

Extension techniques in C^* -cross products by compact group duals and quantum field theory *

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*Dedicated to Eyvind H. Wichmann
on his 70th birthday*

Abstract

Using the methods developed in a previous paper, we consider the problem of extending endomorphisms to cross-product by compact group duals. Then we discuss some other applications to QFT, mainly in connection with PCT symmetries.

1 Introduction

Even if the basic philosophy in local Quantum Field Theory is that all the information is encoded in the observable net \mathcal{A} , the analysis of charged states suggests the introduction of the field net \mathcal{F} . It is then natural to enquire which properties of \mathcal{A} extend to \mathcal{F} and under which conditions. Mathematically this amounts to asking for extension theorems from a C^* -algebra to a cross product by a group dual.

In algebraic QFT, a PCT-symmetry ϑ is defined as an anti-isomorphism of the observable net, whose origin goes back to the particle-antiparticle symmetry. Starting from the seminal papers [1], it has been subject to several investigations ([3, 21, 22] and afterwards [23, 11, 6]) that benefitted from the methods of modular theory advocated in [3, 21, 22].

Motivated by the desire to have a better understanding of the PCT symmetry in AQFT, we used in an essential way the Doplicher-Roberts uniqueness theorem for cross products by compact group duals [14] and subsequent developments [5]. The first simple but important step was to prove an extension theorem for isomorphisms [8, Theorem 2.1]. This allowed us to consider anti-isomorphisms $\mathcal{A} \rightarrow \mathcal{A}_1$ by treating them as isomorphisms $\mathcal{A} \rightarrow \mathcal{A}_1^{\text{opp}}$ (the opposite algebra). As a consequence of our abstract result, we showed that any PCT-symmetry extends to the canonical field net as an anti-automorphism, still retaining (part

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of) its geometrical meaning. More precisely, the extension $\hat{\vartheta}$ is easily seen to act on the local algebras as expected, but showing the right commutation relations with the translations seems to require more assumptions, e.g. $[\hat{\vartheta}, G] = 0$ (and $Z(G)$ discrete), or the full strength of the spectrum condition. The existence of an extension $\hat{\vartheta}$ commuting with the gauge action necessarily implies that $\vartheta \circ \rho \circ \vartheta$ is a conjugate of ρ for every DHR morphism ρ with finite statistics of \mathcal{A} . As to the converse, an affirmative answer is known provided that G satisfies a certain property interesting in its own, concerning the action of $\text{Aut}(G)$ on the dual \hat{G} [9]. The occurrence of such groups, including compact connected Lie groups, is not surprising because of the special role they play in the duality theory for compact groups. But from our point of view, the fact that an order-preserving isomorphism of their duals always originates from an isomorphism between the groups in question has a natural application in the context of an extension to cross products.

2 Extension of endomorphisms

The procedure developed in [8] to extend (anti-)isomorphisms between C^* -algebras to their cross-products by compact group duals can be successfully applied to handle endomorphisms as well. In this section we take this up. The same result has been obtained in [26], §3, but the perspective is somehow different.

We need to recall a few notations and definitions, thus setting the stage for our considerations. Let \mathcal{A} be a C^* -algebra, hereafter assumed to be simple, (Δ, ϵ) a permutation symmetric, specially directed semigroup of unital endomorphisms of \mathcal{A} with unit ι , and \mathcal{T} the full subcategory of $\text{End}(\mathcal{A})$ with objects Δ .

Let ρ be a (necessarily injective) $*$ -endomorphism of \mathcal{A} . Our main task in this section is to show the existence, under suitable assumptions, of an extension $\hat{\rho}$ of ρ as an endomorphism of the cross-product $\mathcal{B} = \mathcal{A} \rtimes \mathcal{T}$, and discuss its properties. The following observation is quite trivial, but it has far reaching consequences. The $*$ -endomorphism ρ , as an isomorphism onto its image, provides immediately another quadruple $\{\mathcal{A}_\rho, \Delta_\rho, \mathcal{T}_\rho, \epsilon_\rho\}$ as above by setting:

$$\begin{aligned} \mathcal{A}_\rho &:= \rho(\mathcal{A}) \\ \Delta_\rho &:= \{\rho\sigma\rho^{-1} ; \sigma \in \Delta\} \subset \text{End}(\mathcal{A}_\rho) \\ \mathcal{T}_\rho &:= \text{full subcategory of } \text{End}(\mathcal{A}_\rho) \text{ with objects } \Delta_\rho \\ \epsilon_\rho(\rho\sigma\rho^{-1}, \rho\tau\rho^{-1}) &:= \rho(\epsilon(\sigma, \tau)), \sigma, \tau \in \Delta \end{aligned}$$

The verification that this quadruple possess the properties mentioned above can be worked out without difficulties and is left to the reader.

Let $\mathcal{B}_\rho := \mathcal{A}_\rho \rtimes \mathcal{T}_\rho$. It is quite clear from the outset that we are in position to apply Theorem 2.1 in [8]. We get the existence of an isomorphism $\tilde{\rho} : \mathcal{B} \rightarrow \mathcal{B}_\rho$, such that $\tilde{\rho}|_{\mathcal{A}} = \rho$. Also, G_ρ may be identified with G and its action is given by $\tilde{\rho}\alpha_g\tilde{\rho}^{-1}$. (Note that, more generally, the same discussion applies if we start

with *any* isomorphism $\phi : \mathcal{A} \rightarrow \mathcal{A}_1$.) What remains to be shown is that $\tilde{\rho}$ is in fact an endomorphism of \mathcal{B} , or, in other words, that $\mathcal{B}_\rho \supset \mathcal{A}_\rho$ can be naturally embedded as a C^* -subalgebra of \mathcal{B} , $\mathcal{A}_\rho \rtimes \mathcal{T}_\rho \subset \mathcal{A} \rtimes \mathcal{T}$. (After this identification we will get that $\hat{\rho} = \tilde{\rho}$.)

To this end, from now on we focus our attention on endomorphisms ρ for which there are unitaries (“two-variable cocycles”) in \mathcal{A} satisfying the conditions (8.10)-(8.12) introduced in [14], namely

$$W_\sigma(\rho) \in (\sigma\rho, \rho\sigma) \quad (2.1)$$

$$W_{\sigma\sigma'}(\rho) = W_\sigma(\rho)\sigma(W_{\sigma'}(\rho)), \quad \sigma, \sigma' \in \Delta \quad (2.2)$$

$$W_{\sigma'}(\rho)TW_\sigma(\rho)^* = \rho(T), \quad T \in (\sigma, \sigma'). \quad (2.3)$$

Then we will infer that $\hat{\rho}(\psi) = W_\sigma(\rho)\psi$, $\psi \in H_\sigma$ ($\sigma \in \Delta$). The essence of the argument goes as follows: if ψ_i is an orthonormal basis in H_σ , i.e., it is a multiplet of isometries in \mathcal{B} implementing σ on \mathcal{A} , then taking into account (2.1), we have

$$\begin{aligned} \sum_i \tilde{\rho}(\psi_i)\rho(A)\tilde{\rho}(\psi_i)^* &= \tilde{\rho}\left(\sum_i \psi_i A \psi_i^*\right) \\ &= \rho(\sigma(A)) \\ &= W_\sigma(\rho)\sigma\rho(A)W_\sigma(\rho)^* \\ &= \sum_i W_\sigma(\rho)\psi_i\rho(A)\psi_i^*W_\sigma(\rho)^* \quad (A \in \mathcal{A}). \end{aligned}$$

Thus $W_\sigma(\rho)\psi_i$ is another multiplet in \mathcal{B} implementing $\rho\sigma\rho^{-1}$ on $\rho(\mathcal{A})$.¹ Consider now the C^* -subalgebra $\tilde{\mathcal{B}}_\rho$ of \mathcal{B} generated by $\rho(\mathcal{A})$ and $W_\sigma(\rho)H_\sigma$, with σ running over all the elements in Δ , endowed with the natural G -action obtained by restricting the action of G on \mathcal{B} . In the next step we exploit the universal property of the cross-product.

Lemma 2.1 *Under the above assumptions there is an isomorphism $\phi_\rho : B_\rho \equiv \mathcal{A}_\rho \rtimes \mathcal{T}_\rho \rightarrow \tilde{\mathcal{B}}_\rho$, that reduces to the identity on \mathcal{A}_ρ .*

Proof. The argument relies on the uniqueness result given in [14], Theorem 5.1 when applied to \mathcal{A}_ρ , G , and $\Delta_\rho \ni \rho\sigma\rho^{-1} \mapsto H_{\rho\sigma\rho^{-1}} \equiv W_\sigma(\rho)H_\sigma$. We check the conditions a) – e) given there:

a) $\mathcal{A}'_\rho \cap \tilde{\mathcal{B}}_\rho = \tilde{\mathcal{B}}_\rho^G \cap \tilde{\mathcal{B}}_\rho$ by the point b).

Since $(\tilde{\mathcal{B}}_\rho, G)$ has full Hilbert spectrum, we are in position to apply Lemma 5.1 in [14]. Therefore the latter relative commutant is generated, as a closed linear space, by $(\rho_H, \iota)_{\mathcal{A}_\rho} H$,² where H are Hilbert G -modules in $\tilde{\mathcal{B}}_\rho$ (thus in \mathcal{B}). However, inspection of the proof of that result shows that in the present situation it is enough to consider H of the form $W_\sigma(\rho)H_\sigma$ with $\sigma \in \Delta$. Now,

¹Formally one might say that $\tilde{\rho}(H_\sigma(\mathcal{A})) = H_{\rho\sigma\rho^{-1}}(\mathcal{A}_\rho) (= W_\sigma(\rho)H_\sigma(\mathcal{A}))$, cf. below.

²We keep the same notation as in the quoted reference.

$\rho(X) \in (\rho_{W_\sigma(\rho)H_\sigma}, \iota)_{\mathcal{A}_\rho} = (\rho\sigma\rho^{-1}, \iota)_{\mathcal{A}_\rho}$ easily implies $X \in (\sigma, \iota)_{\mathcal{A}}$, and thus $X = 0$, whenever $\sigma \neq \iota$, since $\mathcal{A}' \cap \mathcal{B} = \mathbb{C}I$.

b) of course $\rho(\mathcal{A}) \subset \tilde{\mathcal{B}}_\rho^G$. The other inclusion can be deduced with the aid of the following observation: if $H = H_{\rho\sigma\rho^{-1}} = W_\sigma(\rho)H_\sigma$ with $\sigma \in \Delta$, then $H^G = W_\sigma(\rho)H_\sigma^G = W_\sigma(\rho)(\mathbb{C}, H_\sigma)_G \subset W_\sigma(\rho)(\iota_{\mathcal{A}}, \sigma) \subset \rho(\mathcal{A})$ (the first equality follows from $W_\sigma(\rho) \in \mathcal{A}$, while the first inclusion is shown in [13, Lemma 2.4] and the latter one follows by (2.3)). Moreover $H^* \subset \rho(\mathcal{A})K$, where $K = H_{\rho\sigma\rho^{-1}}$: in fact, given an orthonormal basis ψ_i in H and an orthonormal basis φ_j in K , we have that $X = \sum_i \varphi_i \psi_i \in \mathcal{A}$ satisfies $\varphi_j^* X = \psi_j$, and further $X \in \rho(\mathcal{A})$ by combining the fact that (in \mathcal{B}) $H_\sigma H_{\sigma'} = H_{\sigma\sigma'}$ together with ((2.2) and) the above observation.

It follows that $\tilde{\mathcal{B}}_\rho$ is generated, as a closed linear space, by the $\rho(\mathcal{A})H$ with H as above, and therefore $\tilde{\mathcal{B}}_\rho^G$ is generated by $\rho(\mathcal{A})H^G \subset \rho(\mathcal{A})$ (here one can use the conditional expectation provided by taking the mean over G , see e.g. [27, Proposition 6.2]).

c), d) follow by definition and the computation above, respectively;

e) since $\epsilon(\sigma, \tau) = F(H_\sigma, H_\tau)$ where F implements the flip symmetry on $H_\sigma \otimes H_\tau$, $\sigma, \tau \in \Delta$, the conclusion is immediate by a straightforward calculation using (2.2), (2.3). \square

It follows that $\hat{\rho} := \phi_\rho \circ \tilde{\rho}$ is the desired endomorphism of \mathcal{B} extending ρ . Also, $\hat{\rho}$ commutes with the action of G .

As to the opposite direction, if a given ρ has an extension $\hat{\rho}$ commuting with G , then $W_\sigma(\rho) := \sum_{i=1}^d \hat{\rho}(\psi_i) \psi_i^*$, where $\{\psi_i\}_{i=1}^d$ is an orthonormal basis of H_σ , satisfy all the properties (2.1)-(2.3) (see [14, 26]).³

So we have proved the following result.

Theorem 2.2 *Let \mathcal{A} and $\mathcal{B} = \mathcal{A} \rtimes \mathcal{T}$ be as above, and let ρ be an endomorphism of \mathcal{A} . Then there is an extension $\hat{\rho}$ of ρ to \mathcal{B} commuting with G if and only if there are unitaries $W_\sigma(\rho)$, $\sigma \in \Delta$ in \mathcal{A} satisfying the identities (2.1)-(2.3). In this case, $\hat{\rho}$ is uniquely determined by*

$$\hat{\rho}(\psi) = W_\sigma(\rho)\psi, \quad \psi \in H_\sigma, \quad \sigma \in \Delta.$$

In particular, when the conditions are satisfied, $\hat{\rho}$ preserves the \mathbb{Z}_2 -grading. Moreover, $\hat{\rho}(\mathcal{F})^G = \rho(\mathcal{A})$.

Remarks

(i) In general $\hat{\rho}$ will not be irreducible, even if ρ is.

(ii) It is possible to discuss along the same lines G -commuting extensions to \mathcal{B} of actions by semigroups Γ of endomorphisms of \mathcal{A} . One has to require the further hypothesis

$$(2.4) \quad W_\sigma(\rho\rho') = \rho(W_\sigma(\rho'))W_\sigma(\rho), \quad \rho, \rho' \in \Gamma,$$

³The overall discussion should make it clear that in general $\hat{\rho}$ will not be *inner* in \mathcal{B} . However, if $\rho \in \Delta$, then $\hat{\rho}$ is nothing but the endomorphism implemented by the same Hilbert space of isometries as ρ .

see e.g. [26], Lemma 3.10.

Next we discuss some properties of the extensions. In the following we will also make use of

$$(2.5) \quad W_\sigma(\rho')^* T W_\sigma(\rho) = \sigma(T), \quad T \in (\rho, \rho'), \quad \sigma \in \Delta.$$

So let us start with ρ, ρ' both satisfying (2.1) - (2.5) as above. Then ρ and ρ' both extend to \mathcal{B} and it is easy to deduce from (2.5) that

$$(\rho, \rho') \subset (\hat{\rho}, \hat{\rho}').$$

From (2.4) we immediately get that

$$\hat{\rho}\hat{\rho}' = \hat{\rho}'\hat{\rho}.$$

As an application, let $\hat{\rho}$ and $\hat{\bar{\rho}}$ be the extensions of ρ and $\bar{\rho}$ respectively. Then

$$\hat{\bar{\rho}} = \overline{\hat{\rho}}$$

whenever

$$W_\sigma(\bar{\rho})^* \bar{\rho} (W_\sigma(\rho)^*) R = \sigma(R) ; \quad W_\sigma(\rho)^* \rho (W_\sigma(\bar{\rho})^*) \bar{R} = \sigma(\bar{R}), \quad \sigma \in \Delta$$

where $R \in (\iota, \bar{\rho}\rho)$ and $\bar{R} \in (\iota, \rho\bar{\rho})$ are standard solutions of the conjugate equations. Then clearly one has

$$d(\rho) = d(\hat{\rho}).$$

3 On sectors of field nets

Now let \mathcal{A} be an observable net satisfying standard assumptions and $\mathcal{F} = \mathcal{A} \rtimes \mathcal{T}$, where \mathcal{T} is the category whose objects are the DHR morphisms with finite statistics of \mathcal{A} and whose arrows are their intertwiners [16]. Also let ϑ be a PCT-symmetry as in [8], §3. For every DHR morphism ρ of \mathcal{A} we define $\rho_\vartheta := \vartheta\rho\vartheta$. Then ρ_ϑ is a DHR morphism, with the same statistics as ρ . Furthermore, there is an extension $\hat{\vartheta}$ of ϑ to \mathcal{F} [8]. (To simplify things we also assume $\hat{\vartheta}^2 = \text{id}_{\mathcal{F}}$ from now on.)

To define the extension of ρ to \mathcal{F} , a canonical choice is provided by taking $W_\sigma(\rho) = \epsilon(\sigma, \rho)$, the familiar statistical operator. Then all the conditions considered above are satisfied, cf. [25]. In fact for any (relatively) local extension $\mathcal{A} \subset \mathcal{B}$ one gets a unit-preserving monoidal *-functor $\hat{\cdot}$ from the category whose objects are the DHR morphisms of \mathcal{A} to the category whose objects are the DHR morphisms of \mathcal{B} , acting identically on the arrows [10]. We summarize here the relevant properties:

- i) $\hat{\iota}_{\mathcal{A}} = \iota_{\mathcal{B}}$,
- ii) $(\rho_1 \rho_2)^\wedge = \hat{\rho}_1 \hat{\rho}_2$,
- iii) $(\rho_1 \oplus \rho_2)^\wedge = \hat{\rho}_1 \oplus \hat{\rho}_2$,

- iv) $\hat{\rho} = \bar{\rho}$,
- v) $\epsilon(\hat{\rho}, \hat{\sigma}) = \epsilon(\rho, \sigma)$,
- vi) $d(\hat{\rho}) = d(\rho)$,
- vii) $T \in (\rho_1, \rho_2) \Rightarrow T \in (\hat{\rho}_1, \hat{\rho}_2)$.

Actually this functor corresponds to a group homomorphism $h : G_{\mathcal{B}} \rightarrow G_{\mathcal{A}}$, where $G_{\mathcal{A}}$ (resp. $G_{\mathcal{B}}$) is the canonical gauge group for \mathcal{A} (resp. \mathcal{B}), cf. [15], Theorem 6.10. This extension procedure is just an instance of effectiveness of ideas and methods from net cohomology [28].

Another version of this result appears in [2], Sect. 3.3, called ‘‘homomorphism properties.’’ However, that approach is more suitable for low-dimensional QFT’s.

Proposition 3.1 *Let ϑ be a PCT symmetry for \mathcal{A} . Then $(\rho_{\vartheta})^{\sim} = \hat{\vartheta}\hat{\rho}\hat{\vartheta}$.*

Proof. Since $\hat{\rho}(\psi) = \epsilon(\sigma, \rho)\psi$, $\psi \in H_{\sigma}$, we have

$$\begin{aligned} \hat{\vartheta}(\hat{\rho}(\psi)) &= \vartheta(\epsilon(\sigma, \rho))\hat{\vartheta}(\psi) \\ &= \epsilon(\sigma_{\vartheta}, \rho_{\vartheta})\hat{\vartheta}(\psi) \\ &= \hat{\rho}_{\vartheta}(\hat{\vartheta}(\psi)) \end{aligned}$$

where in the last equality we used that $\hat{\vartheta}(\psi) \in H_{\sigma_{\vartheta}}$, see [8], Lemma 3.2. \square

This statement has the following interpretation: if ρ has finite statistics, ρ_{ϑ} is likely to be (equivalent to) a conjugate $\bar{\rho}$ of ρ , so the formula reads as $\hat{\rho} \cong \hat{\rho}_{\vartheta}$; since we know that $\hat{\rho} = \bar{\rho}$ (and $d(\rho) = d(\hat{\rho})$), this means that conjugation by $\hat{\vartheta}$ induces the conjugation on the extended DHR morphisms. Now a natural question is whether the relation $\rho'_{\hat{\vartheta}} \cong \bar{\rho}'$ holds for every DHR morphism ρ' of \mathcal{F} with finite statistics (provided that it holds for \mathcal{A}).

This is tied up with the existence of nontrivial DHR sectors of \mathcal{F} . (For simplicity we assume \mathcal{F} to be Bosonic, but this requirement can be relaxed.) If one can rule out the occurrence of sectors with infinite statistics for \mathcal{A} , then \mathcal{F} has no nontrivial sectors at all (with any statistics) [10], and the conclusion easily follows.

A major open problem in algebraic QFT is indeed to find conditions excluding the occurrence of morphisms with infinite statistics (see e.g. [17]). One may even wonder if some related techniques involving ϑ can be used for that goal, or for showing that \mathcal{F} has no nontrivial sectors.

Returning to our original question, the results allow us to extend to \mathcal{F} morphisms of \mathcal{A} with a priori any statistics. Morphisms with finite statistics will extend to inner morphisms (by definition of the cross product), so that only morphisms with infinite statistics may have nontrivial extension.

We conjecture that if ϑ induces the conjugation on all the DHR morphism ρ of \mathcal{A} with finite statistics (resp. irreducible), then the same is true for $\hat{\vartheta}$.

Furthermore ϑ coincides with the modular conjugation associated with some wedge.

Let $\hat{\vartheta}$ (resp. $\hat{\vartheta}$) be some extension of ϑ to \mathcal{F} (resp. of ϑ to the field net of the field net ⁴ $\mathcal{F}_{\mathcal{F}}$). Of course $\hat{\vartheta}$ should be a reasonable PCT operator as well (see Remark 2 below). Applying the results in [8] we can say that $\hat{\vartheta}\rho'\hat{\vartheta} \cong \overline{\rho'}$ if $[\hat{\vartheta}, H] = 0$, where H is the gauge group of $\mathcal{F}_{\mathcal{F}}$ (the compact group such that $\mathcal{F}_{\mathcal{F}} = \mathcal{F} \rtimes \hat{H}$). For the converse implication we lack the information whether H is quasi-complete [8]. There is an (algebraic) exact sequence $1 \rightarrow H \xrightarrow{i} \mathcal{G} \rightarrow G \rightarrow 1$, where $\mathcal{G} \supset H$ is the (possibly non-compact) “group of symmetries of $\mathcal{F}_{\mathcal{F}}$ extending those of G ” and i denotes the inclusion map. But even if G is known to be quasi-complete, we cannot conclude that H , nor \mathcal{G} , is.

We end this section with some brief observations that are relevant to our discussion. Some of them are new, others are already contained in some form in previous papers.

Remarks.

1. *PCT and (modular) conjugation.* We point out an observation contained in [8], cf. [21, 22]. If ϑ is an anti-automorphism of \mathcal{A} satisfying 1) $\vartheta(\mathcal{A}(\mathcal{O})) = \mathcal{A}(-\mathcal{O})$ for any double cone \mathcal{O} , 2) $\omega \circ \vartheta = \omega^*$ where ω is the vacuum state (hence we can assume that $\vartheta = \text{Ad}(\Theta)$), 3) $\omega(\alpha_{R_1(\pi)}(A)\vartheta(A)) \geq 0$ for any $A \in \mathcal{A}(W_R)''$, then, assuming wedge duality, we have that $\vartheta\rho\vartheta = \overline{\rho}$. In fact, by properties 1)-3) we identify Θ with $V(R_1(\pi))J_{W_R}$ via Araki’s characterization of modular involutions. Under the additional hypothesis of positive energy, Borchers’ theorem [3] yields $\vartheta\alpha_x = \alpha_{-x}\vartheta$. Finally in this case there is an extension of ϑ to \mathcal{F} commuting with the gauge action (implemented by $\hat{V}(R_1(\pi))Z\hat{J}_{W_R}$), and the conclusion follows.

2. *PCT and translations.* We show that a PCT symmetry on the observables has a full fledged extension as a PCT symmetry on the fields: by appealing to the spectrum condition, we get that an extension $\hat{\vartheta}$ of ϑ to the canonical (covariant) field net \mathcal{F}_c has the expected commutation relations with (a suitable extension of) the translations (cf. [8], §3). The main obstacle here is to find some kind of uniqueness result for cocycles. In [24], §1, a similar problem is solved by appealing to the KMS condition. Here we try to keep the use of the modular structure to a minimal amount but again an analyticity requirement is involved. We borrow some ideas from [16], §6. More precisely, given a translation covariant sector ρ , we consider the unique *minimal* covariant representation $\{\rho, U_\rho\}$ and the associated cocycle $W_\rho(x) = U(x)U_\rho(x)^* \in (\rho, \alpha_x\rho\alpha_x^{-1})$. (Accordingly we select the extension $\hat{\alpha}$ of α , corresponding to this choice.) Now, following the pattern outlined in [8], §3, one can easily check that $\vartheta(W_\rho(x)) = W_{\vartheta\rho\vartheta}(-x)$, $x \in \mathbb{R}^4$, from which the conclusion follows.

3. *PCT on intermediate nets.* Since we know that ϑ extends to \mathcal{F} (once again considered to be Bosonic, for simplicity) one may like to consider the action of $\hat{\vartheta}$

⁴At this point we don’t know whether $\mathcal{F}_{\mathcal{F}} = \mathcal{F}$.

on any intermediate subnet $\mathcal{A} \subset \mathcal{B} \subset \mathcal{F}$. Since $\mathcal{B} = \mathcal{F}^L$ for some closed subgroup L of G [10], \mathcal{B} is stable under $\hat{\vartheta}$ if $[\hat{\vartheta}, \alpha_g] = 0$, $g \in L$. (This is obviously true whenever we know that $[\hat{\vartheta}, G] = 0$.) In this situation we expect $\hat{\vartheta}|_{\mathcal{B}}$ to inherit all the properties of a PCT operator from $\hat{\vartheta}$ (in fact from ϑ).

4. *PCT on the net generated by the local charges* (see [7] and references therein). If Λ and $\Lambda' = \text{Ad}(U)\Lambda$ are two anti-isomorphic W^* Standard Split Inclusions via the anti-linear unitary U , then $W_{\Lambda'}U = U \otimes UW_{\Lambda}$ so that $\psi_{\Lambda}(U^*TU) = U^*\psi_{\Lambda'}(T)U$, $T \in B(\mathcal{H})$, and $U\eta_{\Lambda} = \eta_{\Lambda'}$ (ψ_{Λ} is the universal localizing map associated to Λ and η_{Λ} is the product vector [4]).

If $\hat{\vartheta} = \text{Ad}(\hat{\Theta})$ then we get $\psi_{\mathcal{O}, \hat{\Theta}, \Omega}(\hat{\Theta}^*T\hat{\Theta}) = \hat{\Theta}^*\psi_{-\mathcal{O}, -\hat{\Theta}, \Omega}(T)\hat{\Theta}$, $T \in B(\mathcal{H})$. In particular, $\hat{\Theta}\psi_{\mathcal{O}, \hat{\Theta}, \Omega}(V(x))\hat{\Theta}^* = \psi_{-\mathcal{O}, -\hat{\Theta}, \Omega}(V(-x))$, $x \in \mathbb{R}^4$, from which, for the net \mathcal{C}_t generated by the local energy-momentum tensor, it immediately follows that $\hat{\Theta}\mathcal{C}_t(\mathcal{O})\hat{\Theta}^* = \mathcal{C}_t(-\mathcal{O})$. The same argument goes through for the net generated by the local charges.

4 PCT on the field bundle

In this section we start discussing extensions of ϑ in a slightly different context. We present some elementary computations and hints for further investigations.

We consider the ‘‘field bundle operators’’ $\underline{F} = \{\rho, A\}$ where $\rho \in \Delta_t(\mathcal{A})$ (as usual Δ_t denotes the set of all DHR morphisms of \mathcal{A}) and $A \in \mathcal{A}$. These are intrinsic operators that play the role of the unobservable fields, and are quite well suited for treating certain particle aspects of superselection sectors when the field algebra is not available. (This may be the case in theories obtained via a scaling limit procedure, or in lower dimensions.)

Denote by \mathcal{FB} the family of all such operators (see e.g. [12] for a full account of their properties). Then \mathcal{FB} is a bundle of algebras over Δ_t with fiber \mathcal{A} . Suppose we are given an involutive PCT anti-automorphism of \mathcal{A} . Our first aim is to extend it to (an involutive anti-automorphism of) \mathcal{FB} . So we need to explain what we mean by ‘‘extension.’’⁵ Loosely speaking, it is a structure-preserving bijection of \mathcal{FB} coinciding with $(\iota, A) \mapsto (\iota, \theta(A))$ (modulo inners) when restricted to (ι, \mathcal{A}) . Note that $\{\rho, I\}\{\iota, A\} = \{\rho, A\}$. (Moreover, it should be well-behaved with respect to taking conjugates.) We refrain from giving more details, and just notice that, whatever the possibilities are, there is a natural candidate expressed by the formula $\underline{\theta}(\underline{F}) = (\vartheta\rho\vartheta, \vartheta(A))$ (in particular, it is likely that one extension always exists). Let us look at the very first consequences of this definition. We have $\underline{\theta}^2 = \text{id}_{\mathcal{FB}}$, and $\underline{\theta}(\mathcal{FB}) = \mathcal{FB}$. Moreover, this is an antilinear morphism for the natural associative law, namely $\underline{\theta}(E_1E_2) =$

⁵We discuss here only the case of a single (anti-)automorphism, but one-parameter groups or even more general group actions should be treated along the same lines.

$\underline{\theta}(\underline{E}_1)\underline{\theta}(\underline{E}_2)$. In fact

$$\begin{aligned} \underline{\theta}(\{\rho_1, A_1\}\{\rho_2, A_2\}) &= \underline{\theta}(\{\rho_2\rho_1, \rho_2(A_1)A_2\}) \\ &= \{\vartheta\rho_2\rho_1\vartheta, \vartheta(\rho_2(A_1)A_2)\} \\ &= \{\vartheta\rho_2\vartheta\rho_1\vartheta, (\vartheta\rho_2\vartheta)(\vartheta(A_1))\vartheta(A_2)\} \\ &= \underline{\theta}(\{\rho_1, A_1\})\underline{\theta}(\{\rho_2, A_2\}). \end{aligned}$$

In particular $[\underline{E}_1, \underline{E}_2] = 0$ iff $[\underline{\theta}E_1, \underline{\theta}E_2] = 0$.

A net structure is put on \mathcal{FB} by declaring that $\underline{E} = \{\rho, A\} \in \mathcal{FB}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$ if there is a unitary $U \in (\rho, \rho')$ with $\rho' \in \Delta_t(\mathcal{O})$ and $UA \in \mathcal{A}(\mathcal{O})$. Here, as usual, \mathcal{K} denotes the set of open double cones in Minkowski spacetime. Then $\underline{\theta}(\mathcal{FB}(\mathcal{O})) = \mathcal{FB}(-\mathcal{O})$, namely $\underline{E} = \{\rho, A\} \in \mathcal{FB}(\mathcal{O})$ iff $\underline{\theta}(\underline{E}) \in \mathcal{FB}(-\mathcal{O})$. In fact $\vartheta(U) \in (\vartheta\rho\vartheta, \vartheta\rho'\vartheta)$, $\vartheta\rho'\vartheta \in \Delta_t(-\mathcal{O})$ and $\vartheta(U)\vartheta(A) = \vartheta(UA) \in \mathcal{A}(-\mathcal{O})$.

Recall that if $T \in (\rho, \rho')$ then by definition $T \circ \{\rho, A\} = \{\rho', TA\}$. Since $T \in (\rho, \rho')$ iff $\vartheta(T) \in (\vartheta\rho\vartheta, \vartheta\rho'\vartheta)$ we have

$$\begin{aligned} \underline{\theta}(T \circ \{\rho, A\}) &= \{\vartheta\rho'\vartheta, \vartheta(TA)\} \\ &= \vartheta(T) \circ \{\vartheta\rho\vartheta, \vartheta(A)\} \\ &= \vartheta(T) \circ \underline{\theta}(\{\rho, A\}). \end{aligned}$$

Then $\underline{\theta}$ preserves the redundancies present in the field bundle.

If $\underline{E}_1 = \{\rho_1, A_1\} \in \mathcal{FB}(\mathcal{O}_1)$ and $\underline{E}_2 = \{\rho_2, A_2\} \in \mathcal{FB}(\mathcal{O}_2)$ with $\mathcal{O}_1 \subset \mathcal{O}'_2$ then we have $\underline{E}_1\underline{E}_2 = \epsilon(\rho_1, \rho_2) \circ \underline{E}_2\underline{E}_1$. Now $\underline{\theta}(\underline{E}_1) \in \mathcal{FB}(-\mathcal{O}_1)$ and $\underline{\theta}(\underline{E}_2) \in \mathcal{FB}(-\mathcal{O}_2)$, and therefore

$$\underline{\theta}(\underline{E}_1)\underline{\theta}(\underline{E}_2) = \epsilon(\vartheta\rho_1\vartheta, \vartheta\rho_2\vartheta) \circ \underline{\theta}(\underline{E}_2)\underline{\theta}(\underline{E}_1).$$

On the other hand, we have

$$\begin{aligned} \underline{\theta}(\underline{E}_1)\underline{\theta}(\underline{E}_2) &= \underline{\theta}(\underline{E}_1\underline{E}_2) \\ &= \underline{\theta}(\epsilon(\rho_1, \rho_2) \circ \underline{E}_2\underline{E}_1) \\ &= \vartheta(\epsilon(\rho_1, \rho_2)) \circ \underline{\theta}(\underline{E}_2\underline{E}_1). \end{aligned}$$

In other words, in order to extend ϑ to \mathcal{FB} via the aforementioned formula, compatibility with spacelike commutation relations forces the identity $\vartheta(\epsilon(\rho, \sigma)) = \epsilon(\vartheta\rho\vartheta, \vartheta\sigma\vartheta)$ for every $\rho, \sigma \in \Delta_t$. Cf. condition 2) in [8, Proposition 2.6].

If we consider only covariant morphisms, spacetime symmetries are lifted to \mathcal{FB} according to $\underline{\alpha}_L\{\rho, A\} = \{\rho, X_L(\rho)^{-1}\alpha_L(A)\} = \{\rho, U_\rho(L)AU(L)^*\}$, where $X_L(\rho) = U(L)U_\rho(L)^* \in \mathcal{A}(\mathcal{O})' \cap \mathcal{A}(L\mathcal{O})' \subset \mathcal{A}$ if ρ is localized in \mathcal{O} . We expect a relation like $\underline{\theta}\alpha_x = \underline{\alpha}_{-x}\underline{\theta}$ ($x \in \mathbb{R}^4$). We compute

$$\begin{aligned} \underline{\theta}\alpha_x\{\rho, A\} &= \underline{\theta}\{\rho, U_\rho(x)U(x)^*\alpha_x(A)\} = \{\vartheta\rho\vartheta, \vartheta(U_\rho(x)U(x)^*)\vartheta(\alpha_x(A))\}, \\ \underline{\alpha}_{-x}\underline{\theta}\{\rho, A\} &= \underline{\alpha}_{-x}\{\vartheta\rho\vartheta, \vartheta(A)\} = \{\vartheta\rho\vartheta, U_{\vartheta\rho\vartheta}(-x)U(-x)^*\alpha_{-x}(\vartheta(A))\}, \end{aligned}$$

so that we are left with the identity $\vartheta(U_\rho(x)U(x)^*) = U_{\vartheta\rho\vartheta}(-x)U(-x)^*$. Now $U_\rho(x)U(x)^* = X_x(\rho)^* \in (\alpha_x\rho\alpha_{-x}, \rho)$, and thus both $\vartheta(U_\rho(x)U(x)^*)$ and

$U_{\vartheta\rho\vartheta}(-x)U(-x)^*$ are elements of $(\alpha_{-x}\vartheta\rho\vartheta\alpha_x, \vartheta\rho\vartheta)$. If ρ , thus $\vartheta\rho\vartheta$, are irreducible, the two unitaries can only differ by a phase factor. More generally, see the discussion in [8] after Theorem 3.1 and in the previous section.

Restricting ourselves to morphisms with finite statistics, there is a well-defined conjugation on \mathcal{FB} . Let Δ_r be the set of covariant DHR morphisms with finite statistics. Given $\rho \in \Delta_r$ there is $\bar{\rho} \in \Delta_r$, and operators $R \in (\iota, \bar{\rho}\rho)$, $\bar{R} \in (\iota, \rho\bar{\rho})$,⁶ such that $\bar{R}^*\rho(R) = I = R^*\bar{\rho}(\bar{R})$, $\bar{R} = \text{sign}(\lambda_\rho)\epsilon(\bar{\rho}, \rho) \circ R$, and $R^*R = \bar{R}^*\bar{R} = d(\rho)I$. If $\underline{F} = \{\rho, A\}$ with $\rho \in \Delta_r$, set $\underline{F}^\dagger = \{\bar{\rho}, \bar{\rho}(A)^*R\}$. Using the convention that $\bar{\bar{\rho}} = \rho$ and \bar{R} is the intertwiner associated with $\bar{\rho}$, we have $\underline{F}^{\dagger\dagger} = \underline{F}$. Note that $\{\iota, A\}^\dagger = \{\iota, A^*\}$. It is also easy to check that $\underline{F} \in \mathcal{FB}(\mathcal{O})$ iff $\underline{F}^\dagger \in \mathcal{FB}(\mathcal{O})$. We have

$$\begin{aligned} (\underline{F}_1\underline{F}_2)^\dagger &= \{\rho_2\rho_1, \rho_2(A_1)A_2\}^\dagger \\ &= \{\bar{\rho}_2\bar{\rho}_1, \bar{\rho}_2\bar{\rho}_1(\rho_2(A_1)A_2)^*R_{\rho_2\rho_1}\} \\ &= \{\bar{\rho}_2\bar{\rho}_1, \bar{\rho}_2\bar{\rho}_1(A_2^*)\bar{\rho}_2\bar{\rho}_1(\rho_2(A_1^*))R_{\rho_2\rho_1}\} \end{aligned}$$

while

$$\begin{aligned} \underline{F}_2^\dagger\underline{F}_1^\dagger &= \{\bar{\rho}_2, \bar{\rho}_2(A_2)^*R_{\rho_2}\}\{\bar{\rho}_1, \bar{\rho}_1(A_1)^*R_{\rho_1}\} \\ &= \{\bar{\rho}_1\bar{\rho}_2, \bar{\rho}_1(\bar{\rho}_2(A_2)^*R_{\rho_2})\bar{\rho}_1(A_1)^*R_{\rho_1}\} \\ &= \{\bar{\rho}_1\bar{\rho}_2, \bar{\rho}_1\bar{\rho}_2(A_2^*)\bar{\rho}_1(R_{\rho_2}A_1^*)R_{\rho_1}\} \\ &= \{\bar{\rho}_1\bar{\rho}_2, \bar{\rho}_1\bar{\rho}_2(A_2^*)\bar{\rho}_1(\bar{\rho}_2\rho_2(A_1^*)R_{\rho_2})R_{\rho_1}\} \end{aligned}$$

from which the antimultiplicative character of \dagger readily follows⁷.

Now $\vartheta\rho\vartheta$ and $\vartheta\bar{\rho}\vartheta$ are both in Δ_r , $\vartheta(R) \in ((\iota, \vartheta\bar{\rho}\vartheta\vartheta\rho\vartheta))$, $\vartheta(\bar{R}) \in (\iota, \vartheta\rho\vartheta\vartheta\bar{\rho}\vartheta)$, and we easily check the relations⁸

$$\begin{aligned} \vartheta(\bar{R})^*\vartheta\rho\vartheta(\vartheta(R)) &= I = \vartheta(R)^*\vartheta\bar{\rho}\vartheta(\vartheta(\bar{R})), \\ \vartheta(\bar{R}) &= \text{sign}(\lambda_{\vartheta\rho\vartheta})\epsilon(\vartheta\bar{\rho}\vartheta, \vartheta\rho\vartheta) \circ \vartheta(R), \\ \vartheta(R)^*\vartheta(R) &= \vartheta(\bar{R})^*\vartheta(R) = d(\vartheta\rho\vartheta)I, \end{aligned}$$

i.e. $\vartheta\bar{\rho}\vartheta = \overline{\vartheta\rho\vartheta}$. If $\underline{F} = \{\rho, A\}$ we compute

$$(\underline{\theta F})^\dagger = \{\overline{\vartheta\rho\vartheta}, (\overline{\vartheta\rho\vartheta})(\vartheta(A))^*\vartheta(R)\}$$

and

$$\underline{\theta}(\underline{F}^\dagger) = \{\vartheta\bar{\rho}\vartheta, \vartheta(\bar{\rho}(A)^*R)\},$$

from which we conclude that $(\underline{\theta F})^\dagger = \underline{\theta}(\underline{F}^\dagger)$. Summing up, it is tempting to say that $\underline{\theta} \circ \dagger = \dagger \circ \underline{\theta}$ implements an isomorphism of \mathcal{FB} with $\mathcal{FB}^{\text{opp}}$ ($=\mathcal{FB}$ as a set, but with reversed product). The restriction of this isomorphism to \mathcal{A} gives

⁶Sometimes we write R_ρ, \bar{R}_ρ to stress the dependence on ρ .

⁷It is well-known that $\rho_1, \rho_2 \in \Delta_r \Rightarrow \rho_2\rho_1 \in \Delta_r$. We may, and will, assume to have chosen a map $\rho \mapsto \bar{\rho}$ such that $\bar{\rho}_2\bar{\rho}_1 = \bar{\rho}_1\bar{\rho}_2$, $R_{\rho_2\rho_1} = \bar{\rho}_1(R_{\rho_2})R_{\rho_1}$ and $\bar{R}_{\rho_2\rho_1} = R_{\bar{\rho}_2\bar{\rho}_1} = R_{\bar{\rho}_1\bar{\rho}_2} = \rho_2(\bar{R}_{\rho_1})\bar{R}_{\rho_2}$.

⁸Note that $\lambda_{\vartheta\rho\vartheta} = \lambda_\rho$.

the usual isomorphism $\vartheta \circ * : \mathcal{A} \rightarrow \mathcal{A}^{\text{opp}}$. Of course, using this isomorphism we also get an isomorphism between \mathcal{FB} and $\mathcal{FB}(\mathcal{A}^{\text{opp}})$.

So far we have not used the fact that $\vartheta\rho\vartheta$ may express a conjugate of ρ . However this is bound to play a crucial role. For instance in [12, Section VI] a charge conjugation C and a Spin-Statistics Theorem are obtained using Poincaré covariance and positivity of the energy. By combining the properties of \dagger with the information that $\vartheta\rho\vartheta$ is a conjugate of ρ we get a (linear, antimultiplicative) “fiberwise” formula expressing the action of $\underline{\theta}$ on \mathcal{FB} . More precisely, for every $\rho \in \Delta_r$ choose a unitary “cocycle” $X_\vartheta(\rho) \in (\rho, \vartheta\bar{\rho}\vartheta)$. (Notice that even if ρ is irreducible, such an intertwiner is unique only up to the choice of a phase.) Therefore $X_\vartheta(\rho)^* \circ \underline{\theta}(\{\rho, A\}^\dagger) = \{\rho, X_\vartheta(\rho)^*\vartheta(\bar{\rho}(A)^*R)\}$, and we are left with a (non-uniquely defined) fiber-preserving map $\{\rho, A\} \mapsto \{\rho, X_\vartheta(\rho)^*\vartheta(\bar{\rho}(A)^*R)\}$ (in fact an extension of $\vartheta \circ *$). At this point, it is natural to look for a converse and ask to what extent such or similar maps encompass the information that $\vartheta\rho\vartheta$ is actually a conjugate of ρ . So we are faced with the following problem: “What are the relevant properties of the extensions of ϑ to \mathcal{FB} ensuring that $\vartheta\rho\vartheta \simeq \bar{\rho}$ for every ρ with finite statistics?” In the field algebra picture, the answer was the existence of an extension commuting with the gauge action. Making due allowance for the somewhat different setting, we propose as a sufficient condition the existence of a fiber-preserving extension of $\vartheta \circ *$. But probably a new insight is needed.

In low-dimensional QFT, where no canonical description of the field algebra is available yet, an alternative construction is provided by the *reduced field bundle* [19]. It seems worthwhile to give a version of our discussion for that case. We plan to return on this topic elsewhere.

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