the case when $f(u, v) = f(v), g(u, v) = g(u)$ satisfy $(A1)$ and

(A3) there exists $r > 0$ and $0 < \alpha < 1$ such that $h(x) \geq (\frac{x}{r})^{\alpha} h(r)$ for $x \in [0, r]$
    where $h(x) = \min\{f(x) - f(0), g(x) - g(0)\}$,

and establish the existence of at least two positive solutions for certain ranges of λ under some additional conditions.

Note that no sign conditions are required on the reaction terms at the origin and thus allowing the semipositone structure. Also no monotonicity assumptions are required for these results to hold. They establish these result by using fixed point theory in a cone.

Result 3: In [29], the authors establish an existence result for classes of sublinear cooperative semipositone systems in general bounded regions. In particular, they consider the existence of positive solutions to the system

$$-\Delta u_i = \lambda[f_i(u_1, u_2, \dots u_m) - h_i]; \quad \Omega$$
$$u_i = 0; \quad \partial\Omega$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^n$; $n \geq 1$ with a smooth boundary $\partial\Omega$, h_i are nonnegative continuous functions in Ω for $i = 1, 2, \dots, m$ and $f_i : \underbrace{[0, \infty) \times [0, \infty) \times \dots \times [0, \infty)}_{m \text{ times}} \to \mathbb{R}$ are C^1 functions for $i = 1, 2, \dots, m$.

Further, we assume that for each $i = 1, 2, \dots, m$, we have

$$f_i(0, 0, \dots, 0) = 0$$
$$\frac{\partial f_i}{\partial u_j}(z_1, z_2, \dots, z_m) \geq 0, \quad i \neq j \ z_1, z_2, \dots, z_m \in \mathbb{R}$$
$$\frac{\partial f_i}{\partial u_i}(z, z, \dots, z) \geq 0, \quad \forall z \in \mathbb{R}$$
$$\sum_{j=1}^{m} \frac{\partial f_i}{\partial u_j}(0, \dots, 0) > 0$$
$$\lim_{z \to \infty} \frac{f_i(z, \dots, z)}{z} = 0; \quad \text{and} \quad \lim_{z \to \infty} f_i(z, \dots, z) = \infty.$$

Then they establish that there exists $\tilde{\lambda} > 0$ such that for $\lambda > \tilde{\lambda}$, the system has a positive solution $(u_1, u_2, \dots, u_m)$. Further, $u_i(x)/\operatorname{dist}(x, \partial\Omega) = O(\lambda)$ as $\lambda \to \infty$ for $i = 1, 2, \dots, m$. They prove this result by producing a nonnegative sub solution and then applying the method of sub-super solutions. As pointed out earlier, producing nonnegative sub solution is non-trivial in semipositone problems, and this is the important step in the proof of this result. See [8] and [26] where the single equation case of this problem was studied using the method of sub-super solutions. Sub-super solutions are in general hard to apply in the semipositone case since it is hard to construct a nonnegative sub-solution. In fact, in [8] and [26], a non-trivial existence result proved in [36] for a class of semipositone problem with reaction term having "falling zeros", played a crucial

role in the construction of the nonnegative sub solution. However here authors provide a direct method of constructing sub-super solutions.

We note here that semipositone sublinear systems have also been studied in the past in [7]. However, in [7] the coupling was weak so that one could use existence results from the study of single equations case in the construction of the nonnegative sub solution.

Result 4: In [45], the authors prove an existence result for radial solutions in an annulus for classes of quasilinear (including p-Laplacian) systems with superlinear reaction terms. In particular, consider the existence of positive radial solutions for the system

$$
\begin{aligned}
-\operatorname{div}(\alpha(|\nabla u|^2)\nabla u) &= \lambda f(v); \quad a < |x| < b \\
-\operatorname{div}(\alpha(|\nabla v|^2)\nabla v) &= \lambda g(u); \quad a < |x| < b \\
u = v &= 0; \quad |x| \in \{a, b\},
\end{aligned}
$$

where $\phi(s) = \alpha(s^2)s$ is an odd increasing homeomorphism of the real line and λ is a positive parameter. Such radial solutions are solutions to systems of the form

$$
\begin{aligned}
-(r^{n-1}\phi(u'))' &= \lambda r^{n-1} f(v); \ a < r < b \\
-(r^{n-1}\phi(v'))' &= \lambda r^{n-1} g(u); \ a < r < b \\
u(a) = u(b) = 0; &\quad v(a) = v(b) = 0,
\end{aligned}
$$

where $r = |x|$ and n is the dimension of x.

Assume:

(B1) For each $c > 0$ there exist a constant $A_c > 0$ such that $\phi^{-1}(cx) \geq A_c\phi^{-1}(x)$ for all $x \geq 0$ and $A_c \to \infty$ as $c \to \infty$ (which implies the existence of a constant $B_c := 1/A_{1/c} > 0$ such that $\phi^{-1}(cx) \leq B_c\phi^{-1}(x)$ for all $x \geq 0$ and $B_c \to 0$ as $c \to 0$).

(B2) The functions $f, g : [0, \infty) \to \mathbb{R}$ are continuous, and there exists $M > 0$ such that $f(z) \geq -M/2$ and $g(z) \geq -M/2$ for $z \in [0, \infty)$

(B3) $\lim\limits_{z \to \infty} \frac{h^*(z)}{\phi(z)} = \infty$ and $\lim\limits_{z \to \infty} \frac{A_{h^*(z)}}{z} = \infty$ where A_c is as defined in (B1) and $h^*(z) = \inf\limits_{w \geq z} \{\min(f(w), g(w))\}$.

Then they establish that there exists $\lambda^* > 0$ such that for $0 < \lambda < \lambda^*$, the system has a positive solution. This result is an extension to classes of systems including p-Laplacian systems ($\alpha(s^2) = |s|^{p-2}$, $p > 1$) of existence results for superlinear problems discussed in [3], [6], [9], [30], [32], [42], [46], [47] and [48]. In particular, in [47] the authors study the single equation case of this problem for both sublinear and superlinear reaction terms. As noted earlier (see Result 2), in [48], authors again establish such existence results for radial solutions in an annulus but for semilinear elliptic systems. Here authors succeed in establishing

an existence result for quasilinear systems, for a class of superlinear reaction terms. Again here (as in Result 2) no monotonicity assumptions are required on the reaction terms and the semipositone structure is allowed. The result is established via degree theory.

Result 5: In [44], the authors extend the result in [48] for the superlinear case when the region is a ball. In particular, they consider a system of the form

$$-(r^{n-1}u')' = \lambda r^{n-1} f(v); \quad 0 < r < 1$$
$$-(r^{n-1}v')' = \lambda r^{n-1} g(u); \quad 0 < r < 1$$
$$u'(0) = v'(0) = 0; \quad u(1) = v(1) = 0,$$

where $\lambda > 0$ is a parameter, $f, g : [0, \infty) \to \mathbb{R}$ are continuous and

(C1) $\lim\limits_{s \to \infty} \frac{f(s)}{s} = \infty$, $\lim\limits_{s \to \infty} \frac{g(s)}{s} = \infty$

(C2) there exists nonnegative numbers α, β with $\alpha + \beta = n - 2$, and a positive number ϵ such that $nF(s) - \alpha s f(s) \geq \epsilon s f(s)$ and $nG(s) - \beta s f(s) \geq \epsilon s g(s)$ for s large. Here $F(s) = \int_0^s f(t)\, dt$, $G(s) = \int_0^s g(t)\, dt$.

They prove that the system has a positive solution (u_λ, v_λ) for λ small with $|(u_\lambda, v_\lambda)|_\infty \to \infty$ as $\lambda \to 0$ via degree theory.

We note here that the common feature of Results 2, 4 and 5 is that solutions of large supremum norm are obtained and then prove positivity of such solutions (both in single equation and systems case).

Result 6: In [49], again for the superlinear case a non-existence result is proven. In particular, they consider the system

$$-\Delta u = \lambda f(v); \quad \Omega$$
$$-\Delta v = \lambda g(u); \quad \Omega$$
$$u = 0 = v; \quad \partial\Omega,$$

where Ω is a ball or an annulus in $\mathbb{R}^n$, $\lambda > 0$ is a parameter and $f, g : [0, \infty) \to \mathbb{R}$ are continuous, $f(0) < 0$, $g(0) < 0$; $f' \geq 0$, $g' \geq 0$ and

(D1) There exists $K_i > 0$, $M_i > 0$; $i = 1, 2$ such that $f(z) \geq K_1 z - M_1$ and $g(z) \geq K_2 z - M_2$ for $z \geq 0$.

Then they prove that the system has no nonnegative solutions for λ large. They prove the result by analyzing the solutions near the outer boundary and using comparison principles.

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