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A PARTIAL SOLUTION FOR FEYNMAN'S PROBLEM: A NEW DERIVATION OF THE WEYL EQUATION

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ABSTRACT. Associating classical mechanics to a system of PDE, we give a procedure for Feynman type quantization of a "Schrödinger type equation with spin." Mathematically, we construct a "good parametrix" for the Weyl equation with an external electro-magnetic field. Main ingredients are a new interpretation of the matrix structure using superanalysis and a reinterpretation of the method of characteristics as a quantization procedure of Feynman type.

1. Feynman's problem for spin

1.1. **Feynman's path integral representation and his problem.** Feynman proposed to represent solutions of the Schrödinger equation

$$
\begin{gathered}
i\hbar\frac{\partial}{\partial t}u(t,q) = H(q,\partial_q)u(t,q),\\
u(0,q) = \underline{u}(q), \quad q = (q_1,\cdots,q_m) \in \mathbb{R}^m,
\end{gathered}
\tag{1.1}
$$

via the expression, called Feynman's path integral (representation),

$$
F(t,q,q') = \int_{C_{t,q,q'}} d_F\gamma\, e^{i\hbar^{-1}\int_0^t L(\gamma(\tau),\dot\gamma(\tau))d\tau}. \tag{1.2}
$$

Here $H(q,\partial_q)$, the Hamiltonian operator with mass M, is given formally as

$$
H(q,\partial_q) = \frac{-\hbar^2}{2M}\Delta + V(\cdot), \quad \Delta = \sum_{\ell=1}^m \frac{\partial^2}{\partial q_\ell^2},
$$

and $d_F\gamma$ denotes the notorious Feynman measure on the path space

$$
C_{t,q,q'} = \{\gamma(\cdot)\in AC([0,t]:\mathbb{R}^m)\,|\,\gamma(0)=q',\gamma(t)=q\}.
$$

Here, AC denotes absolute continuity. For any path $\gamma \in C_{t,q,q'}$, the classical action $S_t(\gamma)$ is defined by

$$
S_t(\gamma) = \int_0^t d\tau\, L(\gamma(\tau),\dot\gamma(\tau)) \tag{1.3}
$$

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where the Lagrangian function

$$L(\gamma, \dot\gamma) = \frac{M}{2}|\dot\gamma|^2 - V(\gamma) \in C^\infty(T\mathbb{R}^m : \mathbb{R}),$$

corresponds to the Hamiltonian function

$$H(q,p) = \frac{|p|^2}{2M} + V(q) \in C^\infty(T^*\mathbb{R}^m : \mathbb{R}).$$

On the other hand, Feynman noted ([6], page 355) that

> ...... path integrals suffer grievously from a serious defect. They do not permit a discussion of spin operators or other such operators in a simple and lucid way. They find their greatest use in systems for which coordinates and their conjugate momenta are adequate. Nevertheless, spin is a simple and vital part of real quantum-mechanical systems. It is a serious limitation that the half-integral spin of the electron does not find a simple and ready representation. It can be handled if the amplitudes and quantities are considered as quaternions instead of ordinary complex numbers, but the lack of commutativity of such numbers is a serious complication.

[**Problem for system of PDE**]: We regard Feynman's problem as calling for a **new methodology of solving systems of PDE**. A system of PDE has **two non-commutativities**,

(i) one from $[\partial_q, q] = 1$ (Heisenberg relation),

(ii) the other from $[A, B] \neq 0$ (matrix noncommutativity).

Non-commutativity from the Heisenberg relation is nicely controlled by using Fourier transformations (the theory of pseudodifferential operators). Here, we want to give a **new method of treating matrix non-commutativity**; after identifying matrix operations as differential operators and using Fourier transformations, we may develop a theory of pseudodifferential operators for supersmooth functions on superspace $\mathfrak{R}^{m|n}$.

Opinion. For a given system of PDE, if we may reduce that system to scalar PDEs by diagonalization, then we doubt whether it is truly necessary to use a matrix representation. Therefore, if we need to represent some equations using matrices, we should try to treat a system of PDE as it is, **without diagonalization**. (Recall the Witten model, which is represented by two independent-looking equations, has supersymmetry if treated as a system.)

Remark. We may consider the method employed here as an attempt to **extend the "method of characteristics" to PDE with matrix-valued coefficients**.

1.2. **Method of Fujiwara.** Unfortunately, the Feynman measure does not exist. On the other hand, Fujiwara [7, 8] constructed the parametrix and the fundamental solution of (1.1) using Feynman's arguments conversely, that is, he made a part of the argument of Feynman mathematically rigorous.

Let $\sup_{q\in\mathbb{R}^m} |D^\alpha V(q)| \le C_\alpha$, for $|\alpha| \ge 2$. Then there exists a unique path γ_0 in $C_{t,q,q'}$ such that

$$\inf_{\gamma\in C_{t,q,q'}} S_t(\gamma) = S_t(\gamma_0) = S_L(t,q,q'),$$

which gives a solution of the **Hamilton-Jacobi equation**:

$$\frac{\partial}{\partial t}S + H\left(q, \frac{\partial S}{\partial q}\right) = 0.$$

Introducing the **van Vleck determinant**

$$D_L(t, q, q') = \det\left(\partial_{q_i}\partial_{q'_j} S_L(t, q, q')\right),$$

which is a solution of the **continuity equation**

$$\frac{\partial}{\partial t}D_L + \frac{\partial}{\partial q}\left(D_L H_p\left(q, \frac{\partial S}{\partial q}\right)\right) = 0,$$

he defined

$$(F_t u)(q) = (2\pi i\hbar)^{-m/2}\int_{\mathbb{R}^m} dq'\, D_L^{1/2}(t, q, q') e^{i\hbar^{-1}S_L(t,q,q')}u(q'). \tag{1.4}$$

Theorem 1.1 (Fujiwara [7]). *Fix $0 < T < \infty$ arbitrarily. Putting $\mathbb{H} = L^2(\mathbb{R}^m : \mathbb{C})$ and denoting the set of bounded linear operators on $\mathbb{H}$ by $\mathcal{B}(\mathbb{H})$, we have the following:*

(1) F_t defines a bounded linear operator in $\mathbb{H}$:

$$\|F_t u\| \le C\|u\|.$$

(2) For any $u \in L^2(\mathbb{R}^m : \mathbb{C})$, t, s, $t + s \in [-T, T]$,

$$\lim_{t\to 0}\|F_t u - u\| = 0,$$

$$i\hbar\,\frac{\partial}{\partial t}(F_t u)(q)\bigg|_{t=0} = H(q, \partial_q)u(q),$$

$$\|F_{t+s} - F_t F_s\| \le C(t^2 + s^2).$$

(3) Moreover, there exists a limit $\lim_{k\to\infty}(F_{t/k})^k = E_t$ in $\mathcal{B}(\mathbb{H})$, i.e., in the operator norm of $L^2(\mathbb{R}^m : \mathbb{C})$, which satisfies the initial value problem below:

$$i\hbar\frac{\partial}{\partial t}(E_t u)(q) = H(q, \partial_q)(E_t u)(q),$$

$$(E_0 u)(q) = \underline{u}(q).$$

Remarks. (i) In the above, L^2-boundedness of the operator F_t is crucial and is proved using Cotlar's lemma. This usage makes it difficult to prove the boundedness of analogous operators in curved space. Therefore, it is an open problem to "quantize" the Lagrangian on a curved manifold. The above procedure of Fujiwara was modified for the heat equation on a curved manifold by Inoue-Maeda [18] to explain mathematically the origin of the term $(1/12)R$, where R is the scalar curvature of the configuration manifold.
(ii) Though in his papers [7, 8], Fujiwara allowed the time-dependence of $V(t, q)$, but we disregard the time-dependence in Theorem 1.1 for descriptional simplicity.

We call (1.4) a "**good parametrix**" since it has the following properties:
(1) The parametrix has explicit dependence on the classical quantities, i.e., Bohr correspondence is exemplified. As a by-product, we may check easily whether we may derive the heat equation from the Schrödinger equation by replacing $\hbar$ in (1.4) with $-i$.
(2) The infinitesimal generator of the parametrix gives a Hamiltonian operator

$H(q, \partial_q)$ which corresponds to the quantization of the Lagrangian $L(\gamma, \dot{\gamma})$, or the Weyl quantization for the symbol $H(q, p)$.

[**Problem 1**] In (1.2) and (1.4), the Lagrangian is used. How can we connect the above procedure directly to the Hamiltonian without using the Lagrangian? (A partial solution of this problem is now given in [15].)

[**Problem 2**] How we may proceed when
(1) V has singularities like Coulomb potentials? or
(2) there exist many paths connecting points q and q' like the dynamics on the circle?

2. Problem and Result

[**Problem**]: Find a **good representation** of

$$\mathbb{R} \times \mathbb{R}^3 \ni (t, q) \to \psi(t, q) = \begin{pmatrix} \psi_1(t, q) \\ \psi_2(t, q) \end{pmatrix} \in \mathbb{C}^2$$

satisfying the Weyl equation with the time-dependent external electro-magnetic field

$$\begin{gathered} i\hbar \frac{\partial}{\partial t} \psi(t, q) = \mathbb{H}(t)\psi(t, q), \\ \psi(\underline{t}, q) = \underline{\psi}(q), \end{gathered} \tag{2.1}$$

where

$$\mathbb{H}(t) = c \sum_{j=1}^{3} \boldsymbol{\sigma}_j \left(\frac{\hbar}{i} \frac{\partial}{\partial q_j} - \frac{\varepsilon}{c} A_j(t, q) \right) - \varepsilon A_0(t, q),$$

and the Pauli matrices $\{\boldsymbol{\sigma}_j\}$ are represented by

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(i) **What is the classical mechanics corresponding to** (2.1)**?**
(ii) **How about its quantization?**

[**Reduction**]: We introduce **new independent variables**, called **odd** or **fermion variables**, which are used to represent matrix structure (see the appendix on elements of superanalysis).

Then, we may reduce (2.1) to the super Weyl equation on $\mathfrak{R}^{3|2}$:

$$\begin{gathered} i\hbar \frac{\partial}{\partial t} u(t, x, \theta) = \hat{\mathcal{H}}(t) u(t, x, \theta), \\ u(\underline{t}, x, \theta) = \underline{u}(x, \theta), \end{gathered} \tag{2.2}$$

where

$$\hat{\mathcal{H}}(t) = c \sum_{j=1}^{3} \sigma_j \left(\theta, \frac{\partial}{\partial \theta} \right) \left(\frac{\hbar}{i} \frac{\partial}{\partial x_j} - \frac{\varepsilon}{c} A_j(t, x) \right) - \varepsilon A_0(t, x).$$

Remark. Pauli claimed that "**There exists no classical counter-part for a quantum spinning particle.**" In spite of this, we present another representation which exhibits the "**underlying Classical Mechanics.**"

Theorem 2.1 (A good parametrix for the Weyl equation)**.** *Assume that* $\sup|(1+|q|)^{|\alpha|-1}\partial_q^\alpha A_j(t,q)|<\infty$. *Then,*

$$\psi(t,q)=\flat\Big((2\pi i\hbar)^{-3/2}\hbar\iint_{\mathfrak{R}^{3|2}}d\underline{\xi}d\underline{\pi}\mathcal{D}^{1/2}(t,\underline{t};\underline{x},\underline{\theta},\underline{\xi},\underline{\pi}) \times e^{i\hbar^{-1}\mathcal{S}(t,\underline{t};\underline{x},\underline{\theta},\underline{\xi},\underline{\pi})}\mathcal{F}(\#\underline{\psi})(\underline{\xi},\underline{\pi})\Big)\Big|_{\underline{x}_{\mathrm{B}}=q},$$

gives a **good parametrix** *for* (2.1). *Here,* $\mathcal{S}(t,\underline{t};\underline{x},\underline{\theta},\underline{\xi},\underline{\pi})$ *and* $\mathcal{D}(t,\underline{t};\underline{x},\underline{\theta},\underline{\xi},\underline{\pi})$ *are solutions of Hamilton-Jacobi and continuity equations, respectively.* $\mathcal{F}$ *is the Fourier transformation of functions on* $\mathfrak{R}^{3|2}$.

3. Main ingredients

3.1. **How can we regard the method of characteristics as quantization?** On the region Ω in $\mathbb{R}^{m+1}$, we consider the following initial value problem:

$$\begin{gathered}\frac{\partial}{\partial t}u(t,q)+\sum_{j=1}^m a_j(t,q)\frac{\partial}{\partial q_j}u(t,q)=b(t,q)u(t,q)+f(t,q),\\ u(\underline{t},q)=\underline{u}(q).\end{gathered}\tag{3.1}$$

Corresponding characteristics are given by

$$\begin{gathered}\frac{d}{dt}q_j(t)=a_j(t,q(t)),\\ q_j(\underline{t})=\underline{q}_j\quad(j=1,\cdots,m).\end{gathered}$$

We denote the solution by

$$q(t)=q(t,\underline{t};\underline{q})=(q_1(t),\cdots,q_m(t))\in\mathbb{R}^m.$$

The following theorem is well-known.

Theorem 3.1. *Let* $a_j\in C^1(\Omega:\mathbb{R})$ *and* $b,f\in C(\Omega:\mathbb{R})$. *For any point* $(\underline{t},\underline{q})\in\Omega$, *we assume that* $\underline{u}$ *is* C^1 *in a neighborhood of* $\underline{q}$.

Then, in a neighborhood of $(\underline{t},\underline{q})$, *there exists a unique solution* $u(t,q)$ *of* (3.1). *More precisely, putting*

$$U(t,\underline{q})=e^{\int_{\underline{t}}^t d\tau\,B(\tau,\underline{q})}\left\{\int_{\underline{t}}^t ds\,e^{-\int_{\underline{t}}^s d\tau\,B(\tau,\underline{q})}F(s,\underline{q})+\underline{u}(\underline{q})\right\},$$

that solution is represented by

$$u(t,\bar{q})=U(t,y(t,\underline{t};\bar{q}))$$

where $B(t,\underline{q})=b(t,q(t,\underline{t};\underline{q}))$, $F(t,\underline{q})=f(t,q(t,\underline{t};\underline{q}))$ *and* $\underline{q}=y(t,\underline{t};\bar{q})$ *is an inverse function defined through* $\bar{q}=q(t,\underline{t};\underline{q})$.

To understand the above theorem, we take the simplest example:

$$\begin{gathered}i\hbar\frac{\partial}{\partial t}u(t,q)=a\frac{\hbar}{i}\frac{\partial}{\partial q}u(t,q)+bqu(t,q),\\ u(0,q)=\underline{u}(q).\end{gathered}\tag{3.2}$$

From the right-hand side of above, we derive the (Weyl) symbol as a Hamiltonian:

$$H(q,p)=e^{-i\hbar^{-1}qp}\Big(a\frac{\hbar}{i}\frac{\partial}{\partial q}+bq\Big)e^{i\hbar^{-1}qp}=ap+bq.$$

The classical mechanics associated to that Hamiltonian is given by

$$\dot{q}(t) = H_p = a,$$
$$\dot{p}(t) = -H_q = -b$$

with $\begin{pmatrix} q(0) \\ p(0) \end{pmatrix} = \begin{pmatrix} \underline{q} \\ \underline{p} \end{pmatrix}$ which is readily solved as

$$q(s) = \underline{q} + as, \quad p(s) = \underline{p} - bs.$$

Putting $U(t) = u(t, q(t))$ for a solution $u(t,q)$ of (3.2), we have $i\hbar \frac{d}{dt} U(t) = bq(t)U(t)$, therefore

$$U(t, \underline{q}) = \underline{u}(\underline{q}) e^{-i\hbar^{-1}(b\underline{q}t + 2^{-1}abt^2)}.$$

As the inverse function of $\overline{q} = q(t, \underline{q})$ is given by $\underline{q} = y(t, \bar{q}) = \bar{q} - at$, we get

$$u(t, \overline{q}) = U(t, \underline{q})\big|_{\underline{q}=y(t,\bar{q})} = \underline{u}(\overline{q} - at) e^{-i\hbar^{-1}(b\overline{q}t - 2^{-1}abt^2)}.$$

Remark. In the above, we used only the path $q(s)$ moving in the configuration space $\mathbb{R}^m$.

Another point of view from the "Hamiltonian path-integral method": Put

$$S_0(t, \underline{q}, \underline{p}) = \int_0^t ds\, [\dot{q}(s)p(s) - H(q(s), p(s))] = -b\underline{q}t - 2^{-1}abt^2,$$

and

$$S(t, \overline{q}, \underline{p}) = \underline{q}\,\underline{p} + S_0(t, \underline{q}, \underline{p})\big|_{\underline{q}=y(t,\overline{q})} = \overline{q}\underline{p} - a\underline{p}t - b\overline{q}t + 2^{-1}abt^2.$$

Then, $S(t, \bar{q}, \underline{p})$ satisfies the Hamilton-Jacobi equation.

$$\frac{\partial}{\partial t} S + H(\bar{q}, \partial_{\bar{q}} S) = 0,$$
$$S(0, \bar{q}, \underline{p}) = \bar{q}\,\underline{p}.$$

On the other hand, the van Vleck determinant, a scalar in this example, is

$$D(t, \overline{q}, \underline{p}) = \frac{\partial^2 S(t, \overline{q}, \underline{p})}{\partial \overline{q} \partial \underline{p}} = 1.$$

This quantity satisfies the continuity equation:

$$\frac{\partial}{\partial t} D + \partial_{\bar{q}}(DH_p) = 0,$$
$$D(0, \overline{q}, \underline{p}) = 1$$

where $H_p = \frac{\partial H}{\partial p}\big(\bar{q}, \frac{\partial S}{\partial \bar{q}}\big)$.

As an interpretation of Feynman's idea, we understand the transition from classical to quantum mechanics by studying the quantity

$$u(t, \overline{q}) = (2\pi i\hbar)^{-1/2} \int_{\mathbb{R}} d\underline{p}\, D^{1/2}(t, \overline{q}, \underline{p}) e^{i\hbar^{-1}S(t,\overline{q},\underline{p})} \hat{\underline{u}}(\underline{p}).$$

That is, in our case at hand, we should study the quantity defined by

$$\begin{aligned} u(t, \overline{q}) &= (2\pi i\hbar)^{-1/2} \int_{\mathbb{R}} d\underline{p}\, e^{i\hbar^{-1}S(t,\overline{q},\underline{p})} \hat{\underline{u}}(\underline{p}) \\ &= (2\pi i\hbar)^{-1} \iint_{\mathbb{R}^2} d\underline{p} d\underline{q}\, e^{i\hbar^{-1}(S(t,\overline{q},\underline{p}) - \underline{q}\,\underline{p})} \underline{u}(\underline{q}) \\ &= \underline{u}(\overline{q} - at) e^{i\hbar^{-1}(-b\overline{q}t + 2^{-1}abt^2)}. \end{aligned}$$

[**Problem**] Can we extend the above argument to a system of PDEs? For example, Dirac, Weyl or Pauli equations, quantum mechanical equations with spin.

Remark. Using the representation theory of the Heisenberg group, the procedure of geometric quantization produces the operator $e^{-i\hbar^{-1}t(a\boldsymbol{P}+b\boldsymbol{Q}+c\boldsymbol{I})}$ corresponding to the Hamiltonian $ap+bq+c$.

3.2. **A new look for the matrix structure.** The fact "Clifford algebra is represented on Grassmann algebra" is embodied as follows (here, we restrict ourselves to 2×2 matrices):

We decompose a 2×2 matrix A as follows:

$$\begin{aligned}
A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
&= \frac{a+d}{2}\mathbb{I}_2 + \frac{a-d}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{b+c}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{b-c}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \frac{a+d}{2}\mathbb{I}_2 + \frac{a-d}{2}\boldsymbol{\sigma}_3 + \frac{b+c}{2}\boldsymbol{\sigma}_1 + \frac{b-c}{2}i\boldsymbol{\sigma}_2.
\end{aligned}$$

where the Pauli matrices $\{\boldsymbol{\sigma}_j\}$ satisfies the **Clifford relation**: $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_j+\boldsymbol{\sigma}_j\boldsymbol{\sigma}_i=2\delta_{ij}$.

Using the **identification**

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \underset{\flat}{\overset{\sharp}{\rightleftarrows}} u_0+u_1\theta_1\theta_2 = u(\theta)$$

we have, for example,

$$\flat\Big(\theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2}\Big)\sharp\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \flat(u_1+u_0\theta_1\theta_2) = \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = \boldsymbol{\sigma}_1\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Therefore, the action of A on a vector ${}^t(u_0,u_1)$ is regarded as the action of a differential operator

$$\mathcal{A}\Big(\theta,\frac{\partial}{\partial\theta}\Big) = \frac{a+d}{2} + \frac{a-d}{2}\Big(1-\theta_1\frac{\partial}{\partial\theta_1}-\theta_2\frac{\partial}{\partial\theta_2}\Big) + c\theta_1\theta_2 - b\frac{\partial^2}{\partial\theta_1\partial\theta_2}$$

on a function $u(\theta)=u_0+u_1\theta_1\theta_2$. Moreover, we may associate the complete Weyl symbol $\mathcal{A}(\theta,\pi)$ as

$$\mathcal{A}(\theta,\pi) = \frac{a+d}{2} - i\frac{a-d}{2}\hbar^{-1}\big(\theta_1\pi_1+\theta_2\pi_2\big) + c\theta_1\theta_2 + b\hbar^{-2}\pi_1\pi_2.$$

This is our interpretation of the fact above. (Though we may define $\mathcal{A}(\theta,\partial_\theta)$ and the Fourier transformations w.r.t. odd variables such that the symbol $\mathcal{A}(\theta,\pi)$ doesn't contain $\hbar$, we take this form in this paper.)

Remark. $\sigma_j(\theta,\partial_\theta)$ are taken not only to be even operators but also to annihilate the set $\{v_1\theta_1+v_2\theta_2\}$.

Remark. We may regard $\theta_j\sim dz_j$ and $\frac{\partial}{\partial\theta_j}\sim\frac{\partial}{\partial z_j}\rfloor$, i.e., for $j=1,2$,

$$\begin{aligned}
\theta_j(u_0+u_1\theta_1\theta_2) = u_0\theta_j &\sim dz_j\wedge(u_0+u_1dz_1\wedge dz_2) = u_0dz_j, \\
\frac{\partial}{\partial\theta_1}(u_0+u_1\theta_1\theta_2) = u_1\theta_2 &\sim \frac{\partial}{\partial z_1}\rfloor(u_0+u_1dz_1\wedge dz_2) = u_1dz_2, \\
\frac{\partial}{\partial\theta_2}(u_0+u_1\theta_1\theta_2) = -u_1\theta_1 &\sim \frac{\partial}{\partial z_2}\rfloor(u_0+u_1dz_1\wedge dz_2) = -u_1dz_1.
\end{aligned}$$

4. Outline of our procedure

For the sake of simplicity, we outline the case when $\mathbb{H}(t)$ is independent of time t.

(I) We identify a "spinor" $\psi(t,q) = {}^t(\psi_1(t,q), \psi_2(t,q)) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}^2$ with an even supersmooth function $u(t,x,\theta) = u_0(t,x)+u_1(t,x)\theta_1\theta_2 : \mathbb{R}\times\mathfrak{R}^{3|2} \to \mathfrak{C}_{\mathrm{ev}}$. Here, $u_0(t,x)$, $u_1(t,x)$ are the Grassmann continuation of $\psi_1(t,q)$, $\psi_2(t,q)$, respectively.

(II) We represent the matrices $\{\boldsymbol{\sigma}_j\}$, which act on $u(t,x,\theta)$ as follows:

$$\begin{gathered}
\sigma_1\Big(\theta,\frac{\partial}{\partial\theta}\Big) = \theta_1\theta_2 - \frac{\partial^2}{\partial\theta_1\partial\theta_2},\\
\sigma_2\Big(\theta,\frac{\partial}{\partial\theta}\Big) = i\Big(\theta_1\theta_2 + \frac{\partial^2}{\partial\theta_1\partial\theta_2}\Big),\\
\sigma_3\Big(\theta,\frac{\partial}{\partial\theta}\Big) = 1 - \theta_1\frac{\partial}{\partial\theta_1} - \theta_2\frac{\partial}{\partial\theta_2}.
\end{gathered}$$

(III) Therefore, we may correspond the differential operator given by

$$\mathcal{H}\Big(x,\frac{\hbar}{i}\frac{\partial}{\partial x},\theta,\frac{\partial}{\partial\theta}\Big) = c\sum_{j=1}^{3}\sigma_j\Big(\theta,\frac{\partial}{\partial\theta}\Big)\Big(\frac{\hbar}{i}\frac{\partial}{\partial x_j} - \frac{\varepsilon}{c}A_j(x)\Big) + \varepsilon A_0(x) \tag{4.1}$$

which yields the superspace version of the Weyl equation

$$\begin{gathered}
i\hbar\frac{\partial}{\partial t}u(t,x,\theta) = \mathcal{H}\Big(x,\frac{\hbar}{i}\frac{\partial}{\partial x},\theta,\frac{\partial}{\partial\theta}\Big)u(t,x,\theta),\\
u(0,x,\theta) = u(x,\theta).
\end{gathered} \tag{4.2}$$

Moreover, the "complete Weyl symbol" of (4.1) is given by

$$\begin{aligned}
\mathcal{H}(x,\xi,\theta,\pi) =& c(\eta_1+i\eta_2)\theta_1\theta_2 + c\hbar^{-2}(\eta_1-i\eta_2)\pi_1\pi_2\\
&- ic\hbar^{-1}\eta_3(\theta_1\pi_1+\theta_2\pi_2) + \varepsilon A_0(x),
\end{aligned}$$

where $\eta_j = \xi_j - (\varepsilon/c)A_j(x)$.

(IV) We consider the classical mechanics corresponding to $\mathcal{H}(x,\xi,\theta,\pi)$ given by

$$\begin{gathered}
\frac{d}{dt}x_j = \frac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\xi_j},\ \frac{d}{dt}\xi_k = -\frac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial x_k},\\
\frac{d}{dt}\theta_l = -\frac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\pi_l},\ \frac{d}{dt}\pi_m = -\frac{\partial\mathcal{H}(x,\xi,\theta,\pi)}{\partial\theta_m}.
\end{gathered}$$

Proposition 4.1. *There exists a unique global solution $(x(t),\xi(t),\theta(t),\pi(t))$ of above ODE with initial data $(x(0),\xi(0),\theta(0),\pi(0)) = (\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) \in \mathfrak{R}^{6|4} = \mathcal{T}^*\mathfrak{R}^{3|2}$.*

Proposition 4.2. *For any fixed $(t,\underline{\xi},\underline{\pi})$, the map defined by*

$$(\underline{x},\underline{\theta}) \to (\underline{x} = x(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}), \underline{\theta} = \theta(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}))$$

gives a supersmooth diffeomorphism from $\mathfrak{R}^{3|2} \to \mathfrak{R}^{3|2}$. Therefore, there exists the inverse map given by

$$(\underline{x},\underline{\theta}) \to (\underline{x} = y(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}), \underline{\theta} = \omega(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})),$$

which satisfies

$$\begin{gathered}
\underline{x} = x(t,y(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\xi},\omega(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\pi}),\\
\underline{\theta} = \theta(t,y(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\xi},\omega(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\pi}),\\
\underline{x} = y(t,x(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\xi},\theta(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\pi}),\\
\underline{\theta} = \omega(t,x(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\xi},\theta(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\underline{\pi}).
\end{gathered}$$

Now, we put

$$\mathcal{S}_0(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = \int_0^t ds\,\{\langle\dot{x}(s)|\xi(s)\rangle + \langle\dot{\theta}(s)|\pi(s)\rangle - \mathcal{H}(x(s),\xi(s),\theta(s),\pi(s))\},$$

and

$$\mathcal{S}(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = \langle\underline{x}|\underline{\xi}\rangle + \langle\underline{\theta}|\underline{\pi}\rangle + \mathcal{S}_0(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})\Big|_{\underline{x}=y(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}),\ \underline{\theta}=\omega(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})}$$

Proposition 4.3. *$\mathcal{S}(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$ satisfies the Hamilton-Jacobi equation:*

$$\frac{\partial}{\partial t}\mathcal{S}(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) + \mathcal{H}\Big(x,\frac{\partial\mathcal{S}}{\partial\bar{x}},\bar{\theta},\frac{\partial\mathcal{S}}{\partial\bar{\theta}}\Big) = 0,$$
$$\mathcal{S}(0,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = \langle\bar{x}|\underline{\xi}\rangle + \langle\bar{\theta}|\underline{\pi}\rangle.$$

Remark. This process of constructing solution of H-J equation is essentially due to Jacobi.

Now, we put

$$\mathcal{D}(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = \mathrm{sdet}\begin{pmatrix} \frac{\partial^2\mathcal{S}}{\partial\bar{x}\,\partial\underline{\xi}} & \frac{\partial^2\mathcal{S}}{\partial\bar{x}\,\partial\underline{\pi}} \\ \frac{\partial^2\mathcal{S}}{\partial\bar{\theta}\,\partial\underline{\xi}} & \frac{\partial^2\mathcal{S}}{\partial\bar{\theta}\,\partial\underline{\pi}} \end{pmatrix}.$$

Then, we get

Proposition 4.4. *$\mathcal{D}(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$ satisfies the following continuity equation:*

$$\frac{\partial}{\partial t}\mathcal{D} + \frac{\partial}{\partial\bar{x}}\Big(\mathcal{D}\frac{\partial\mathcal{H}}{\partial\xi}\Big) + \frac{\partial}{\partial\bar{\theta}}\Big(\mathcal{D}\frac{\partial\mathcal{H}}{\partial\pi}\Big) = 0,$$
$$\mathcal{D}(0,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) = 1.$$

In the above, the argument of $\mathcal{D}$ is $(t,\underline{x},\underline{\xi},\underline{\theta},\underline{\pi})$, and those of $\frac{\partial\mathcal{H}}{\partial\xi}$ and $\frac{\partial\mathcal{H}}{\partial\pi}$ are $(\bar{x},\frac{\partial\mathcal{S}}{\partial\bar{x}},\bar{\theta},\frac{\partial\mathcal{S}}{\partial\bar{\theta}})$, respectively.

From here, we change the order of variables

$$(\underline{x},\underline{\xi},\underline{\theta},\underline{\pi}) \to (\underline{x},\underline{\theta},\underline{\xi},\underline{\pi}).$$

We define an operator

$$(\mathcal{U}(t)u)(\bar{x},\bar{\theta}) = (2\pi i\hbar)^{-3/2}\hbar\iint d\underline{\xi}d\underline{\pi}\,\mathcal{D}^{1/2}(t,\underline{x},\underline{\theta},\underline{\xi},\underline{\pi})e^{i\hbar^{-1}\mathcal{S}(t,\underline{x},\underline{\theta},\underline{\xi},\underline{\pi})}\mathcal{F}u(\underline{\xi},\underline{\pi}).$$

The function $u(t,\bar{x},\bar{\theta}) = (\mathcal{U}(t)\underline{u})(\bar{x},\bar{\theta})$ will be shown to be a desired good parametrix for (2.2), identical with (4.2).

(V) On the other hand, using Fourier transformation, we have readily that

$$\mathcal{H}\Big(x,\frac{\hbar}{i}\frac{\partial}{\partial x},\theta,\frac{\partial}{\partial\theta}\Big) = \hat{\mathcal{H}}$$

where $\hat{\mathcal{H}}$ is a (Weyl type) pseudo-differential operator with symbol $\mathcal{H}(x,\xi,\theta,\pi)$, that is,

$$(\hat{\mathcal{H}}u)(x,\theta)$$
$$= (2\pi i\hbar)^{-3}\hbar^2\iint d\xi d\pi dy d\omega\, e^{i\hbar^{-1}(\langle x-y|\xi\rangle+\langle\theta-\omega|\pi\rangle)}\mathcal{H}\Big(\frac{x+y}{2},\xi,\frac{\theta+\omega}{2},\pi\Big)u(y,\omega).$$

Theorem 4.5. *(1) For $t \in \mathbb{R}$, $\mathcal{U}(t)$ is a bounded operator in $\mathcal{L}^2_{SS}(\mathfrak{R}^{3|2})$ such that for $|t|, |s| \ll 1$,*

$$\|\mathcal{U}(t)\mathcal{U}(s) - \mathcal{U}(t+s)\|_{\mathbb{B}(\mathcal{L}^2_{SS}(\mathfrak{R}^{3|2}), \mathcal{L}^2_{SS}(\mathfrak{R}^{3|2}))} \le C(|t|^2 + |s|^2).$$

(2) $\mathbb{R} \ni t \to \mathcal{U}(t) \in \mathbb{B}(\mathcal{L}^2_{SS}(\mathfrak{R}^{3|2}), \mathcal{L}^2_{SS}(\mathfrak{R}^{3|2}))$ is continuous.
(3) There exists the limit $\mathcal{E}(t) = \lim_{n\to\infty} \mathcal{U}(t/n)$ for any $|t| < \infty$ in the uniform operator topology.
(4) Put $u(t, \underline{x}, \underline{\theta}) = (\mathcal{E}(t)\underline{u})(\underline{x}, \underline{\theta})$, for $\underline{u} \in \mathcal{C}_{SS}(\mathfrak{R}^{3|2})$ with $\underline{u}_0(q)$, $\underline{u}_1(q)$ being compactly supported. Then

$$i\hbar \frac{\partial}{\partial t} u(t, \underline{x}, \underline{\theta}) = \hat{\mathcal{H}} u(t, \underline{x}, \underline{\theta}),$$
$$u(0, \underline{x}, \underline{\theta}) = \underline{u}(\underline{x}, \underline{\theta}).$$

(VI) We interpret the above theorem using the identification maps

$$L^2(\mathbb{R}^3 : \mathbb{C}^2) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} \mathcal{L}^2_{SS}(\mathfrak{R}^{3|2}).$$

That is, remarking $\flat\hat{\mathcal{H}}\sharp = \mathbb{H}$ and putting $\mathbb{E}(t) = \flat\mathcal{E}(t)\sharp$, we have

Theorem 4.6. *(1) For $t \in \mathbb{R}$, $\mathbb{E}(t)$ is a well defined unitary operator in $L^2(\mathbb{R}^3 : \mathbb{C}^2)$.*
(2) (i) $\mathbb{R} \ni t \to \mathbb{E}(t) \in \mathbb{B}(L^2(\mathbb{R}^3 : \mathbb{C}^2), L^2(\mathbb{R}^3 : \mathbb{C}^2))$ is continuous.
(ii) $\mathbb{E}(t)\mathbb{E}(s) = \mathbb{E}(t+s)$ for any $t, s \in \mathbb{R}$.
(iii) Put $\psi(t, q) = \flat\big(\mathcal{E}(t)\sharp\underline{\psi}\big)\big|_{x_{\mathrm{B}}=q}$, for $\underline{\psi} \in C_0^\infty(\mathbb{R}^3 : \mathbb{C}^2)$. Then

$$i\hbar \frac{\partial}{\partial t} \psi(t, q) = \mathbb{H}\psi(t, q),$$
$$\psi(0, q) = \underline{\psi}(q).$$

Corollary 4.7. *$\mathbb{H}$ is an essentially self-adjoint operator in $L^2(\mathbb{R}^3 : \mathbb{C}^2)$.*

Appendix A. Elements from superanalysis

In this appendix, we gather notion from superanalysis without proof. As is well-known, we use very heavily Taylor expansion, integration by parts, change of variables under integral sign and Fourier transformations when we perform real analysis on $\mathbb{R}^m$. After defining non-commutative numbers $\mathfrak{R}$ and $\mathfrak{C}$, called supernumbers, we develop such tools on superspace $\mathfrak{R}^{m|n}$. We must emphasize the reason why Rogers [34] used the Banach-Grassmann algebra B_∞, which is obtained from the sequence space ℓ^1 by defining the product with Grassmann relations. It is because not only she but also many others try to apply the general theory of differential calculus on Banach space, which is not valid on Fréchet space in general. On the other hand, de Witt [5] introduced his non-Hausdorff topology (called coarse topology) on his supermanifolds, which is scarcely used by mathematicians, because it seems difficult to find a firm foundation of differential calculus. Our topology is just between coarse and Banach topology, which guarantees to develop that calculus by the help of grading inherited from Grassmann relations, i.e., we develop the theory of differential calculus on the Frechét space with Grassmann grading.

A.1. **Supernumbers, superspaces and linear algebra.**

A.1.1. *Supernumbers.* For symbols $\{\sigma_j\}_{j=1}^{\infty}$ satisfying the Grassmann relation

$$\sigma_j\sigma_k + \sigma_k\sigma_j = 0, \quad j,k = 1,2,\cdots,$$

we put

$$\mathfrak{C} = \{X = \sum_{I\in\mathcal{I}} X_I\sigma^I \mid X_I \in \mathbb{C}\}$$

where

$$\mathcal{I} = \{I = (i_k) \in \{0,1\}^{\mathbb{N}} \mid |I| = \sum_k i_k < \infty\},$$

$$\sigma^I = \sigma_1^{i_1}\sigma_2^{i_2}\cdots, \quad I = (i_1, i_2, \cdots), \quad \sigma^{\tilde{0}} = 1, \quad \tilde{0} = (0,0,\cdots) \in \mathcal{I}.$$

Besides trivially defined linear operations of sums and scalar multiplications, we have a product operation in $\mathfrak{C}$: For

$$X = \sum_{J\in\mathcal{J}} X_J\sigma^J, \quad Y = \sum_{K\in\mathcal{I}} Y_K\sigma^K,$$

we put

$$XY = \sum_{I\in\mathcal{I}} (XY)_I\sigma^I \quad \text{with} \quad (XY)_I = \sum_{I=J+K} (-1)^{\tau(I;J,K)} X_J Y_K.$$

Here, $\tau(I;J,K)$ is an integer defined by

$$\sigma^J\sigma^K = (-1)^{\tau(I;J,K)}\sigma^I, \quad I = J+K.$$

Proposition A.1 ([21, 10])**.** *$\mathfrak{C}$ forms an ∞-dimensional Fréchet-Grassmann algebra over $\mathbb{C}$, that is, an associative, distributive and non-commutative ring with degree, which is endowed with the Fréchet topology.*

Remark. (1) We call this $\mathfrak{C}$ as super(complex)numbers. Degree in $\mathfrak{C}$ is defined by introducing subspaces

$$\mathfrak{C}^{[j]} = \{X = \sum_{I\in\mathcal{I}, |I|=j} X_I\sigma^I\} \quad \text{for} \quad j = 0,1,\cdots$$

which satisfy

$$\mathfrak{C} = \oplus_{j=0}^{\infty}\mathfrak{C}^{[j]}, \quad \mathfrak{C}^{[j]}\cdot\mathfrak{C}^{[k]} \subset \mathfrak{C}^{[j+k]}.$$

(2) Define

$$\mathrm{proj}_I(X) = X_I \quad \text{for} \quad X = \sum_{I\in\mathcal{I}} X_I\sigma^I \in \mathfrak{C}.$$

The topology in $\mathfrak{C}$ is given by $X \to 0$ in $\mathfrak{C}$ if and only if $\mathrm{proj}_I(X) \to 0$ in $\mathbb{C}$, for any $I \in \mathcal{I}$.

This topology is equivalent to the one introduced by the metric $\mathrm{dist}(X,Y) = \mathrm{dist}(X-Y)$ where $\mathrm{dist}(X)$ is defined by

$$\mathrm{dist}(X) = \sum_{I\in\mathcal{I}} \frac{1}{2^{r(I)}} \frac{|\,\mathrm{proj}_I(X)|}{1+|\,\mathrm{proj}_I(X)|} \quad \text{with} \quad r(I) = 1 + \frac{1}{2}\sum_{k=1}^{\infty} 2^k i_k \quad \text{for} \quad I \in \mathcal{I}.$$

(3) We introduce parity in $\mathfrak{C}$ by setting

$$p(X) = \begin{cases} 0 & \text{if } X = \sum_{I\in\mathcal{I},|I|=\mathrm{ev}} X_I\sigma^I, \\ 1 & \text{if } X = \sum_{I\in\mathcal{I},|I|=\mathrm{od}} X_I\sigma^I, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$X \in \mathfrak{C}$ is called homogeneous if it satisfies $p(X) = 0$ or $= 1$. We put

$$\mathfrak{C}_{\mathrm{ev}} = \oplus_{j=0}^{\infty} \mathfrak{C}^{[2j]} = \{X \in \mathfrak{C} \,|\, p(X) = 0\},$$
$$\mathfrak{C}_{\mathrm{od}} = \oplus_{j=0}^{\infty} \mathfrak{C}^{[2j+1]} = \{X \in \mathfrak{C} \,|\, p(X) = 1\},$$
$$\mathfrak{C} \cong \mathfrak{C}_{\mathrm{ev}} \oplus \mathfrak{C}_{\mathrm{od}} \cong \mathfrak{C}_{\mathrm{ev}} \times \mathfrak{C}_{\mathrm{od}}.$$

Analogous to $\mathfrak{C}$, we define, super(real)numbers as

$$\mathfrak{R} = \{X \in \mathfrak{C} \,|\, \pi_{\mathrm{B}} X \in \mathbb{R}\},\ \mathfrak{R}^{[j]} = \mathfrak{R} \cap \mathfrak{C}^{[j]},$$
$$\mathfrak{R}_{\mathrm{ev}} = \mathfrak{R} \cap \mathfrak{C}_{\mathrm{ev}}, \quad \mathfrak{R}_{\mathrm{od}} = \mathfrak{R} \cap \mathfrak{C}_{\mathrm{od}} = \mathfrak{C}_{\mathrm{od}},$$
$$\mathfrak{R} \cong \mathfrak{R}_{\mathrm{ev}} \oplus \mathfrak{R}_{\mathrm{od}} \cong \mathfrak{R}_{\mathrm{ev}} \times \mathfrak{R}_{\mathrm{od}}.$$

We introduced the body (projection) map π_{B} by

$$\pi_{\mathrm{B}} X = \mathrm{proj}_{\tilde{0}}(X) = X_{\tilde{0}} = X^{[0]} = X_{\mathrm{B}} \quad \text{for any} \quad X \in \mathfrak{C},$$

and the soul part X_{S} of X as

$$X_{\mathrm{S}} = X - X_{\mathrm{B}} = \sum_{|I| \geq 1} X_I \sigma^I.$$

A.1.2. *Superspaces.* We define the (real) superspace $\mathfrak{R}^{m|n}$ by

$$\mathfrak{R}^{m|n} = \mathfrak{R}_{\mathrm{ev}}^{m} \times \mathfrak{R}_{\mathrm{od}}^{n}.$$

The distance between $X, Y \in \mathfrak{R}^{m|n}$ is defined by,

$$\mathrm{dist}_{m|n}(X, Y) = \mathrm{dist}_{m|n}(X - Y)$$

with

$$\mathrm{dist}_{m|n}(X) = \sum_{j=1}^{m} \left(\sum_{I \in \mathcal{I}} \frac{1}{2^{r(I)}} \frac{|\,\mathrm{proj}_I(x_j)|}{1 + |\,\mathrm{proj}_I(x_j)|} \right) + \sum_{k=1}^{n} \left(\sum_{I \in \mathcal{I}} \frac{1}{2^{r(I)}} \frac{|\,\mathrm{proj}_I(\theta_k)|}{1 + |\,\mathrm{proj}_I(\theta_k)|} \right).$$

We use the following notation:

$$X = (X_A)_{A=1}^{m+n} = (x, \theta) \in \mathfrak{R}^{m|n} \quad \text{with}$$
$$x = (X_A)_{A=1}^{m} = (x_j)_{j=1}^{m} \in \mathfrak{R}^{m|0}, \quad \theta = (X_A)_{A=m+1}^{m+n} = (\theta_k)_{k=1}^{n} \in \mathfrak{R}^{0|n}, \text{etc.}$$

We generalize the body map π_{B} from $\mathfrak{R}^{m|n}$ or $\mathfrak{R}^{m|0}$ to $\mathbb{R}^m$ by putting,

$$X = (x, \theta) \in \mathfrak{R}^{m|n} \longrightarrow \pi_{\mathrm{B}} X = X_{\mathrm{B}} = (x_{\mathrm{B}}, 0) \cong x_{\mathrm{B}} = \pi_{\mathrm{B}} x = (\pi_{\mathrm{B}} x_1, \cdots, \pi_{\mathrm{B}} x_m) \in \mathbb{R}^m.$$

We call $x_j \in \mathfrak{R}_{\mathrm{ev}}$ and $\theta_k \in \mathfrak{R}_{\mathrm{od}}$ as even and odd (alias bosonic and ferminionic) variable, respectively.

Remark. Though the differential calculus on Fréchet spaces has some difficulties in general (e.g. Yamamuro [47]), such calculus on Fréchet-Grassmann algebra holds safely in our case, because of the grading of Grassmann generators. For example, the implicit and inverse function theorems, and the chain rule for differentiation. See, Inoue and Maeda [21], Inoue [10, 13, 17].

A.1.3. *Elementary Linear Algebra.*

Definition A.1. *A rectangular array M, whose cells are indexed by pairs consisting of a row number and a column number, is called a supermatrix and denoted by $M \in \mathrm{Mat}((m|n)\times(r|s):\mathfrak{C})$, if it satisfies the following:*

1. *A $(m+n)\times(r+s)$ matrix M is decomposed blockwisely as $M=\begin{bmatrix} A & C \\ D & B\end{bmatrix}$ where A, B, C and D are $m\times r$, $n\times s$, $m\times s$ and $n\times r$ matrices with elements in $\mathfrak{C}$, respectively.*
2. *One of the following conditions is satisfied: Either*
   - *$p(M)=0$, that is, $p(A_{jk})=0=p(B_{uv})$ and $p(C_{jv})=1=p(D_{uk})$ or*
   - *$p(M)=1$, that is, $p(A_{jk})=1=p(B_{uv})$ and $p(C_{jv})=0=p(D_{uk})$.*

We call M is even denoted by $\mathrm{Mat}_{\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})$ (resp. odd denoted by $\mathrm{Mat}_{\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})$) if $p(M)=0$ (resp. $p(M)=1$). Therefore, we have

$$\mathrm{Mat}((m|n)\times(r|s):\mathfrak{C})=\mathrm{Mat}_{\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})\oplus\mathrm{Mat}_{\mathrm{od}}((m|n)\times(r|s):\mathfrak{C}).$$

Moreover, we may decompose M as $M=M_{\mathrm{B}}+M_{\mathrm{S}}$ where

$$M_{\mathrm{B}}=\begin{cases}\begin{bmatrix} A_{\mathrm{B}} & 0 \\ 0 & B_{\mathrm{B}}\end{bmatrix} & \text{when } p(M)=0,\\[2ex] \begin{bmatrix} 0 & C_{\mathrm{B}} \\ D_{\mathrm{B}} & 0\end{bmatrix} & \text{when } p(M)=1.\end{cases}$$

The summation of two matrices in $\mathrm{Mat}_{\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})$ or in $\mathrm{Mat}_{\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})$ is defined as usual, but the sum of $\mathrm{Mat}_{\mathrm{ev}}((m|n)\times(r|s):\mathfrak{C})$ and $\mathrm{Mat}_{\mathrm{od}}((m|n)\times(r|s):\mathfrak{C})$ is not defined in itself except at least one of them being zero matrix.

It is clear that if M is the $(m+n)\times(r+s)$ matrix and N is the $(r+s)\times(p+q)$ matrix, then we may define the product MN and its parity $p(MN)$ as

$$(MN)_{ij}=\sum_k M_{ik}N_{kj},\quad p(MN)=p(M)+p(N)\mod 2.$$

For notational simplicity, we put $\mathrm{Mat}[m|n:\mathfrak{C}]=\mathrm{Mat}((m|n)\times(m|n):\mathfrak{C})$.

Definition A.2. *Let $M=\begin{bmatrix} A & C \\ D & B\end{bmatrix}\in\mathrm{Mat}[m|n:\mathfrak{C}]$. We define the supertrace of M by*

$$\operatorname{str} M=\operatorname{tr} A-(-1)^{p(M)}\operatorname{tr} B.$$

We get

Proposition A.2. *(a) Let $M,N\in\mathrm{Mat}[m|n:\mathfrak{C}]$ such that $p(M)+p(N)\equiv 0 \bmod 2$. Then, we have*

$$\operatorname{str}(M+N)=\operatorname{str} M+\operatorname{str} N.$$

(b) M is a matrix of size $(m+n)\times(r+s)$ and N is a matrix of size $(r+s)\times(m+n)$. Then,

$$\operatorname{str}(MN)=(-1)^{p(M)p(N)}\operatorname{str}(NM).$$

If $M\in\mathrm{Mat}[m|n:\mathfrak{C}]$ is even, denoted by $M\in\mathrm{Mat}_{\mathrm{ev}}[m|n:\mathfrak{C}]$, then M acts on $\mathfrak{R}^{m|n}$ linearly. Denoting this by T_M, we call it super linear transformation on $\mathfrak{R}^{m|n}$ and M is called the representative matrix of T_M.

Proposition A.3. *Let* $M \in \mathrm{Mat}_{\mathrm{ev}}[m|n : \mathfrak{C}]$ *and assume* $\det M_{\mathrm{B}} \neq 0$. *Then, for given* $Y \in \mathfrak{R}^{m|n}$,

$$T_M X = Y$$

has the unique solution $X \in \mathfrak{R}^{m|n}$, *which is denoted by* $X = M^{-1}Y$.

Definition A.3. $M \in \mathrm{Mat}_{\mathrm{ev}}[m|n : \mathfrak{C}]$ *is called invertible or non-singular if* M_{B} *is invertible, i.e.,* $\det A_{\mathrm{B}} \det B_{\mathrm{B}} \neq 0$, *and denoted by* $M \in \mathrm{GL}_{\mathrm{ev}}[m|n : \mathfrak{C}]$.

Definition A.4. *Let* $B = (B_{jk})$ *be* $(\ell \times \ell)$*-matrix with elements in* $\mathfrak{C}_{\mathrm{ev}}$, *denoted by,* $B \in \mathrm{Mat}[\ell : \mathfrak{C}_{\mathrm{ev}}]$. *As* $\mathfrak{C}_{\mathrm{ev}}$ *is a commutative ring, we may define* $\det B$ *as usual:*

$$\det B = \sum_{\rho \in \wp_\ell} \mathrm{sgn}(\rho) B_{1\,\rho(1)} \cdots B_{\ell\,\rho(\ell)}.$$

Then, we have, as ordinary case,

$$\det(AB) = \det A \det B,\ \det(\exp A) = \exp(\mathrm{tr}\, A) \text{ for } A, B \in \mathrm{Mat}[\ell : \mathfrak{C}_{\mathrm{ev}}]. \tag{A.1}$$

Definition A.5. *Let* $M \in \mathrm{Mat}_{\mathrm{ev}}[m|n : \mathfrak{C}]$ *be given. When* $\det B_{\mathrm{B}} \neq 0$, *we put*

$$\mathrm{sdet}\, M = (\det(A - CB^{-1}D))(\det B)^{-1}$$

and call it superdeterminant or Berezinian of M.

Corollary A.4. *When* $\det B_{\mathrm{B}} \neq 0$ *and* $\mathrm{sdet}\, M \neq 0$, *then* $\det A_{\mathrm{B}} \neq 0$.

Theorem A.5. *Let* $M, N \in \mathrm{Mat}_{\mathrm{ev}}[m|n : \mathfrak{C}]$.

(1) If M *is invertible, then we have* $\mathrm{sdet}\, M \neq 0$. *Moreover, if* A *is nonsingular, then*

$$(\mathrm{sdet}\, M)^{-1} = (\det A)^{-1}(\det(B - DA^{-1}C)).$$

(2) Multiplicativity of sdet *on* $\mathrm{GL}_{\mathrm{ev}}[m|n : \mathfrak{C}]$*:*

$$\mathrm{sdet}(MN) = \mathrm{sdet}\, M\, \mathrm{sdet}\, N.$$

(3) str *and* sdet *are matrix invariants on* $\mathrm{GL}_{\mathrm{ev}}[m|n : \mathfrak{C}]$*: That is, for* $M,\ N \in \mathrm{GL}_{\mathrm{ev}}[m|n : \mathfrak{C}]$,

$$\mathrm{str}\, M = \mathrm{str}\, NMN^{-1}$$
$$\mathrm{sdet}\, M = \mathrm{sdet}\, NMN^{-1}.$$

(4) Moreover, we have

$$\exp(\mathrm{str}\, M) = \mathrm{sdet}(\exp M) \quad \textit{for } M \in \mathrm{GL}_{\mathrm{ev}}[m|n : \mathfrak{C}].$$

A.2. **Elementary analysis I. Differential Calculus.**

A.2.1. *Supersmooth functions.*

Definition A.6. *For any* $f(q) \in C^\infty(\mathbb{R}^m : \mathbb{C})$, *we put,*

$$\tilde{f}(x) = \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \partial_q^\alpha f(x_{\mathrm{B}}) x_{\mathrm{S}}^\alpha \in \mathfrak{C}_{\mathrm{ev}} \quad \textit{for } x = x_{\mathrm{B}} + x_{\mathrm{S}} \in \mathfrak{R}^{m|0}$$

which is called the Grassmann continuation (or extension) of $f(q)$, *and denoted simply by* $f(x)$.

Moreover, if $g(q) = \sum_{I \in \mathcal{I}} g_I(q)\sigma^I \in C^\infty(\mathbb{R}^m : \mathfrak{C})$, *we define* $g(x) = \sum_{I \in \mathcal{I}} g_I(x)\sigma^I$ *where* $g_I(x)$ *is the Grassmann continuation of* $g_I(q)$.

Definition A.7. *We define functions $u \in \mathcal{C}_{\mathrm{SS,ev}}(\mathfrak{R}^{m|n})$, or $u \in \mathcal{C}_{\mathrm{SS}}(\mathfrak{R}^{m|n} : \mathfrak{C})$ by*

$$u(X) = u(x,\theta) = \sum_{|a|\le n} \tilde{u}_a(x)\theta^a = \text{or simply} \sum_{|a|\le n} u_a(x)\theta^a,$$

which are called supersmooth functions on $\mathfrak{R}^{m|n}$. Here $u_a(x)$ is the Grassmann continuation of $u_a(q) \in C^\infty(\mathbb{R}^m : \mathbb{C})$, or $u_a(q) \in C^\infty(\mathbb{R}^m : \mathfrak{C})$.

Example. For $\xi = (\xi_1, \cdots, \xi_m) \in \mathfrak{R}^{m|0} = \mathfrak{R}^m_{\mathrm{ev}}$, we define $|\xi| \in \mathfrak{R}_{\mathrm{ev}}$ as follows: Putting

$$|\xi| = |\xi|_{\mathrm{B}} + |\xi|_{\mathrm{S}} \quad \text{with} \quad |\xi|_{\mathrm{S}} = \sum_{|I|=\mathrm{even}\ge 2} |\xi|_I \sigma^I, \quad |\xi|_{\mathrm{B}} \ge 0,\ |\xi|_I \in \mathbb{R},$$

we should have

$$|\xi|^2 = \sum_{j=1}^m (\xi_{j,\mathrm{B}} + \xi_{j,\mathrm{S}})(\xi_{j,\mathrm{B}} + \overline{\xi_{j,\mathrm{S}}}) = \sum_{j=1}^m \xi_{j,\mathrm{B}}^2 + \sum_{j=1}^m \xi_{j,\mathrm{B}}(\xi_{j,\mathrm{S}} + \overline{\xi_{j,\mathrm{S}}}) + \sum_{j=1}^m \xi_{j,\mathrm{S}}\overline{\xi_{j,\mathrm{S}}},$$

$$\xi_{j,\mathrm{S}} = \sum_{|I|=\mathrm{even}\ge 2} \xi_{j,I}\sigma^I,\ \overline{\xi_{j,\mathrm{S}}} = \sum_{|I|=\mathrm{even}\ge 2} \overline{\xi_{j,I}}\sigma^I$$

with $\overline{\xi_{j,I}}$ being the complex conjugate of $\xi_{j,I}$ in $\mathbb{C}$. Therefore, $|\xi|_{\mathrm{B}} = \{\sum_{j=1}^m \xi_{j,\mathrm{B}}^2\}^{1/2}$ and

$$\begin{aligned} 2|\xi|_K|\xi|_{\mathrm{B}} &+ \sum_{I+J=K} |\xi|_I \overline{|\xi|_J}(-1)^{\tau(K;I,J)} \\ &= \sum_{j=1}^m 2\xi_{j,\mathrm{B}}\, \Re\xi_{j,K} + \sum_{I+J=K}\sum_{j=1}^m \xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)} \end{aligned}$$

which are solved by induction with respect to the length $|K|$. For example, if $|K| = 2$, we have

$$|\xi|_K = |\xi|_{\mathrm{B}}^{-1} \sum_{j=1}^m \xi_{j,\mathrm{B}}\, \Re\xi_{j,K}.$$

If $|K| = 4$,

$$\begin{aligned} 2|\xi|_K = |\xi|_{\mathrm{B}}^{-1}\big(2\sum_{j=1}^m \xi_{j,\mathrm{B}}\, \Re\xi_{j,K} &+ \sum_{I+J=K}\sum_{j=1}^m \xi_{j,I}\overline{\xi_{j,J}}(-1)^{\tau(K;I,J)} \\ &- \sum_{I+J=K}\sum_{j=1}^m |\xi|_I|\xi|_J(-1)^{\tau(K;I,J)}\big), \quad \text{etc.} \end{aligned}$$

Now, we define $\sin|\xi|$ and $\cos|\xi|$ as

$$\sin|\xi| = \sum_{n=0}^\infty \frac{1}{n!}\sin\big(|\xi|_{\mathrm{B}} + \frac{n\pi}{2}\big)|\xi|_{\mathrm{S}}^n, \quad \cos|\xi| = \sum_{n=0}^\infty \frac{1}{n!}\cos\big(|\xi|_{\mathrm{B}} + \frac{n\pi}{2}\big)|\xi|_{\mathrm{S}}^n.$$

We may characterize these supersmooth functions as follows:

Definition A.8. *A set $U = \pi_{\mathrm{B}}(U) \times \mathfrak{R}^n_{\mathrm{od}}$ is called a superdomain if $\pi_{\mathrm{B}}(U)$ is a domain in $\mathbb{R}^m$. Let a function f from a superdomain $U \subset \mathfrak{R}^{m|n}$ to $\mathfrak{C}$ be given.*

(i) It is called F-differentiable at $X = (X_A)_{A=1}^{m+n}$ in the direction $Y = (Y_A)_{A=1}^{m+n}$ if there exist

$$\frac{d}{dt} f(X + tY)\Big|_{t=0} = f'_F(X;Y).$$

We denote $f'_F(X;Y_A) = f'_{X_A}(X;Y_A)$ *for* $A = 1, \cdots, m+n$ *and are called Fréchet derivatives.*

(ii) f *is called G-differentiable at* $X = (x, \theta)$ *if there exist* $F_A(X) \in \mathfrak{C}$ *and* $R_A(X,Y) \in \mathfrak{C}$ *such that*

$$f(X+Y) - f(X) = \sum_{A=1}^{m+n} Y_A F_A(X) + \sum_{A=1}^{m+n} Y_A R_A(X,Y),$$

which satisfy

$$d(R_A(X,Y), 0) \to 0 \quad \textit{when} \quad d_{m|n}(Y, 0) \to 0.$$

Proposition A.6. *Let* f *be a function from a superdomain* $U \subset \mathfrak{R}^{m|n}$ *to* $\mathfrak{C}$. *Then, the following are equivalent:*
(i) f *be supersmooth, denoted by* $\mathcal{C}_{\rm SS}(U : \mathfrak{C})$,
(ii) f *is G-differentiable,*
(iii) f *is F-differentiable and there exist functions* $F_A(X)$ *such that*

$$f'_{X_A}(X;Y_A) = Y_A F_A(X) \quad \textit{for any} \quad A = 1, \cdots, m+n,$$

(iv) f *is F-differentiable and satisfies*

$$Z_A f'_{X_A}(X;Y_A) - (-1)^{p(Y_A)p(Z_A)} Y_A f'_{X_A}(X;Z_A) = 0 \quad \textit{for} \quad p(X_A) = p(Y_A) = p(Z_A),$$
$$f'_{X_A}(X;Y_A Z_A) = Z_A f'_{X_A}(X;Y_A) \quad \textit{for} \quad p(Z_A) = 0 \quad \textit{and} \quad p(X_A) = p(Y_A).$$

To understand the meaning of supersmoothness, we consider the dependence with respect to the 'coordinate' more precisely.

Proposition A.7. *Let* $f = \sum_I f_I(X)\sigma^I \in \mathcal{C}_{\rm SS}(U : \mathfrak{C})$ *where* U *is a superdomain in* $\mathfrak{R}^{m|n}$. *Let* $X = (X_A)_{A=1}^{m+n}$ *be represented by* $X_A = \sum_I X_{A,I}\sigma^I$ *where* $A = 1, \cdots, m+n$, $X_{A,I} \in \mathbb{C}$ *for* $|I| \neq 0$ *and* $X_{A,0} \in \mathbb{R}$. *Then,* $f(X)$, *considered as a function of countably many variables* $\{X_{A,I}\}$ *with values in* $\mathfrak{C}$, *satisfies the following (Cauchy-Riemann type) equations.*

$$\frac{\partial}{\partial X_{A,I}} f(X) = \sigma^I \frac{\partial}{\partial X_{A,0}} f(X) \quad \textit{for} \quad 1 \le A \le m,\ |I| = \textit{even}, \tag{A.2}$$

$$\sigma^K \frac{\partial}{\partial X_{A,J}} f(X) + \sigma^J \frac{\partial}{\partial X_{A,K}} f(X) = 0 \textit{ for } m+1 \le A \le m+n \ |J| = |k| = \textit{ odd.}$$

Here, we define

$$\frac{\partial}{\partial X_{A,I}} f(X) = \frac{d}{dt} f(X + tY_{(A,I)})\big|_{t=0}, \tag{A.3}$$

with $Y_{(A,I)} = (\overbrace{0, \cdots, 0, 1}^{A}, 0, \cdots, 0)\sigma^I \in \mathfrak{R}^{m|n}$ *and* $Y_{(A,0)} = (\overbrace{0, \cdots, 0, 1}^{A}, 0, \cdots, 0) \in \mathfrak{R}^{m|n}$. *Conversely, let a function* $f(X) = \sum_I f_I(X)\sigma^I$ *be given such that* $f_I(X + tY) \in C^\infty([0,1] : \mathbb{C})$ *for each fixed* $X, Y \in U$ *and* $f(X)$ *satisfies the following equations:*

$$\frac{d}{dt} f(X + tY_{(A,I)})\bigg|_{t=0} = \frac{d}{dt} f(X + tY_{(A,0)})\bigg|_{t=0} \sigma^I \quad \textit{for} \quad 1 \le A \le m,\ |J| = \textit{even}$$

$$\sigma^K \frac{d}{dt} f(X + tY_{(A,J)})\bigg|_{t=0} + \sigma^J \frac{d}{dt} f(X + tY_{(A,K)})\bigg|_{t=0} = 0$$
$$\textit{for} \quad m+1 \le A \le m+n,\ |J| = \textit{odd} = |K|. \tag{A.4}$$

Then, $f \in \mathcal{C}_{\rm SS}(U : \mathfrak{C})$.

A.2.2. *Derivations.* For a given supersmooth function $u(X)$ on $\mathfrak{R}^{m|n}$, we define its derivatives as follows: For $j = 1, 2, \cdots, m$ and $k = 1, 2, \cdots, n$, we put

$$U_j(X) = \sum_{|a|\le n} \partial_{x_j} u_a(x)\theta^a,$$
$$U_{k+m}(X) = \sum_{|a|\le n} (-1)^{l_k(a)} u_a(x)\theta_1^{a_1}\cdots\theta_k^{a_k-1}\cdots\theta_n^{a_n}$$

where $l_k(a) = \sum_{j=1}^{k-1} a_j$ and $\theta_k^{-1} = 0$. $U_\kappa(X)$ are called the partial derivatives of u with respect to X_κ at $X = (x, \theta)$ and are denoted by

$$U_j(X) = \frac{\partial}{\partial x_j} u(x,\theta) = \partial_{x_j} u(x,\theta) \quad \text{for} \quad j = 1, 2, \cdots, m,$$
$$U_{m+s}(X) = \frac{\partial}{\partial \theta_s} u(x,\theta) = \partial_{\theta_s} u(x,\theta) \quad \text{for} \quad s = 1, 2, \cdots, n$$

or simply by

$$U_\kappa(X) = \partial_{X_\kappa} u(X) \quad \text{for} \quad \kappa = 1, \cdots, m+n.$$

For

$$\mathfrak{a} = (\alpha, a),\ \alpha = (\alpha_1, \cdots, \alpha_m) \in \mathbf{N}^m,\ a = (a_1, \cdots, a_n) \in \{0,1\}^n,$$
$$|\alpha| = \sum_{j=1}^m \alpha_j,\ \ |a| = \sum_{k=1}^n a_k,\ \ |\mathfrak{a}| = |\alpha| + |a|,$$

we put

$$\partial_X^{\mathfrak{a}} = \partial_x^\alpha \partial_\theta^a \quad \text{with} \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_m}, \quad \partial_\theta^a = \partial_{\theta_1}^{a_1}\cdots\partial_{\theta_n}^{a_n}.$$

Example. $\partial_{\theta_2}\theta_1\theta_2\theta_3 = -\theta_1\theta_3$, $\quad \partial_{\theta_1}\partial_{\theta_3}\theta_1\theta_2\theta_3 = \theta_2 \ne -\theta_2 = \partial_{\theta_3}\partial_{\theta_1}\theta_1\theta_2\theta_3$, etc.

Remarks for the need of ∞ number of Grassmann generators.

(i) Though $\mathfrak{C}$ does not form a field because $X^2 = 0$ for any $X \in \mathfrak{C}_{\rm od}$, but if $X, Y \in \mathfrak{C}$ satisfy $XY = 0$ for any $Y \in \mathfrak{C}_{\rm od}$, then $X = 0$. This property holds only when the number of generators is infinite. By this, we may determine the derivative $\partial_X^{\mathfrak{a}} u(X)$ uniquely.

(ii) In general, we need at least countable number of operations in doing analysis. If the number of Grassmann generators is finite, then the effect of odd variables may vanish after finitely many operations.

(iii) Rothstein [41] claims that the super Lie algebra $Der(\mathcal{C}_{\rm SS}(\mathcal{U}))$ is the free module of $\mathcal{C}_{\rm SS}(\mathcal{U})$ only when the number of the Grassmann generators is infinite or null. (Though Rothstein used the Banach-Grassmann algebra introduced by Rogers [34], but his argument is also applicable in our situation.)

A.2.3. *Taylor expansions and Implicit function theorem.*

Definition A.9. *For a supersmooth function f, we define df by*

$$df(X) = d_X f(X) = \sum_{\kappa=1}^{m+n} dX_\kappa \frac{\partial f(X)}{\partial X_\kappa},$$

or

$$df(x,\theta) = \sum_{j=1}^m dx_j \frac{\partial f(x,\theta)}{\partial x_j} + \sum_{s=1}^n d\theta_s \frac{\partial f(x,\theta)}{\partial \theta_s}.$$

From Definition A.9, we get readily

Proposition A.8. *Let U be a superdomain in $\mathfrak{R}^{m|n}$. For $f, g \in \mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$, the product fg belongs to $\mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$ and the differentials $d_X f(X)$ and $d_X g(X)$ may be regarded as continuous linear mappings from $\mathfrak{R}^{m|n}$ into $\mathfrak{C}^{m+n}$. Moreover, they satisfy the following:*

(1) For any homogeneous elements λ, $\mu \in \mathfrak{C}$, we have

$$d_X(\lambda f + \mu g)(X) = (-1)^{p(\lambda)p(X)} \lambda d_X f(X) + (-1)^{p(\mu)p(X)} \mu d_X g(X). \tag{A.5}$$

(2) (Leibnitz formula)

$$\partial_{X_\kappa}[f(X)g(X)] = (\partial_{X_\kappa} f(X))g(X) + (-1)^{p(X_\kappa)p(f(X))} f(X)(\partial_{X_\kappa} g(X)). \tag{A.6}$$

Proposition A.9 (Taylor's formula)**.** *Let $X = (x, \theta), Y = (y, \omega) \in U \subset \mathfrak{R}^{m|n}$ satisfying $Y + t(X - Y) \in U$ for $0 \le t \le 1$. For $f \in \mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$, Taylor's formula holds. That is, for any positive integer p, we have*

$$f(x, \theta) - \sum_{|\alpha|+|a| \le p,\ |a| \le n} \frac{1}{\alpha!}(x - y)^\alpha (\theta - \omega)^a \partial_x^\alpha \partial_\theta^a f(y, \omega) = \tau_p(X, Y) \tag{A.7}$$

where

$$\begin{aligned} \tau_p(X, Y) \quad &= \sum_{|\alpha|+|a| = p+1,\ |a| \le n} (x - y)^\alpha (\theta - \omega)^a \\ &\times \int_0^1 dt\, \frac{1}{p!}(1 - t)^p \partial_x^\alpha \partial_\theta^a f(y + t(x - y), \omega + t(\theta - \omega)). \end{aligned} \tag{A.8}$$

Definition A.10. *Let $U \subset \mathfrak{R}^{m|n}$ and $U' \subset \mathfrak{R}^{m'|n'}$ be superdomains and let φ be a continuous mapping from U to U', denoted by*

$$\varphi(X) = (\varphi_1(X), \cdots, \varphi_{m'}(X), \varphi_{m'+1}(X), \cdots, \varphi_{m'+n'}(X)) \in \mathfrak{R}^{m'|n'}.$$

φ is called a supersmooth mapping from U to U' if each $\varphi_\kappa(X) \in \mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$ for $\kappa = 1, \cdots, m' + n'$ and $\varphi(U) \subset U'$.

Proposition A.10 (Composition of supersmooth mappings)**.** *Let $U \subset \mathfrak{R}^{m|n}$ and $U' \subset \mathfrak{R}^{m'|n'}$ be superdomains and let $\Phi : U \to U'$ and $\Phi' : U' \to \mathfrak{R}^{m''|n''}$ be supersmooth mappings.*

Then, the composition $\Psi = \Phi' \circ \Phi : U \to \mathfrak{R}^{m''|n''}$ gives a supersmooth mapping and

$$d_X \Psi(X) = [d_Y \Phi'(Y)]\big|_{Y = \Phi(X)} [d_X \Phi(X)]. \tag{A.9}$$

Definition A.11. *Let $U \subset \mathfrak{R}^{m|n}$ and $U' \subset \mathfrak{R}^{m'|n'}$ be superdomains and let $\varphi : U \to U'$ be a supersmooth mapping represented by $\varphi(X) = (\varphi_1(X), \cdots, \varphi_{m'+n'}(X))$ with $\varphi_\kappa(X) \in \mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$.*
(1) φ is called a supersmooth diffeomorphism if (i) φ is a homeomorphism between U and U' and (ii) φ and φ^{-1} are supersmooth mappings.
(2) For any $f \in \mathcal{C}_{\mathrm{SS}}(U' : \mathfrak{C})$, $(\varphi^* f)(X) = (f \circ \varphi)(X) = f(\varphi(X))$, called the pull back of f, is well-defined and belongs to $\mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$.

Remarks. (1) It is easy to see that if φ is a supersmooth diffeomorphism, then $\varphi_{\mathrm{B}} = \pi_{\mathrm{B}} \circ \varphi$ is an (ordinary) C^∞ diffeomorphism from U_{B} to U'_{B}.
(2) If we introduce the topologies in $\mathcal{C}_{\mathrm{SS}}(U' : \mathfrak{C})$ and $\mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$ properly, φ^* gives a continuous linear mapping from $\mathcal{C}_{\mathrm{SS}}(U' : \mathfrak{C})$ to $\mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$. Moreover, if $\varphi : U \to U'$ is a supersmooth diffeomorphism, then φ^* defines an automorphism from $\mathcal{C}_{\mathrm{SS}}(U' : \mathfrak{C})$ to $\mathcal{C}_{\mathrm{SS}}(U : \mathfrak{C})$.

Proposition A.11 (Inverse function theorem). *Let U be a superdomain in $\mathfrak{R}^{m|n}$ and let $G(X) : U \subset \mathfrak{R}^{m|n} \to \mathfrak{R}^{m|n}$ be a supersmooth mapping. We assume the super matrix $[d_X G(X)]$ is invertible at $X = \tilde{X}_{\mathrm{B}} \in \pi_{\mathrm{B}}(U)$.*

Then, there exists a superdomain U', a neighborhood of $\tilde{Y} = G(\tilde{X})$ and a unique supersmooth mapping F satisfying $F(G(X)) = X$ and we have

$$d_Y F(Y) = (d_X G(X))^{-1}\big|_{X=F(Y)} \quad in \quad U'. \tag{A.10}$$

Moreover, we have

Proposition A.12 (Implicit function theorem). *Let $\Phi(X,Y) : U \times U' \to \mathfrak{C}^{m'|n'}$ be a supersmooth mapping and $(\tilde{X},\tilde{Y}) \in U \times U'$, where U and U' are superdomains of $\mathfrak{R}^{m|n}$ and $\mathfrak{R}^{m'|n'}$, respectively. Suppose $\Phi(\tilde{X},\tilde{Y}) = 0$ and $\partial_Y \Phi = [\partial_{y_j}\Phi, \partial_{\omega_r}\Phi]$ is a continuous and invertible supermatrix at $(\tilde{X}_{\mathrm{B}}, \tilde{Y}_{\mathrm{B}}) \in \pi_{\mathrm{B}}(U) \times \pi_{\mathrm{B}}(U')$.*

Then, there exist a superdomain $V \subset U$ satisfying $\tilde{X}_{\mathrm{B}} \in \pi_{\mathrm{B}}(V)$ and a unique supersmooth mapping $Y = f(X)$ on V such that $\tilde{Y} = f(\tilde{X})$ and $\Phi(X, f(X)) = 0$ in V. Moreover, we have

$$\partial_X f(X) = -\,[\partial_Y \Phi(X,Y)]^{-1}[\partial_X \Phi(X,Y)]\big|_{Y=f(X)}. \tag{A.11}$$

A.3. **Elementary analysis II. Integral Calculus.**

A.3.1. *Integration (even case).* Now, we define the integration of a supersmooth function $u(x)$ on an even superdomain $U_{\mathrm{ev}} \subset \mathfrak{R}^{m|0}$, which is similar to the integral of holomorphic functions on a complex domain. (See, Rogers [34, 39].)

Definition A.12. *Let $u(x)$ be a supersmooth function defined on a even super domain $U_{\mathrm{ev}} \subset \mathfrak{R}^{1|0}$. Let $\lambda = \lambda_{\mathrm{B}} + \lambda_{\mathrm{S}}$, $\mu = \mu_{\mathrm{B}} + \mu_{\mathrm{S}} \in U_{\mathrm{ev}}$ and let a continuous and piecewise C^1-curve $c : [\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] \to U_{\mathrm{ev}}$ be given such that $c(\lambda_{\mathrm{B}}) = \lambda$, $c(\mu_{\mathrm{B}}) = \mu$. We define*

$$\int_c dx\, u(x) = \int_{\lambda_{\mathrm{B}}}^{\mu_{\mathrm{B}}} dt\, u(c(t))\dot{c}(t) \in \mathfrak{C} \tag{A.12}$$

and call it the integral of u *along the curve c.*

Using the integration by parts, we get the following fundamental result (see [5]).

Proposition A.13. *Let $u(t) \in C^\infty([\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] : \mathfrak{C})$ and let $u(x)$ be the Grassmann continuation of $u(t)$. Suppose that there exists a function $U(t) \in C^\infty([\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] : \mathfrak{C})$ satisfying $U'(t) = u(t)$ on $[\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}]$.*

Then, for any continuous and piecewise C^1-curve $c : [\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] \to U_{\mathrm{ev}} \subset \mathfrak{R}^{1|0}$ such that $c(\lambda_{\mathrm{B}}) = \lambda$, $c(\mu_{\mathrm{B}}) = \mu$, we have

$$\int_c dx\, u(x) = U(\lambda) - U(\mu). \tag{A.13}$$

Corollary A.14. *Let $u(x)$ be a supersmooth function defined on a even superdomain $U_{\mathrm{ev}} \subset \mathfrak{R}^{1|0}$ into $\mathfrak{C}$. Let c_1, c_2 be continuous and piecewise C^1-curves from $[\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] \to U_{\mathrm{ev}}$ such that $\lambda = c_1(\lambda_{\mathrm{B}}) = c_2(\lambda_{\mathrm{B}})$ and $\mu = c_1(\mu_{\mathrm{B}}) = c_2(\mu_{\mathrm{B}})$. If c_1 is homotopic to c_2, then*

$$\int_{c_1} dx\, u(x) = \int_{c_2} dx\, u(x). \tag{A.14}$$

Thus, if $[\lambda_{\mathrm{B}}, \mu_{\mathrm{B}}] \subset \pi_{\mathrm{B}}(U_{\mathrm{ev}})$, *we have*

$$\int_{\lambda}^{\mu} dx\, u(x) = \int_{\lambda_{\mathrm{B}}}^{\mu_{\mathrm{B}}} dt\, u(t). \tag{A.15}$$

Because of (A.15), we have

Definition A.13. *(1) Let* I_{ev} *be a even superdomain in* $\mathfrak{R}^{m|0}$ *such that* $\pi_{\mathrm{B}}(I_{\mathrm{ev}}) = \prod_{j=1}^{m}(a_j, b_j) \subset \mathbb{R}^m$ *with* $-\infty < a_j < b_j < \infty$, *which is called a even supercube. For* $u \in \mathcal{C}_{\mathrm{SS}}(I_{\mathrm{ev}} : \mathfrak{C})$, *we define*

$$\int_{I_{\mathrm{ev}}} dx\, u(x) = \int_{a_1}^{b_1} dq_1 \cdots \int_{a_m}^{b_m} dq_m\, u(q_1, \cdots, q_m) = \int_{\pi_{\mathrm{B}}(I_{\mathrm{ev}})} dx_{\mathrm{B}}\, u(x_{\mathrm{B}}). \tag{A.16}$$

(2) For any even superdomain $U_{\mathrm{ev}} \subset \mathfrak{R}^{m|0}$ *such that* $\pi_{\mathrm{B}}(U_{\mathrm{ev}})$ *is of definite area, we may put*

$$\int_{U_{\mathrm{ev}}} dx\, u(x) = \int_{\pi_{\mathrm{B}}(U_{\mathrm{ev}})} dx_{\mathrm{B}}\, u(x_{\mathrm{B}}) \tag{A.17}$$

for $u \in \mathcal{C}_{\mathrm{SS}}(U_{\mathrm{ev}} : \mathfrak{C})$.

A.3.2. *Integration (odd case).* It seems natural to put formally

$$d\theta_j = \sum_{I \in \mathcal{I}, |I| = \mathrm{od}} d\theta_{j,I}\sigma^I \quad \text{for} \quad \theta_j = \sum_{I \in \mathcal{I}, |I| = \mathrm{od}} \theta_{j,I}\sigma^I.$$

Therefore, we have $d\theta_j \wedge d\theta_k = d\theta_k \wedge d\theta_j$ for $j \neq k$. This suggests the integration w.r.t. odd variables is quite different from the one w.r.t. ordinary variables. In fact, it is defined as follows:

Let v be a polynomial of odd variables $\theta = (\theta_1, \cdots, \theta_n) \in \mathfrak{R}^n_{\mathrm{od}}$ such that

$$v(\theta_1, \cdots, \theta_n) = \sum_{|b| \leq n} v_b \theta^b \quad \text{with homogeneous } v_b\theta^b \in \mathfrak{C} \text{ for each } b.$$

Denote by $P_n(\mathfrak{C})$ the set of all v as above.

Definition A.14. *For* $v \in P_n(\mathfrak{C})$, *we put*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = \int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, v(\theta_1, \cdots, \theta_n) = (\partial_{\theta_n} \cdots \partial_{\theta_1} v)(0)$$

and we call it the integral of v *on* $\mathfrak{R}^{0|n}$.

Especially for odd integration, we have the following curious looking but well-known relations

$$\int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, \theta_1 \cdots \theta_n = 1 \quad \text{and} \quad \int_{\mathfrak{R}^{0|n}} d\theta_n \cdots d\theta_1\, 1 = 0 \quad \text{(Berezin integral).}$$

Moreover, we have

Proposition A.15. *Given* v, $w \in P_n(\mathfrak{C})$, *we have the following:*

1. *($\mathfrak{C}$-linearity) For any homogeneous* λ, $\mu \in \mathfrak{C}$,
$$\int_{\mathfrak{R}^{0|n}} d\theta(\lambda v + \mu w)(\theta) = (-1)^{np(\lambda)}\lambda \int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) + (-1)^{np(\mu)}\mu \int_{\mathfrak{R}^{0|n}} d\theta\, w(\theta).$$
2. *(Translational invariance) For any* $\rho \in \mathfrak{R}^{0|n}$, *we have*
$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta + \rho) = \int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta).$$

3. *(Integration by parts) For $v \in P_n(\mathfrak{C})$ such that $p(v) = 1$ or 0, we have*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta)\partial_{\theta_s} w(\theta) = -(-1)^{p(v)} \int_{\mathfrak{R}^{0|n}} d\theta\, (\partial_{\theta_s} v(\theta)) w(\theta).$$

4. *(Linear change of variables) Let $A = (A_{jk})$ with $A_{jk} \in \mathfrak{R}_{\rm ev}$ be invertible. Then,*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = (\det A)^{-1} \int_{\mathfrak{R}^{0|n}} d\omega\, v(A\cdot\omega).$$

5. *(Iteration of integrals)*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = \int_{\mathfrak{R}^{0|n-k}} d\theta_n \cdots d\theta_{k+1} \left(\int_{\mathfrak{R}^{0|k}} d\theta_k \cdots d\theta_1\, v(\theta_1, \cdots, \theta_k, \theta_{k+1}, \cdots, \theta_n) \right).$$

6. *(Odd change of variables) Let $\theta = \theta(\omega)$ be an odd change of variables such that $\theta(0) = 0$ and $\det \left.\dfrac{\partial\theta(\omega)}{\partial\omega}\right|_{\omega=0} \neq 0$. Then, for any $v \in P_n(\mathfrak{C})$,*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, v(\theta) = \int_{\mathfrak{R}^{0|n}} d\omega\, v(\theta(\omega)) \det{}^{-1} \frac{\partial\theta(\omega)}{\partial\omega}.$$

7. *For $v \in P_n(\mathfrak{C})$ and $\omega \in \mathfrak{R}^{0|n}$,*

$$\int_{\mathfrak{R}^{0|n}} d\theta\, (\theta_1 - \omega_1)\cdots(\theta_n - \omega_n) v(\theta) = v(\omega).$$

A.3.3. *Integration (mixed case).* Finally, we define

Definition A.15. *Let $U = U_{\rm ev} \times \mathfrak{R}^n_{\rm od} \subset \mathfrak{R}^{m|n}$ be a superdomain and let $u \in \mathcal{C}_{\rm SS}(U : \mathfrak{C})$, that is, $u(x,\theta) = \sum u_a(x)\theta^a$ with $u_a(x) \in \mathcal{C}_{\rm SS}(U_{\rm ev} : \mathfrak{C})$. Then, we define*

$$\begin{aligned}
\int_{\mathfrak{R}^{m|n}} dx d\theta\, u(x,\theta) &= \int_{\mathfrak{R}^{m|0}} dx \left\{ \int_{\mathfrak{R}^{0|n}} d\theta\, u(x,\theta) \right\} \\
&= \int_{\mathbb{R}^m} dX_{\rm B} (\partial_{\theta_n} \cdots \partial_{\theta_1} u)(X_{\rm B}) \quad (\pi_{\rm B}(\mathfrak{R}^{m|0}) = \mathbb{R}^m) \\
&= \int_{\mathfrak{R}^{0|n}} d\theta \left\{ \int_{\mathfrak{R}^{m|0}} dx\, u(x,\theta) \right\} = \int_{\mathfrak{R}^{m|n}} d\theta dx\, u(x,\theta).
\end{aligned}$$

A.3.4. *Change of variables under integral sign.*

Theorem A.16. *Let*

$$x = x(y,\omega), \quad \theta = \theta(y,\omega)$$

be a supersmooth diffeomorphism from $\mathfrak{R}^{m|n}_Y$ to $\mathfrak{R}^{m|n}_X$. Putting

$$M = \begin{bmatrix} A & C \\ D & B \end{bmatrix}, \quad A = \frac{\partial x}{\partial y}, \quad B = \frac{\partial\theta}{\partial\omega}, \quad C = \frac{\partial x}{\partial\omega}, \quad D = \frac{\partial\theta}{\partial y},$$

we assume that either $\det A|_{\omega=0}$ and $\det(B - DA^{-1}C)|_{\omega=0}$, or $\det B|_{\omega=0}$ and $\det(A - CB^{-1}D)|_{\omega=0}$, are non-zero for all y. Then, for any function $f \in \mathcal{C}_{\rm SS}(\mathfrak{R}^{m|n}_X : \mathfrak{C})$ with compact support, we have the change of variables formula

$$\int_{\mathfrak{R}^{m|n}_X} dx d\theta\, f(x,\theta) = \int_{\mathfrak{R}^{m|n}_Y} dy d\omega\, f(x(y,\omega), \theta(y,\omega)) ({\rm sdet} M)(y,\omega).$$

Remark. In case when we consider the integral on a superdomain with boundary and functions with support intersecting with that boundary, there occurs some difficulty which is exemplified as follows: Let $I = \widetilde{(0,1)} \times \mathfrak{R}^2_{\mathrm{od}} \subset \mathfrak{R}^{1|2}$, where $\widetilde{(0,1)} = \{x \in \mathfrak{R}_{\mathrm{ev}} \mid \pi_{\mathrm{B}}(x) \in (0,1)\} \subset \mathfrak{R}_{\mathrm{ev}}$. We have

$$\int_I dy d\omega_2 d\omega_1 \, y = 0.$$

On the other hand, using the change of coordinates for $0 \neq \alpha \in \mathfrak{R}_{\mathrm{ev}}$ and for $j = 1, 2$

$$y = x + \alpha\theta_1\theta_2, \ \omega_1 = \theta_1, \ \omega_2 = \theta_2,$$

which maps I to I and yields $\mathrm{sdet}(\frac{\partial(y,\omega)}{\partial(x,\theta)}) = 1$, we have

$$\int_I dx d\theta_2 d\theta_1 \, (x + \alpha\theta_1\theta_2) = \int_{\widetilde{(0,1)}} dx \alpha = \alpha.$$

The resolution of this difficulty is due to Rothstein [42], but we don't mention it here (see also Zirnbauer [48], Martellini and Teofilatto [32], Inoue and Nomura [23]).

A.4. **A few elements from real analysis.**

A.4.1. *Scalar products and norms.* Following [10], we introduce

$$\mathcal{C}_{\mathrm{SS,ev}}(\mathfrak{R}^{m|n}) = \{u(X) = \sum_{|a|=\mathrm{even}\le n} u_a(x)\theta^a \,\big|\, u_a(q) \in C^\infty(\mathbb{R}^m : \mathbb{C}) \quad \text{for any} \quad a\},$$

$$\mathcal{D}_{\mathrm{SS,ev}}(\mathfrak{R}^{m|n}) = \{u(X) = \sum_{|a|=\mathrm{even}\le n} u_a(x)\theta^a \,\big|\, u_a(q) \in \mathcal{D}(\mathbb{R}^m : \mathbb{C}) \quad \text{for any} \quad a\},$$

$$\mathcal{S}_{\mathrm{SS,ev}}(\mathfrak{R}^{m|n}) = \{u(X) = \sum_{|a|=\mathrm{even}\le n} u_a(x)\theta^a \,\big|\, u_a(q) \in \mathcal{S}(\mathbb{R}^m : \mathbb{C}) \quad \text{for any} \quad a\}, \text{ etc.}$$

Let another set of odd variables $\{\bar\theta_j\}_{j=1}^n$ satisfy $\theta_j\theta_k + \theta_k\theta_j = \theta_j\bar\theta_k + \bar\theta_k\theta_j = \bar\theta_j\bar\theta_k + \bar\theta_k\bar\theta_j = 0$. We define the conjugation $\overline{u(x,\theta)} = \sum_a \overline{u_a(x)}\,\overline{\theta^a}$ where $\overline{\theta^a} = \bar\theta_n^{a_n}\cdots\bar\theta_1^{a_1}$ and $\overline{u_a(x)}$ being the complex conjugate of $u_a(x)$. Then, we define

$$(u,v) = \int_{\mathfrak{R}^{m|2n}} dx d\theta \, d\bar\theta \, e^{\langle\bar\theta|\theta\rangle}\overline{u(x,\theta)}v(x,\theta) = \sum_{|a|\le n}\int_{\mathfrak{R}^{m|0}} dx \, \overline{u_a(x)}v_a(x),$$

$$(\!(u,v)\!)_k = \sum_{|\mathfrak{a}|\le k}(\partial_X^{\mathfrak{a}}u, \partial_X^{\mathfrak{a}}v) = \sum_{|\alpha|+|a|\le k}(\partial_x^\alpha u_a, \partial_x^\alpha v_a),$$

$$(\!(\!(u,v)\!)\!)_k = \sum_{|\mathfrak{a}|+l\le k}((1+|X_{\mathrm{B}}|^2)^{l/2}\partial_X^{\mathfrak{a}}u, (1+|X_{\mathrm{B}}|^2)^{l/2}\partial_X^{\mathfrak{a}}v)$$

with

$$\|u\|^2 = (u,u), \quad \|u\|_k^2 = (\!(u,u)\!)_k, \quad |\!|\!|u|\!|\!|_k^2 = (\!(\!(u,u)\!)\!)_k.$$

The space $\mathcal{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{m|n})$ is the completion of $\mathcal{D}_{\mathrm{SS,ev}}(\mathfrak{R}^{m|n})$ in the norm $\|\cdot\|$.

Generally, we may identify vectors $L^2(\mathbb{R}^m : \mathbb{C}^r)$ with supersmooth functions $\mathcal{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{m|n})$ with suitably related r and n. Without specifying this relation, we consider only the case

$$L^2(\mathbb{R}^3 : \mathbb{C}^2) \underset{\flat}{\overset{\sharp}{\rightleftarrows}} \mathcal{L}^2_{\mathrm{SS,ev}}(\mathfrak{R}^{3|2}).$$

(See, more precisely, [10, 13].)

A.4.2. *Fourier transformations.* For $\hbar \in \mathbb{R}^\times$ and $k \in \mathbb{C}^\times$, $v(x), w(\xi) \in \mathcal{S}_{\mathrm{SS,ev}}(\mathfrak{R}^{m|n})$ and $v(\theta), w(\pi) \in P_n(\mathfrak{C})$, we put

$$(F_e v)(\xi) = (2\pi i\hbar)^{-m/2} \int_{\mathfrak{R}^{m|0}} dx\, e^{-i\hbar^{-1}\langle x|\xi\rangle} v(x),$$
$$(\bar{F}_e w)(x) = (2\pi i\hbar)^{-m/2} \int_{\mathfrak{R}^{m|0}} d\xi\, e^{i\hbar^{-1}\langle x|\xi\rangle} w(\xi),$$
$$(F_o v)(\pi) = k^{n/2}\iota_n \int_{\mathfrak{R}^{0|n}} d\theta\, e^{-ik^{-1}\langle\theta|\pi\rangle} v(\theta),$$
$$(\bar{F}_o w)(\theta) = k^{n/2}\iota_n \int_{\mathfrak{R}^{0|n}} d\pi\, e^{ik^{-1}\langle\theta|\pi\rangle} w(\pi)$$

where

$$\langle x|\xi\rangle = \sum_{j=1}^m x_j\xi_j, \quad \langle\theta|\pi\rangle = \sum_{k=1}^n \theta_k\pi_k, \quad \iota_n = e^{-\pi i n(n-2)/4}.$$

We put

$$(\mathcal{F}u)(\xi,\pi) = c_{m,n} \int_{\mathfrak{R}^{m|n}} dX\, e^{-i\hbar^{-1}\langle X|\Xi\rangle} u(X) = \sum_a [(F_e u_a)(\xi)][(F_o\theta^a)(\pi)],$$
$$(\bar{\mathcal{F}}v)(x,\theta) = c_{m,n} \int_{\mathfrak{R}^{m|n}} d\Xi\, e^{i\hbar^{-1}\langle X|\Xi\rangle} v(\Xi) = \sum_a [(\bar{F}_e v_a)(x)][(\bar{F}_o\pi^a)(\theta)]$$

where

$$\langle X|\Xi\rangle = \langle x|\xi\rangle + \hbar k^{-1}\langle\theta|\pi\rangle \in \mathfrak{R}_{\mathrm{ev}}, \quad c_{m,n} = (2\pi i\hbar)^{-m/2} k^{n/2}\iota_n.$$

Using these Fourier transformations, we may prove the Plancherel formula and define pseudo-differential operators, Fourier integral operators analogously as the standard cases. In §4 before, we take $k = \hbar$.

We introduce useful constants $e(a,b)$ and $e(a)$ as follows:

$$\theta^a\theta^b = (-1)^{e(a,b)}\theta^{a+b}, \quad \int_{\mathfrak{R}^{0|n}} d\theta\, e^{-ik^{-1}\langle\theta|\pi\rangle}\theta^a = (-ik^{-1})^{n-|a|}(-1)^{e(a)}\pi^{\tilde{1}-a},$$

with $e(a) \equiv n|a| - |a| - 1 + [(n-1-|a|)/2] + e(\bar{1}-a, a) \mod 2$ and $\bar{1} = (1,\cdots,1) \in \{0,1\}^n$.

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