Fourth Mississippi State Conference on Differential Equations
and Computational Simulations, Electronic Journal of Differential Equations,
Conference 03, 1999, pp 63–73.
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Uniqueness implies existence for discrete fourth order Lidstone boundary-value problems *

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Abstract

We study the fourth order difference equation

$$u(m+4) = f(m, u(m), u(m+1), u(m+2), u(m+3))\,,$$

where $f : \mathbb{Z}\times\mathbb{R}^4 \to \mathbb{R}$ is continuous and the equation $u_5 = f(m, u_1, u_2, u_3, u_4)$ can be solved for u_1 as a continuous function of u_2, u_3, u_4, u_5 for each $m \in \mathbb{Z}$. It is shown that the uniqueness of solutions implies the existence of solutions for Lidstone boundary-value problems on $\mathbb{Z}$. To this end we use shooting and topological methods.

1 Introduction

For notation, $\mathbb{R}$ denotes the real numbers, $\mathbb{Z}$ denotes the integers, $\mathbb{N}$ denotes the natural numbers, and given $a < b$ in $\mathbb{Z}$, intervals are used to denote discrete sets such as $[a, b] = \{a, a+1, \dots, b\}$, $(a, b] = \{a+1, \dots, b\}$, $[a, +\infty) = \{a, a+1, \dots\}$, etc. This paper is devoted to establishing the existence of unique solutions of the fourth order finite difference equation,

$$u(m+4) = f(m, u(m), u(m+1), u(m+2), u(m+3)), \tag{1}$$

satisfying boundary conditions

$$\begin{aligned} u(m_1) &= u_1, \\ \Delta^2 u(m_2) &= u_2, \\ \Delta^2 u(m_3) &= u_3, \\ u(m_4) &= u_4, \end{aligned} \tag{2}$$

where $m_i \in \mathbb{Z}$ and $u_i \in \mathbb{R}$ for $1 \le i \le 4$, and $m_1 \le m_2 < m_3 - 2 \le m_4 - 4$.

Such boundary-value problems are called *Lidstone boundary-value problems* because of their analogy to Lidstone problems in the continuous case.

We assume throughout that the following condition holds:

* *Mathematics Subject Classifications:* 39A10, 34B10, 34B15.
Key words: Difference equation, uniqueness, existence.

Published July 10, 2000.

(A) $f(m, x_1, x_2, x_3, x_4) : \mathbb{Z} \times \mathbb{R}^4 \to \mathbb{R}$ is continuous and the equation, $x_5 = f(m, x_1, x_2, x_3, x_4)$ can be solved for x_1 as a continuous function of x_2, x_3, x_4, x_5.

Remark 1.1 *We note that (A) implies solutions of initial value problems for (1) are unique and exist on all of* $\mathbb{Z}$.

The problems which will be considered will be classified as 2-point, 3-point, and 4-point. Unique solutions are obtained by using a shooting method. It will be clarified later as to what is meant by uniqueness of solutions.

Uniqueness of solutions implying their existence for boundary value problems for ordinary differential equations enjoys quite a history. Motivation for considering a uniqueness result is that this result may imply the existence of solutions of the boundary-value problem. In the case where $n = 2$, Lasota and Luczyński [12] have shown that with respect to the ordinary differential equation

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}), \tag{3}$$

if the following conditions are satisfied, then each right $(1, 1)$-focal point boundary-value problem has a solution on $I = \{x \in \mathbb{R} \mid a < x < b\}$:

(i) $f(x, y_1, y_2, \ldots, y_n)$ is continuous on $I \times \mathbb{R}^n$.

(ii) Solutions of initial value problems for (3) are unique.

(iii) Solutions of initial value problems for (3) extend to I.

(iv) Each right $(1, 1, \ldots, 1)$-focal point boundary value problem for (3) on I has at most one solution.

In regard to k-point conjugate boundary-value problems, $2 \le k \le n$, for (3) on I, Hartman [2] and Klaasen [11] have proven with conditions (i) - (iii) and the following conditions that each n-point conjugate boundary-value problem for (3) has a solution on I:

(v) Each n-point conjugate boundary valued problem for (3) on I has a most one solution.

(vi) If $\{y_k(x)\}$ is a sequence of solutions of (3) and K is a compact subinterval of I such that $\{y_k(x)\}$ is uniformly bounded on K, then there exists a subsequence $\{y_{k_j}(x)\}$ such that $\{y_{k_j}^{(i)}(x)\}$ converges uniformly on K, $0 \le i \le n - 1$.

Hartman [3] has also proven that if equation (3) satisfies conditions (i)–(iii) and (v), and if each n-point conjugate boundary-value problem for (3) has a solution on I, then all k-point conjugate boundary-value problems, $2 \le k \le n$, for (3) have unique solutions. Henderson [7, 8] also proved uniqueness implies existence for solutions of right focal boundary-value problems.

For finite difference equations of arbitrary order, Henderson [5, 6, 9] showed that uniqueness of solutions implies their existence for conjugate boundary-value problems as well as for right focal boundary value problems. Then in a very recent work, Davis and Henderson [1] obtained uniqueness implies existence results for fourth order Lidstone boundary-value problems for ordinary differential equations. It is these later works which are the primary motivation for this paper.

In Section 2, all preliminary results will be stated. These results include the discrete Rolle's theorem by Hartman [2], the Brouwer invariance of domain theorem, two compactness conditions, and a continuous dependence on initial values result. A definition of generalized zero is also stated.

In Section 3, the existence of the unique solution of the 2-point problem is established. Shooting methods are used in conjunction with topological methods involving the connectedness of the real line. In Section 4, each of the 3-point problems will be considered. We make use of the solution of the 2-point problem obtained in Section 3 and employ shooting methods again. In Section 5, the existence of a solution of the 4-point problem will be established. The existence of solutions of the 3-point problems will be used in obtaining the unique solution of the 4-point problem. Again, shooting methods are used.

2 Preliminaries

In this section, we state a definition and auxiliary results which will be fundamental in the development of later sections. The definition extends the idea of a zero of a function, and was introduced in Hartman's [4] landmark paper.

Definition Suppose $u : \mathbb{Z} \to \mathbb{R}$. Then u has a *generalized zero* (g.z.) at $m_0 \in \mathbb{Z}$ if $u(m_0) = 0$, or if there exists $j \in \mathbb{N}$ such that $(-1)^j u(m_0 - j)u(m_0) > 0$, and if $j > 1$,

$$u(m_0 - j + 1) = \cdots = u(m_0 - 1) = 0\,.$$

Hartman [4] also proved the following discrete Rolle's Theorem.

Theorem 2.1 *If $u(m)$ has a g.z. at a and at b $(a < b)$ in $\mathbb{Z}$, then $\Delta u(m) = u(m+1) - u(m)$ has a g.z. on $[a, b)$.*

In view of the above definition, we now state our fundamental uniqueness assumption on boundary-value problems for (1).

(B) For any $m_1, m_2, m_3, m_4 \in \mathbb{Z}$, if $\phi(m)$ and $\psi(m)$ are solutions of (1) such that $\phi(m) - \psi(m)$ has a g.z. at m_1, $\Delta(\phi(m) - \psi(m))$ has a g.z. at m_2, $\Delta^2(\phi(m) - \psi(m))$ has a g.z. at m_3, and $\Delta^3(\phi(m) - \psi(m))$ has a g.z. at m_4, then $\phi(m) \equiv \psi(m)$ on $\mathbb{Z}$.

Remark 2.2 *If (B) holds, we say that (1) is* disfocal *on $\mathbb{Z}$. It follows from (B) and repeated applications of Theorem 2.1 that solutions of any conjugate, right focal, or Lidstone boundary-value problems are unique when such solutions exist.*

Many of our future results rely on continuous dependence arguments. The next theorem from Hurewicz and Wallman [10] provides a powerful tool in those arguments.

Theorem 2.3 (Brouwer Invariance of Domain) *Suppose $G \subset \mathbb{R}^n$ is open and $\phi : G \to \mathbb{R}^n$ is continuous and one-to-one. Then $\phi(G) \subset \mathbb{R}^n$ is open and $\phi : G \to \phi(G)$ is an open mapping; that is, ϕ is a homeomorphism.*

The next two theorems from Henderson [7] are also essential in the continuous dependence arguments.

Theorem 2.4 *Assume that with respect to (1), conditions (A) and (B) are satisfied. Given a solution $u(m)$ of (1) on $\mathbb{Z}$, points $s_1 < s_2 < s_3 < s_4$ belonging to $\mathbb{Z}$, an interval $[s_1, b]$, where $b \geq s_4$, and $\epsilon > 0$, there exists a solution $v(m)$ of (1) satisfying $v(s_i) = y_i$, $1 \leq i \leq 4$, and $|v(m) - u(m)| < \epsilon$, for all $m \in [s_1, b]$.*

Theorem 2.5 *Assume that (A) and (B) are satisfied and suppose that given $m_1 \in \mathbb{Z}$ and for some $m_2, \ldots, m_4 \in \mathbb{N}$, there exist unique solutions of (1) satisfying*

$$u(s_i) = y_i, \quad 1 \leq i \leq 4,$$

where $s_1 = m_1, s_2 = s_1 + m_2, s_3 = s_2 + m_3, s_4 = s_3 + m_4$. If there exist a sequence $\{y_k(m)\}$ of solutions of (1) and an $M > 0$ such that $|y_k(s_i)| \leq M$, $1 \leq i \leq n$, for all $k \in \mathbb{N}$, then there exists a subsequence $\{y_{k_j}(m)\}$ that converges pointwise on $\mathbb{Z}$. In particular, for this subsequence, if $\lim_j y_{k_j}(s_i) = y_i$, $1 \leq i \leq 4$, then $\{y_{k_j}(m)\}$ converges pointwise on $\mathbb{Z}$ to the solution of the conjugate boundary value problem for (1) satisfying

$$y(s_i) = y_i, \quad 1 \leq i \leq 4.$$

In addition, from uniqueness of solutions of initial value problems for (1), we also have continuous dependence of solutions on initial values. The proof of the following is straightforward.

Theorem 2.6 (Continuous Dependence on Initial Values) *Suppose $f(m, u_1, u_2, u_3, u_4) : \mathbb{Z} \times \mathbb{R}^4 \to \mathbb{R}$ is continuous. Let $u(m; m_0, u_0, u_1, u_2, u_3)$ be the solution of (1) satisfying the initial conditions,*

$$\begin{aligned} u(m_0) &= u_0, \\ u(m_0+1) &= u_1, \\ u(m_0+2) &= u_2, \\ u(m_0+3) &= u_3, \end{aligned}$$

on $\mathbb{Z}$ where $m_0 \in \mathbb{Z}$, $u_i \in \mathbb{R}$ for $0 \leq i \leq 3$. Then given $\epsilon > 0$, $k \in \mathbb{N}$, there exists a $\delta(\epsilon, m_0, k, u_0, u_1, u_2, u_3) > 0$ such that if $|u_i - v_i| < \delta$, $0 \leq i \leq 3$, then

$$|u(m; m_0, u_0, u_1, u_2, u_3) - u(m; m_0, v_0, v_1, v_2, v_3)| < \epsilon,$$

for every $m \in [m_0, m_0 + k]$.

3 Existence of Two-Point Problems

In this section, we consider the two-point problem for (1) satisfying

$$u(m_1) = u_1, \Delta^2 u(m_1) = u_2, \Delta^2 u(m_2) = u_3, u(m_2) = u_4, \tag{4}$$

where $m_1 < m_2 - 2$.

Theorem 3.1 *Suppose that with respect to (1), conditions (A) and (B) are satisfied. Then given $m_1 < m_2 - 2$ in $\mathbb{Z}$ and given $u_i \in \mathbb{R}, 1 \leq i \leq 4$, there exists a unique solution of (1), (4).*

Proof Let $m_1 < m_2 - 2$ in $\mathbb{Z}$ and $u_1, u_2, u_3, u_4 \in \mathbb{R}$ be given. Let $y(m)$ solve (1) subject to the conjugate boundary conditions,

$$\begin{aligned} y(m_1) &= u_1, \\ y(m_2) &= u_4, \\ \Delta y(m_2) &= 0, \\ \Delta^2 y(m_2) &= u_3. \end{aligned}$$

By Remark 2.2, $y(m)$ is unique. Define $S \subseteq \mathbb{R}$ by

$$S = \{\Delta^2 v(m_1) | v(m) \text{ is a solution of (1), and } v(m_1) = y(m_1) = u_1, \\ \Delta^2 v(m_2) = \Delta^2 y(m_2) = u_3, v(m_2) = y(m_2) = u_4\}.$$

Note that $S \neq \phi$, since $\Delta^2 y(m_1) \in S$.

We first show that S is an open subset of $\mathbb{R}$. Fix any $s \in S$. By the definition of S, there exists a solution $y_s(m)$ of (1) such that

$$\begin{aligned} y_s(m_1) &= y(m_1), \\ \Delta^2 y_s(m_1) &= s, \\ \Delta^2 y_s(m_2) &= \Delta^2 y(m_2), \\ y_2(m_2) &= y(m_2). \end{aligned}$$

Let $G = \mathbb{R}^4$. Now fix $t_0 \in \mathbb{Z}$. Define $\phi : G \to \mathbb{R}^4$ by

$$\phi((c_1, c_2, c_3, c_4)) = (p(m_1), \Delta^2 p(m_1), \Delta^2 p(m_2), p(m_2))$$

where $p(m)$ is the solution of the initial value problem for (1) satisfying

$$\Delta^{i-1} p(t_0) = c_i, 1 \leq i \leq 4.$$

By Theorem 2.6, solutions are continuous with respect to initial conditions and so ϕ is continuous. To see that ϕ is also one-to-one, suppose $\phi((c_1, c_2, c_3, c_4)) = \phi((d_1, d_2, d_3, d_4))$. Then there are solutions p and q of (1) such that

$$\begin{aligned} \phi((c_1, c_2, c_3, c_4)) &= (p(m_1), \Delta^2 p(m_1), \Delta^2 p(m_2), p(m_2)) \\ &= (q(m_1), \Delta^2 q(m_1), \Delta^2 q(m_2), q(m_2)) \\ &= \phi((d_1, d_2, d_3, d_4)), \end{aligned}$$

where $\Delta^{i-1}p(t_0) = c_i$ $1 \le i \le 4$, and $\Delta^{i-1}q(t_0) = d_i$, $1 \le i \le 4$. But by the uniqueness of solutions of Lidstone boundary-value problems, $p(m) = q(m)$ for every m in $\mathbb{Z}$. Then $(c_1, c_2, c_3, c_4) = (d_1, d_2, d_3, d_4)$, and thus ϕ is one-to-one. Hence, by Theorem 2.4, it follows that $\phi(G) \subseteq \mathbb{R}^4$ is open, and $\phi : G \to \phi(G)$ is a homeomorphism. Thus ϕ^{-1} is continuous on $\phi(G)$. So there exists $\delta_0 > 0$ such that for every δ, where $0 < |\delta| < \delta_0$, there is a solution $y_\delta(m)$ of (1) which satisfies

$$\begin{aligned} y_\delta(m_1) &= y_s(m_1) \\ \Delta^2 y_\delta(m_1) &= \Delta^2 y_s(m_1) + \delta = s + \delta \\ \Delta^2 y_\delta(m_2) &= \Delta^2 y_s(m_2) \\ y_\delta(m_2) &= y_s(m_2). \end{aligned}$$

This follows because $\phi(G)$ is open, ϕ^{-1} is continuous, and $(y_s(t_0), \Delta y_s(t_0), \Delta^2 y_2(t_0), \Delta^3 y_s(t_0)) \in G$. Thus $(s - \delta_0, s + \delta_0) \subset S$. Thus S is an open subset of $\mathbb{R}$.

We next show that S is also a closed subset of $\mathbb{R}$. Assume otherwise. Then we have the existence of a limit point $n_0 \in \bar{S} \backslash S$ and a strictly monotone sequence $\{n_k\}_{k=1}^{\infty} \subset S$ such that $n_k \to n_0$. Assume without loss of generality that $n_k \uparrow n_0$. By the definition of S, for each $k \ge 1$, there exists a solution $y_k(m)$ of (1) such that

$$\begin{aligned} y_k(m_1) &= y(m_1) = u_1 \\ \Delta^2 y_k(m_1) &= n_k \\ \Delta^2 y_k(m_2) &= \Delta^2 y(m_2) = u_3 \\ y_k(m_2) &= y(m_2) = u_4. \end{aligned}$$

By Theorem 2.1 and the disfocality of (1), it follows that for all $k \ge 1$, we have $\Delta^2 y_{k+1}(m) > \Delta^2 y_k(m)$ on $(-\infty, m_2)$. Since $y_k(m_1) = y_{k+1}(m_1)$ and $y_k(m_2) = y_{k+1}(m_2)$, we must have that $y_k(m) < y_{k+1}(m)$ on $(-\infty, m_2] \backslash \{m_1\}$, for all $k \ge 1$. We also claim that $\{y_k(m)\}_{k=1}^{\infty}$ is not bounded on any finite subset of $\mathbb{Z}$ having at least four points. To see this, assume otherwise. If $[c, d] \subset (a, b)$ and $M > 0$ are such that $|y_k(m)| \le M$ for all $m \in [c, d]$ and for all $k \ge 1$, then by Theorem 2.5, there is a subsequence $\{y_{k_j}(m)\}_{j=1}^{\infty}$ such that $\{y_{k_j}(m)\}_{j=1}^{\infty}$ converges uniformly on each finite subset of $\mathbb{Z}$. Also by Theorem 2.5, we have that this subsequence converges uniformly on finite subsets of $\mathbb{Z}$ to the solution $z(m)$ of (1) where

$$\begin{aligned} z(m_1) &= u_1 \\ \Delta^2 z(m_1) &= n_0 \\ \Delta^2 z(m_2) &= u_3 \\ z(m_2) &= u_4. \end{aligned}$$

But this implies that $n_0 \in S$ which is a contradiction. So it must be that $\{y_k(m)\}_{k=1}^{\infty}$ is not uniformly bounded on any finite subset of $\mathbb{Z}$ having at least four points.

Now, let w denote the unique solution of the conjugate boundary value problem consisting of (1) with boundary conditions

$$\begin{aligned} w(m_1) &= y_k(m_1), \\ \Delta w(m_1) &= 0, \\ \Delta^2 w(m_1) &= n_0, \\ w(m_2) &= y_k(m_2). \end{aligned}$$

(We note that $\Delta^2 w(m_1) > \Delta^2 y_k(m_1)$, for all $k \geq 1$.) It follows that for every $k \geq 1$, $w(m) > y_k(m)$ at $m = m_1+1$ or $m = m_1+2$. Since $\{y_k(m)\}_{k=1}^{\infty}$ is unbounded above on each finite subset of $(-\infty, m_2)\backslash\{m_1\}$, so also is $\{\Delta^2 y_k(m)\}_{k=1}^{\infty}$ at points of (m_1, m_2). Then there exists integers α and β where $m_1 < \alpha < \beta < m_2$ such that $(w - y_k)(m)$ has a g.z. at β and $\Delta^2(w - y_k)(m)$ has a g.z. at α. Moreover, it follows from $\Delta^2 w(m_1) > \Delta^2 y_k(m_1)$ and Theorem 2.1 that there exists $\gamma \in [a, \beta)$ such that $\Delta(w - y_k)(m)$ has a g.z. at γ. Also, there exists a $\rho \in [\gamma, m_2)$ such that $\Delta^2(w - y_k)(m)$ has a g.z. at ρ. Thus, we have that $(w - y_k)(m)$ has a g.z. at m_1 and m_2 and that $\Delta^2(w - y_k)(m)$ has a g.z. at α and ρ. So it must be that $w(m) = y_k(m)$, for all $m \in \mathbb{Z}$, by the uniqueness of Lidstone boundary-value problems. But $\Delta^2 q(m_1) = n_0 > \Delta^2 y_k(m_1)$ for all $k \geq 1$. This is a contradiction. Hence S is a closed set.

Since $S \subseteq \mathbb{R}$ is both open and closed, we must have that $S = \mathbb{R}$. Thus, choosing $u_2 \in S$, the corresponding solution is the desired unique solution of (1), (4). ♢

4 Existence of Three-Point Problems

In this section, we establish the existence of solutions of the Three-point Lidstone boundary-value problems. There are two such problems for (1), those satisfying either

$$\begin{aligned} u(m_1) &= u_1, \\ \Delta^2 u(m_1) &= u_2, \\ \Delta^2 u(m_2) &= u_3, \\ u(m_3) &= u_4, \end{aligned} \tag{5}$$

where $m_1 < m_2 - 2 < m_3 - 4$, or

$$\begin{aligned} u(m_1) &= u_1, \\ \Delta^2 u(m_2) &= u_2, \\ \Delta^2 u(m_3) &= u_3, \\ u(m_3) &= u_4. \end{aligned} \tag{6}$$

where $m_1 < m_2 < m_3 - 2$.
We will make the argument for solutions of only (1), (5).

Theorem 4.1 *Suppose that with respect to (1), conditions (A) and (B) are satisfied. Then, given $m_1 < m_2 - 2 < m_3 - 4$ in $\mathbb{Z}$ and given $u_1, u_2, u_3, u_4 \in \mathbb{R}$, there exists a unique solution of (1), (5).*

Proof Let integers $m_1 < m_2 - 2 < m_3 - 4$ be given, and let $u_1, u_2, u_3, u_4 \in \mathbb{R}$. Let $y(m)$ be the solution of (1), (4) given by Theorem 3.1 which satisfies

$$\begin{aligned} y(m_1) &= u_1, \\ \Delta^2 y(m_1) &= u_2, \\ \Delta^2 y(m_2) &= u_3, \\ y(m_2) &= 0. \end{aligned}$$

Define a set $S \subseteq \mathbb{R}$ by

$$S = \{v(m_3) \mid v \text{ is a solution of (1), and } v(m_1) = y(m_1) = u_1, \\ \Delta^2 v(m_1) = \Delta^2 y(m_1) = u_2, \text{ and } \Delta^2 v(m_2) = \Delta^2 y(m_2) = u_3\}.$$

Note that $S \neq \phi$, since $y(m_3) \in S$. Along the lines of the argument in Theorem 3.1, it follows that S is also open.

We next show that S is also a closed subset of $\mathbb{R}$. We assume otherwise. Then there exist a limit point $n_0 \in \bar{S} \backslash S$ and a monotone sequence $\{n_k\}_{k=1}^{\infty} \subset S$ such that $n_k \to n_0$. Assume without loss of generality that $\{n_k\}_{k=1}^{\infty}$ is increasing. By definition of S, for $k \geq 1$, there exists a solution $y_k(m)$ of (1) such that

$$\begin{aligned} y_k(m_1) &= y(m_1) \\ \Delta^2 y_k(m_1) &= \Delta^2 y(m_1) \\ \Delta^2 y_k(m_2) &= \Delta^2 y(m_2) \\ y_k(m_3) &= n_k. \end{aligned}$$

It then follows from Theorem 2.1 and the disfocality of (1) that for every $k \geq 1$, $y_k(m) < y_{k+1}(m)$ on (m_1, ∞). Also, $\{y_k(m)\}_{k=1}^{\infty}$ is unbounded above on each finite subinterval of $[m_2, \infty) \backslash \{m_3\}$. Let w denote the unique solution of the conjugate boundary-value problem consisting of (1) with boundary conditions

$$\begin{aligned} w(m_1) &= y_k(m_1), \\ \Delta w(m_1) &= 0, \\ \Delta^2 w(m_1) &= \Delta^2 y_k(m_1), \\ w(m_3) &= n_0. \end{aligned}$$

(Note that $w(m_3) > y_k(m_3)$, for all $k \geq 1$.) Since $\{y_k(m_3-1)\}_{k=1}^{\infty}$ and $\{y_k(m_3+1)\}_{k=1}^{\infty}$ are not bounded above, but $y_k(m_3) < y(m_3)$ for all k, there exists a $k \geq 1$ such that $(w - y_k)(m)$ has a g.z. at m_3 and a g.z. at $m_3 + 1$. We also have that $\Delta(w - y_k)(m)$ has a g.z. at m_3. Thus, there exists a $v \in [m_1, m_3)$ such that $\Delta(w - y_k)(m)$ has a g.z. at v. So there must be an $r \in [m_1, v-1]$ such that $\Delta^2(w - y_k)(m)$ has a g.z. at r. But $w(m_1) = y_k(m_1)$ and $\Delta^2 w(m_1) = \Delta^2 y_k(m)$. Thus, utilizing the points $m_1, m_3 + 1$, r, it follows by the uniqueness of 3-point Lidstone boundary value problems that we must have $w(m) = y_k(m)$ for all $m \in \mathbb{Z}$. But $w(m_3) = n_0$ where $n_0 > y_k(m_3)$ for all $k \geq 1$. This is a contradiction. Thus S must be a closed subset of $\mathbb{R}$.

Since $S \subseteq \mathbb{R}$ is both open and closed, we must have that $S = \mathbb{R}$. Thus choosing $u_4 \in S$, the corresponding solution is the desired unique solution of (1), (5). $\diamondsuit$

Since the argument for solutions of (1), (6) is completely analogous, we state the next theorem without proof.

Theorem 4.2 *Suppose that with respect to (1) conditions (A) and (B) are satisfied. Then, given* $m_1 < m_2 < m_3 - 2$ *and given* $u_1, u_2, u_3, u_4 \in \mathbb{R}$*, there exists a unique solution of (1) satisfying (6).*

5 Existence of Four-Point Problems

In this section, we establish the existence of solutions of the Four-point Lidstone boundary-value problems. We will utilize the result from Theorem 4.1, to obtain solutions of (1) satisfying

$$\begin{aligned} u(m_1) &= u_1, \\ \Delta^2 u(m_2) &= u_2, \\ \Delta^2 u(m_3) &= u_3, \\ u(m_4) &= u_4, \end{aligned} \tag{7}$$

where $m_1 < m_2 < m_3 - 2 < m_4 - 4$.

Theorem 5.1 *Suppose that with respect to (1) conditions (A) and (B) are satisfied. Then for any integers* $m_1 < m_2 < m_3 - 2 < m_4 - 4$ *and any* $u_1, u_2, u_3, u_4 \in \mathbb{R}$*, there exists a unique solution of (1), (7).*

Proof Let integers $m_1 < m_2 < m_3 - 2 < m_4 - 4$ be given and let $u_1, u_2, u_3, u_4 \in \mathbb{R}$. Let $y(m)$ be the solution of (1), (5) given by Theorem 4.1 which satisfies

$$\begin{aligned} y(m_2) &= 0 \\ \Delta^2 y(m_2) &= u_2 \\ \Delta^2 y(m_3) &= u_3 \\ y(m_4) &= u_4. \end{aligned}$$

Define a set $S \subseteq \mathbb{R}$ by

$$\begin{aligned} S &= \{v(m_1) \mid v \text{ is a solution of (1), and} \\ \Delta^2 v(m_2) &= \Delta^2 y(m_2) = u_2, \Delta^2 v(m_3) = \Delta^2 y(m_3) = u_3, v(m_4) = y(m_4) = u_4\}. \end{aligned}$$

Notice that $S \neq \phi$, since $y(m_1) \in S$. Again S is an open subset of $\mathbb{R}$.

We again show that S is also a closed subset of $\mathbb{R}$. We assume otherwise. Then there exists a limit point $n_0 \in \bar{S} \backslash S$ and a monotone sequence $\{n_k\}_{k=1}^{\infty} \subset S$ such that $n_k \to n_0$. Without loss of generality, assume $\{n_k\}_{k=1}^{\infty}$ is increasing. By the definition of S, for $k \geq 1$, there exists a solution $y_k(m)$ of (1) such that

$$\begin{aligned} y_k(m_1) &= n_k \\ \Delta^2 y_k(m_2) &= \Delta^2 y(m_2) \\ \Delta^2 y_k(m_3) &= \Delta^2 y(m_3) \\ y_k(m_4) &= y(m_4). \end{aligned}$$

It then follows from Theorem 2.1 and the disfocality of (1) that for all $k \geq 1$, $y_k(m) < y_{k+1}(m)$ on $(-\infty, m_4)$. Also, $\{y_k(m)\}_{k=1}^{\infty}$ is unbounded above on each finite subset of $(-\infty, m_2]\backslash\{m_1\}$. Let w denote the unique solution of the 3-point Lidstone boundary-value problem consisting of (1) with

$$\begin{aligned} w(m_1) &= n_0 \\ \Delta^2 w(m_2) &= \Delta^2 y(m_2) = u_2 \\ \Delta^2 w(m_3) &= \Delta^2 y(m_3) = u_3 \\ w(m_3) &= 0. \end{aligned}$$

(Here $w(m_1) > y_k(m_1)$, for all $k \geq 1$.) Since $\{y_k(m_1 - 1)\}_{k=1}^{\infty}$ and $\{y_k(m_1 + 1)\}_{k=1}^{\infty}$ are unbounded above, but $y_k(m_1) < w(m_1)$ for all k, there exists a $k \geq 1$ such that $(w - y_k)(m)$ has a g.z. at m_1 and has a g.z. at $m_1 + 1$. Then $\Delta(w - y_k)(m)$ has a g.z. at m_1. Also, $\Delta^2(w - y_k)$ has a g.z. at m_2 and m_3, so there exists an $\omega \in (m_2, m_3)$ such that $\Delta^3(w - y_k)(m)$ has a g.z. at ω. Hence, utilizing the points m_1, m_2, and ω, it follows by the uniqueness of focal boundary-value problems that $w(m) = y_k(m)$ for all $m \in \mathbb{Z}$. But $w(m_1) = n_0 > y_k(m_1)$ for all $k \geq 1$. This is a contradiction. Thus S must be a closed subset of $\mathbb{R}$.

Since $S \subseteq \mathbb{R}$ is both open and closed, we must have that $S = \mathbb{R}$. Thus, choosing $u_1 \in S$, the corresponding solution is the desired unique solution of (1), (7).

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