THE MATHEMATICS OF SUSPENSIONS: KAC WALKS AND ASYMPTOTIC ANALYTICITY

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ABSTRACT. Of concern are suspension flows. These combine directed and random motions and are typically modelled by parabolic partial differential equations. Sometimes they can be better modelled (in terms of fitting the data generated by certain blood flow experiments) by hyperbolic equations, such as the telegraph equation, which have parabolic (or analytic) asymptotics.

1. Motivation

Engineering models for practical suspension flows depict the erratic motions of the individual particles with the mathematics of Brownian motion and diffusion. The particles in such flows include blood cells that move through the body's vessels or the passages of an artificial kidney or lung, the pulpy material in fruit juice that is being processed or sipped through a straw, and bits of coal in a coal-oil slurry being fed to a burner in an electrical power generation plant. These particles are far from small, usually are not present as a dilute species, and often exhibit large-scale erratic motions due to interaction with one another in the shearing flow. Despite these differences from true Brownian motion, the engineering models are functional, as Acrivos [1] noted in his recent review. But he also noted that aspects remain to be described well. In particular, the question of scaling for the strength of the erratic motion is open - there is no direct equivalent of the temperature / thermal energy that is linked to the motion of Brownian particles.

Such models depend on experimental measurements of effective properties to represent the diffusion coefficient and the viscosity. The methods and representations used to compile and reduce data for such measurements originate in basic mathematics and physics. Typically, experimental methods quantify the erratic motion by timing displacements of known extents i.e., by observing the jumps in the walk or related measures. Two particular random walks are of interest. One is the simple drunkard's walk that well is linked to the diffusion equation; the other is the persistent random walk, which is linked to the telegraphers' equation. The first expects that the particle is essentially stationary at a position and will go in either

direction; the next position depends only on the current position. The latter walk assumes that the particle is moving with a velocity and the probabilistic aspect is whether the sign of the velocity changes; it depends on the current and previous position. Other aspects of the walks and the models are described in two other papers related to our work [3, 13]. One paper explores the physical rationale for describing the random motion in particular ways, and the other describes the initial experimental results and an alternative means to obtain effective coefficients. In this paper linkages between the two random walk models and their related differential equations and semigroups are explored. To motivate the treatments below, a few aspects of suspension flows are reviewed below; more material appears in the other papers and in a thesis by Leggas [12].

The small particles are coupled to the general flow by viscous tractions acting on their surfaces. The Reynolds number that provides a fractional measure of the inertia and viscous traction on the particle is typically of the order 0.001 or less, which reflects small sizes and the small relative velocities between particles and the local average motion of the suspensions. The axial speed then provides a reasonable, but not perfect, indication of lateral position. Reasons for the imperfections include that particles can tumble temporarily in groups as faster and slower bodies pass one another. Also, many small inertial events either individually or cumulatively act and lead to differences between the motion of points in an ideal continuum fluid and the actual motion of suspended, bodies among many molecules that are much closer to the point approximation.

The experimental method involves tracking identifiable bodies among equivalent unmarked bodies. An identifiable body is found at a time when it has a velocity similar to that of the reference frame in which the observations are being made. In the initial studies, this reference frame speed is approximately the average speed of the flow and the suspension flow occurs in a rectangular channel. The time increments needed for the body to move selected net distances in either axial direction are measured. These first passage times are used to estimate the stochastic process that is associated with the suspension flow. As noted above, the change of apparent axial speed occurs because the particle changes its lateral location in a related fashion.

The form of the experiments is similar to a model of Taylor dispersion devised by Van den Broeck [18]. He assumed that the identifiable body would jump among parallel tracks of known velocities and that the body would jump at times that followed a Poisson distribution. The experiments collect observations of particle motions while moving along the flow axis as if one of the Van den Broeck tracks were being followed. The measured time increments are a means of approximating the first passages to other tracks by a continuous time random walk. The particle waits on the initial track until it jumps to some other track. The jump occurs instantly with a change of velocity that is sufficient to make it match the speed of the new track and provide an overall continuity of flow. A kind of dilute condition is implicit in this analogy because a landing space always exists in the receiving track.

One basic question is whether the velocity-based walk is a more fundamental representation of the events than the more commonly used drunkard's walk. The drunkard's walk implies a kind of local equilibrium; the history of the past motion is not applicable. Use of the velocity-based requires extra initial knowledge, which is reflected in its initial conditions. Improved engineering models of the complex

events in suspension flows may require that such detailed considerations be a part of the measurements of effective properties that are entered into the model. This paper explores the connections among the two random walks and especially focuses on the incorporation of initial and boundary conditions in them and their duals.

2. The Kac Walk

We begin with the random walk model leading to the telegraph equation. The idea of this model originated with G. I. Taylor [16]. It was developed by S. Goldstein [7]. The connection with the Poisson process was noted by M. Kac [11]. Kac was the main pioneer in using stochastic processes to help in understanding hyperbolic partial differential equations, so we like to refer to this model as a "Kac walk".

A particle starts at the origin $0 \in \mathbb{R}$ on the real line. After each time interval of length $\Delta t > 0$ it will move, either to the left or right, a distance $\Delta x > 0$. The speed is $c = \Delta x / \Delta t$. The first step is determined by the flip of a fair coin; one moves either to the left or to the right with probability $1/2$. On each subsequent step, the direction of the move is determined by the flip of a weighted coin. Let a be a positive constant so that $a\Delta t < 1$. The probability of reversing direction is $a\Delta t$; the probability of continuing in the same direction is $1 - a\Delta t$. Let S_n be the position of the random walk after n steps, or, at time $n\Delta t$. We want to compute $E(f(S_n))$ for arbitrary functions f.

We shall reproduce Kac's calculation. We do this to persuade the reader that the surprising appearance of the telegraph equation and the Poisson process in the description of the solutions is really elementary.

We now define the coin flips probabilistically. Let ξ be $+1$ or -1, with probability $1 - a\Delta t$ or $a\Delta t$, respectively. Let $\xi_1, \xi_2, \ldots$ be independent identically distributed random variables, distributed as ξ. Suppose for definiteness that the first step is to the *right*. Then

$$S_n = S_n^+ = c\Delta t(1 + \xi_1 + \xi_1\xi_2 + \ldots + \xi_1 \ldots \xi_{n-1}).$$

If the first step were to the left, we would have

$$S_n = S_n^- = -S_n^+.$$

Let us now write S_n in place of S_n^+ and consider

$$u_n^{\pm}(x) = E(f(x \pm S_n))$$

for $x \in \mathbb{R}$. Conditioning on ξ_1, we have

$$
\begin{aligned}
u_n^+(x) &= E\{f(x + c\Delta t + c\Delta t\xi_1(1 + \xi_2 + \ldots + \xi_2 \ldots \xi_{n-1}))\} \\
&= E\{f(x + \ldots)|\xi_1 = -1\}P\{\xi_1 = -1\} \\
&\quad + E\{f(x + \ldots)|\xi_1 = 1\}P\{\xi_1 = 1\} \\
&= a\Delta t \ u_{n-1}^-(x + c\Delta t) + (1 - a\Delta t)u_{n-1}^+(x - c\Delta t). \quad (2.1)
\end{aligned}
$$

Similarly

$$u_n^-(x) = a\Delta t \ u_{n-1}^+(x - c\Delta t) + (1 - a\Delta t)u_{n-1}^-(x - c\Delta t). \quad (2.2)$$

Equation (2.1) leads to

$$
\begin{aligned}
\frac{u_n^+(x) - u_{n-1}^+(x)}{\Delta t} &= \frac{u_{n-1}^+(x + c\Delta t) - u_{n-1}^+(x)}{\Delta t} \\
&\quad - au_{n-1}^+(x + c\Delta t) + au_{n-1}^-(x + c\Delta t).
\end{aligned}
$$

Taking the limit as $\Delta t, \Delta x \to 0$ with $a > 0$ and $c = \Delta x / \Delta t$ fixed and with $n\Delta t \to t$, we obtain functions $u^{\pm}(x, t)$ satisfying

$$\frac{\partial u^+}{\partial t} = c\frac{\partial u^+}{\partial x} + au^+ + au^-.$$

Similarly (2.2) leads to

$$\frac{\partial u^-}{\partial t} = -c\frac{\partial u^-}{\partial x} + au^+ - au^-.$$

Letting $u = \frac{u^+ + u^-}{2}$, $v = \frac{u^+ - u^-}{2}$, we can add and subtract the above differential equations to obtain

$$\frac{\partial u}{\partial t} = c\frac{\partial v}{\partial x}, \quad \frac{\partial v}{\partial t} = -c\frac{\partial u}{\partial x} - 2av. \tag{2.3}$$

If we take $\partial/\partial t$ of the first equation in (2.3), $\partial/\partial x$ of the second, and eliminate $\partial^2 v/\partial t\partial x$ we finally obtain the telegraph equation

$$\frac{1}{c}\frac{\partial^2 u}{\partial t^2} = c\frac{\partial^2 u}{\partial x^2} - \frac{2a}{c}\frac{\partial u}{\partial t},$$

or

$$\frac{\partial^2 u}{\partial t^2} + 2a\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}. \tag{2.4}$$

The initial conditions are

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t}(x, 0) = 0. \tag{2.5}$$

The first is clear; the second is perhaps not, but it is a result of the assumption that the first step determined the direction to move by flipping a fair coin.

When $a = 0$, each ξ_i is 1 and so

$$u^{\pm}(x) = f(x \pm nc\Delta t),$$

whence

$$u(x, t) = \frac{f(x + ct) + f(x - ct)}{2}.$$

This is the unique solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \tag{2.6}$$

(which is (2.4) with $a = 0$) with initial conditions (2.5). This helps explain the second initial condition in (2.5).

If one lets $\Delta t, \Delta x \to 0$ in such a way that $a \to \infty, c \to \infty, n\Delta t \to t$ and $\frac{c^2}{a} \to D > 0$, then the heat equation

$$\frac{\partial u}{\partial t} = \frac{D}{2}\frac{\partial^2 u}{\partial x^2}$$

would emerge, as it did in the classical (independent) work of Einstein, Bachelier, and Smoluchowski. This gives the first formal connection between the telegraph equation and the heat equation; more on this later.

Kac's next calculation is especially interesting. Redo the random walk analysis in continuous time. Thus we study one dimensional continuous motion with constant speed c, which changes direction in a time interval of length dt with probability adt (and maintains the same direction in this time interval with probability $1 - adt$). This leads to a Poisson process $\{N(t) = N_a(t) : t \geq 0\}$ with intensity a. That is,

$N(t)$ takes values in $\mathbb{N}_0 = \{0, 1, 2, \dots\}, N(0) = 0, P\{N(t) = k\} = e^{-at}(at)^k/k!$ for $k \in \mathbb{N}_0$, and for $0 \le t_1 < t_2 < \dots$,

$$N(t_2) - N(t_1), N(t_3) - N(t_2), N(t_4) - N(t_3), \dots$$

are independent. Then the number of sign changes in the velocity process up to time t is $(-1)^{N(t)}$. Hence the position (or displacement) is

$$S(t) = \int_0^t v(\tau)d\tau = c\int_0^t (-1)^{N(\tau)}d\tau,$$

which is the continuous analogue of S_n. This led Kac to derive (and prove) that the solution of the telegraph equation problem (2.4), (2.5) is

$$
\begin{aligned}
u(x) &= \frac{1}{2}\{E[f(x + c\int_0^t (-1)^{N(\tau)}d\tau)] + E[f(x - c\int_0^t (-1)^{N(\tau)}d\tau)]\} \\
&= E[v(x, \int_0^t (-1)^{N(\tau)}d\tau)]
\end{aligned}
$$

where v is the unique solution of the wave equation (2.6), (2.5). Thus to get the solution $u(x,t)$ of the telegraph equation, take the corresponding solution $v(x,\sigma)$ of the wave equation, but evalutate it at a random time $\sigma = \int_0^t (-1)^{N(\tau)}d\tau$ determined by a Poisson process (with intensity a) and then average. For more on random walk models, see [18, 20].

We view the Kac calculations as a precise version of some heuristic reasoning based on some old (1930) but very interesting discussion of Uhlenbeck and Ornstein [17]. They were interested in the kinetic theory of gases and Brownian motion. Consider the formula [17, p.826]

$$\beta s^2 = \alpha_1 t - 1 - e^{-\alpha_2 t} \tag{2.7}$$

where $\beta = f_1^2/2mkT, \alpha_j = f_j/m_j$, represents distance and t time, f is for friction and m for mass. This equation is due to Fürth [4] and Ornstein [14]. For large t, this is approximately $\beta s^2 = \alpha t$, which is Einstein's (1905) equation

$$\beta s^2 = 2Dk = \frac{2kT}{f_1}t. \tag{2.8}$$

This corresponds to a free particle of mass m_1, where the friction coefficient f_1 only depends on the surrounding medium. (Here T is the absolute temperature and k is Boltzmann's constant.) Suppose we consider a tagged particle in a suspension, and look at its associated parameter $\alpha_2 = f_2/m_2$. Thus $f_2 = f_1$ (since the surrounding medium is unchanged) but $m_2 > m_1$ since the particle carries some of the surrounding fluid with it and thus has an effective mass exceeding its free mass. Thus we take $\alpha_1 > \alpha_2$, while the earlier authors took $\alpha_1 = \alpha_2$. For small t, we replace $e^{-\alpha_2 t}$ by it second order Taylor expansion $1 - \alpha_2 t + \alpha_2^2 t^2/2$, and (2.7) becomes

$$\beta s^2 = (\alpha_1 - \alpha_2)t + \frac{\alpha_2^2}{2}t^2. \tag{2.9}$$

Now think of s [resp. t] as representing $\frac{\partial}{\partial x}$ [resp. $\frac{\partial}{\partial t}$]. Then (2.8), (2.9) are formally the first and second order (in time) equations

$$\beta\frac{\partial^2 u}{\partial x^2} = \alpha\frac{\partial u}{\partial t}, \tag{2.10}$$

$$\beta\frac{\partial^2 u}{\partial x^2} = (\alpha_1 - \alpha_2)\frac{\partial u}{\partial t} + \frac{\alpha_2^2}{2}\frac{\partial^2 u}{\partial t^2}. \tag{2.11}$$

Velocities play no role in (2.10) (Brownian particles have infinite speed), but they play a crucial role in the telegraph equation (11). (In the Ornstein-Uhlenbeck stochastic process, particles have finite velocity but infinite acceleration.)

3. THE COSINE FUNCTION VERSION

The initial value problem for

$$v''(t) = Av(t) \ \ (t \geq 0), \tag{3.1}$$

$$v(0) = f, \ \ v'(0) = 0 \tag{3.2}$$

(for $v'' = d^2v/dt^2$, $v : (0, \infty) \to X$, and A a linear operator on a Banach space X) is well posed if and only if A generates a cosine function $C = \{C(t) : t \geq 0\}$ on X, in which case the unique solution of (3.1), (3.2) is given by

$$v(t) = C(t)f;$$

it is a strong solution if $f \in Dom(A)$ and a mild solution otherwise (Cf. [5].) If we replace (3.2) by

$$v(0) = f, \quad v'(0) = g, \tag{3.3}$$

and if A is injective, then the corresponding solution of (3.1) is

$$v(t) = C(t)f + \frac{d}{dt}C(t)A^{-1}g = C(t)f + \int_0^t C(s)gds. \tag{3.4}$$

It is not difficult to show that the unique solution to the abstract telegraph equation

$$u''(t) + 2au(t) = Au(t) \tag{3.5}$$

with (3.2) is

$$u(t) = E\left[v\left(\int_0^t (-1)^{N(\tau)}d\tau\right)\right], \tag{3.6}$$

where $v(t) = C(t)f$ is the solution of the corresponding abstract wave equation. As before $\{N(t) : t \geq 0\}$ is a Poission process of intensity $a > 0$.

Continue to suppose that A is injective. The unique solution of (3.5), (3.3) is given by

$$\begin{aligned}
u(t) &= E\{C\left(\int_0^t (-1)^{N(\tau)}d\tau\right)(f - 2aA^{-1}g) \\
&\quad + \int_0^t C\left(\int_0^s (-1)^{N(\tau)}d\tau\right)gds + 2aA^{-1}g\}.
\end{aligned} \tag{3.7}$$

To see this, let v satisfy

$$v'' + 2av' = Av, \ \ v(0) = h, \ \ v'(0) = g.$$

Then $w = v'$ satisfies

$$w'' + 2aw' = Aw, \ \ w(0) = g, \ \ w'(0) = Ah - 2ag.$$

If $h = 2aA^{-1}g$, then

$$w(t) = \left[C\left(\int_0^t (-1)^{N(\tau)}d\tau\right)g\right].$$

Then, $z = u - v$ satisfies $z'' + 2az' = Az$,

$$z(0) = f - 2aA^{-1}g, \quad z'(0) = 0;$$

and so $u = z + v$ and (3.7) follows. $\diamondsuit$

Note that this reduces to our previous result (3.4) for the wave equation where $a = 0$.

Kac [11] gave a rigorous proof of the representation formula (3.6) for the solution of (3.5) with $u(0) = f$, $u'(0) = 0$ and $A = c^2 d^2/dx^2$ on $L^2(\mathbb{R})$ using Laplace transforms. The proof uses power series constructions. The same proof works when A has enough analytic vectors. The power series for $cos(ta)$ is $\sum_{n=0}^{\infty}(-1)^n(ta)^{2n}/(2n)!$, and this function satisfies $u'' = -a^2 u$ (as a function of t). Thus we expect $C(t)f$, the solution of $u'' = Au$, $u(0) = f$, $u'(0) = 0$ to be given by

$$C(t)f = \sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}A^n f}{(2n)!},$$

at least for all f for which the above series converges nicely. We define f to be an *entire vector* for A if the series $\sum_{n=0}^{\infty} t^n A^n f/n!$ has an infinite radius of convergence (in the complex t-plane). Let $E(A)$ be the set of such vectors. Let $\widetilde{E}(A) = \{f \in X : \sum_{n=0}^{\infty} t^{2n}A^n f/(2n)!$ has an infinite radius of convergence$\}$. Here we clearly are assuming that X is a complex Banach space. Clearly $E(A) \subset \widetilde{E}(A)$. If $\widetilde{E}(A)$ is dense in X, then Kac's arguments [11] establish the validity of (3.6).

Now let A be a normal operator on a Hilbert space $\mathcal{H}$. By the spectral theorem,

$$A = UM_\alpha U^{-1}$$

where $U : L^2(\Omega, \mu) \to \mathcal{H}$ is unitary and (Ω, μ) represents some measure space; and $\alpha : \Omega \to \mathbb{C}$ is measurable, and $M_\alpha u(x) = \alpha(x)u(x)$ for $x \in \Omega$, $Dom(M_\alpha) = \{u : u, \alpha u \in L^2(\Omega, \mu)\}$. The case when A generates a cosine function corresponds to the range of α being (essentially) contained in a parabolic region of the form

$$\text{conv}\{x + iy \in \mathbb{C} : x = -c_1(y - c_2)^2 + c_3\}$$

(conv = convex hull) for any positive constants c_1, c_2, c_3. Let $B_n = \{z \in \mathbb{C} : |z| \leq n\}$, $\chi_n =$ the characteristic function of B_n,

$$E_n = U(\text{Range}(\chi_n))U^{-1} \subset X;$$

then $E = \bigcup_{n=1}^{\infty} E_n$ is a dense set of entire vectors for A.

This completes the proof of (3.6) (and also of (3.7)) for the only cases we shall consider. For more general information see [8, 9, 10, 15, 6].

4. Remarks on Higher Dimensions

In the previous section we showed how the Kac ideas involving the Poisson process to represent solutions of the abstract telegraph equation work in great generality. But the random walk model is usually given in just one dimension. We make a few remarks here about extensions.

Griego and Hersh [8] and Pinsky [15] considered the first order hyperbolic system

$$\frac{\partial u_i}{\partial t} = c_i \frac{\partial u_i}{\partial x} + \sum_{j=1}^{n} q_{ij}u_j,$$

$$u_i(x, 0) = f_i(x), \tag{4.1}$$

where $u_i : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ $(i = 1, \ldots, n)$, $Q = (q_{ij})$ generates an n-state Markov chain (i.e. $P(t) = e^{tQ}$ is an $n \times n$ matrix of nonnegative numbers and $P(t)\mathbf{1} = \mathbf{1}$ where $\mathbf{1} = (1, 1, \ldots 1)^{tr}$; equivalently, $q_{ij} \geq 0$ for $i \neq j$ and the row sums $\sum_{i=1}^{n} q_{ij}$ are all zero), and $c_i \in \mathbb{R}$. Let $p_{ij}(t)$ be the ijth entry of $P(t) = e^{tQ}$, and let $\xi(t)$ be the position at time t of a particle whose velocity $v(t)$ at time t switches from c_i to c_j with probability $p_{ij}(t)$. Then we have the representation

$$
\begin{aligned}
u_i(x, t) &= E_{x,i}[f_{v(t)}(\xi(t))] \\
&= E\{f_{v(t)}(\xi(t)) : \xi(0) = x, \xi'(0) = i\}.
\end{aligned} \tag{4.2}
$$

If $n = 2$, $Q = \left(\begin{smallmatrix} -a & a \\ a & -a \end{smallmatrix} \right)$, and $c_2 = -c_1 = -c$, then the system (4.1) reduces to the single equation

$$
u_{tt} + 2au_t = c^2 u_{xx}. \tag{4.3}
$$

If we replace (4.1) by

$$
\frac{du_i}{dt} = A_i u_i + \sum_{i=1}^{n} q_{ij} u_j
$$

with $A_1 = -A_2 = A$, a (C_0) group generator, then (4.3) becomes

$$
u_{tt} + 2au_t = A^2 u,
$$

and the representation formula (4.2) can be shown to imply the Kac formula (3.6) for the solution of (3.5).

Now let $x = (x_1, \ldots, x_n)$ vary over $\mathbb{R}^n$, and let e_i be the unit vector in the positive direction of the ith coordinate axis L_i, $1 \leq i \leq n$. Perform the Kac construction on each line L_i. Then the unique solution of

$$
\partial^2 u_i / \partial t^2 + 2a_i \, \partial u_i / \partial t = c_i^2 \partial^2 u_i / \partial x_i^2,
$$

$$
u_i(x_i, 0) = f_i(x_i), \quad \partial u_i(x_i, 0) / \partial t = 0
$$

is

$$
u_i(x_i, t) = E[v_i(x, \int_0^t (-1)^{N_i(s)} ds)]
$$

where v_i satisfies the same problem but with a_i replaced by zero. If $N_1 = \ldots = N_n = N$, so that there is only one Poisson process of intensity $a = a_1 = \ldots = a_n$, then

$$
U(x, t) = E[V(x, \int_0^t (-1)^{N(s)} ds)]
$$

satisfies

$$
U_{tt} + 2aU_t = \sum_{i=1}^{n} c_i^2 U_{x_i x_i},
$$

$$
U(x, 0) = \sum_{i=1}^{n} f_i(x_i), \quad U_t(x, 0) = 0,
$$

where V satisfies the same problem but with u replaced by zero. Thus a very special case of an n-dimensional telegraph equation is governed directly by the Kac random walk, but the initial function $F(x) = \sum_{i=1}^{n} f_i(x_i)$ must be very special. Nevertheless, the conclusion of Section 3 shows that this representation formula holds even for general F in $L^2(\mathbb{R}^n)$.

5. Asymptotic Analyticity

Of concern is the problem

$$u_{tt} + 2au_t + A^2 u = 0, \tag{5.1}$$

where a is a positive constant and $A = A^* \geq 0$ is a nonnegative selfadjoint operator on a Hilbert space $\mathcal{H}$. By the spectral theorem [5], the general solution of (5.1) is given by

$$u(t) = \sum_{j=1}^{2} T_j(t) f_j$$

where

$$T_j(t) = e^{tA_j},$$

$$A_{1,2} = -aI \pm (a^2 I - A^2)^{\frac{1}{2}};$$

here the subscript 1 [resp. 2] goes with the $+$ [resp. $-$] sign. Recall that by the spectral theorem,

$$A = U M_m U^{-1}$$

where $U : \mathcal{H} \to L^2(\Omega, \mu)$ is a unitary operator from $\mathcal{H}$ to an L^2 space on some measure space, and M_m is the operation of multiplication by the μ-measurable function

$$m : \Omega \to [0, \infty).$$

Hence for any Borel function g on $(0, \infty)$,

$$g(A) = U M_{g(m)} U^{-1}.$$

Thus $A_{1,2}$ and $T_j(t)$ are all well defined (for $t \geq 0$).

Let

$$F(x) = -1 + (1-x)^{\frac{1}{2}}$$

for $0 < x < 1$. Then $F'(x) = -\frac{1}{2}(1-x)^{-\frac{1}{2}}$, $F''(x) = -\frac{1}{4}(1-x)^{-\frac{3}{2}}$, and so (since $F(0) = 0$, $F'(0) = -\frac{1}{2}$, $F''(0) = -\frac{1}{4}$)

$$
\begin{aligned}
F(x) &= -\frac{x}{2} + \bigcirc(x^2) \\
&= -\frac{x}{2} - \frac{x^2}{8} + \bigcirc(x^3)
\end{aligned}
$$

as $x \to 0$, by Taylor's theorem. Using the spectral theorem, $U^{-1} T_1(t) U$ can be approximated by the multiplication operator

$$
\begin{aligned}
e^{t(-a + (a^2 - m^2)^{\frac{1}{2}})} &= \exp\{-ta(1 - (1 - \frac{m^2}{a^2})^{\frac{1}{2}})\} \\
\cong e^{-ta\frac{m^2}{2a^2}} &= e^{-tm^2/2a},
\end{aligned}
$$

which implies that $T_1(t)$ is approximately $e^{-tA^2/2a}$.

Write $A = \int_0^{\infty} \lambda \, dP(\lambda)$, so that $P(B) = \int_B dP(\lambda)$ orthogonally projects onto the maximal invariant closed subspace of $\mathcal{H}$ for A in which A has spectrum contained in B, for B any Borel set in $[0, \infty)$.

Let $0 < \epsilon < a$. Then any solution u of (5.1) is equal to

$$e^{t(-a + (a^2 - A^2)^{\frac{1}{2}})} f = \bigcirc(e^{-t(a^2 - \epsilon^2)^{\frac{1}{2}}})$$

(as $t \to \infty$) for some f in $P([0, \epsilon])$. In that sense, the solution is given by an analytic semigroup plus a small error in the asymptotic limit, $t \to \infty$. By the analysis given above, this analytic semigroup solution is approximately

$$e^{-tA^2/2a} f,$$

which is a solution of the variant of equation (5.1) with the u'' term missing.

This analysis becomes most transparent when A has an orthonormal basis $\{\varphi_n\}$ of eigenvalues. Thus $A\varphi_n = \lambda_n \varphi_n$, $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \quad \leq \lambda_n \to \infty$. Suppose M is such that $\lambda_M < a \leq \lambda_{M+1}$. Then any solution u of (5.1) satisfies, for suitable constant c_j,

$$u(t) = \sum_{j=1}^{M} \exp\{t(-a + (a^2 - \lambda_j^2)^{\frac{1}{2}})\} c_j \varphi_j + \bigcirc(e^{-ta}),$$

which approximately equals

$$\sum_{j=1}^{M} e^{-t\lambda_1^2/2a} c_j \varphi_j + \bigcirc(e^{-ta}).$$

The above calculations are correct but not "canonical", and so they are nonoptimal.

If our original solution of (5.1) is

$$u(t) = e^{tA_1} h_1 + e^{tA_2} h_2,$$

then we approximate it by

$$e^{t(-a+(a^2-A^2)^{\frac{1}{2}})} P([0, \epsilon]) h,$$

with an error of $\bigcirc(\exp[t(-a + (a^2 - \epsilon^2)^{\frac{1}{2}})])$, and this in turn can be approximated by

$$e^{-tA^2/2a} P([0, \epsilon]) h_1.$$

But there is no natural way of choosing ϵ. In the orthonormal basis case, if λ_1 is a simple eigenvalue, we can write

$$\begin{aligned} u(t) &= \exp\{t(-a + (a^2 - \lambda_1^2)^{\frac{1}{2}})\} c_1 \varphi_1 \\ &\cong e^{-t\lambda_1^2/2a} c_1 \varphi_1 \end{aligned}$$

with an error of $\bigcirc(\exp[-t(a - (a^2 - \lambda_2^2)^{\frac{1}{2}})])$. This gives all relevant ergodic information, but says very little in the case when $c_1 = 0$.

We summarize the above results.

Theorem 5.1. *Let $A = A^*$ be a nonnegative selfadjoint operator on a Hilbert space $\mathcal{H}$ and let a be a positive constant. Then the unique solution u of*

$$u'' + 2au' + A^2 u = 0, \quad u(0) = f \quad u'(0) = g$$

satisfies, for any given $\epsilon > 0$,

$$u(t) = \exp\{t(-a + (a^2 - A^2)^{\frac{1}{2}})\} \ell + \delta(t)$$

where

$$\delta(t) = \bigcirc(e^{-t(a^2 - \epsilon^2)^{\frac{1}{2}}})$$

as $t \to \infty$. This solution is approximately equal to

$$e^{-tA^2/2a} \ell.$$

This choice of ℓ depends on ϵ and so is not made in a canonical fashion. We expand on this point.

Clearly solutions of the abstract telegraph equation (5.1) are asymptotically given by an analytic semigroup, but there is not a unique way to make this association. If the initial conditions for (5.1) are $u(0) = h, u'(0) = k$, then

$$u(t) = e^{tA_1}f_1 + e^{tA_2}f_2$$

where f_1, f_2 case be explicitly computed in terms of h and k. We focus on f_1 and throw f_2 into the error term. In projecting f_1 onto $P([0,\epsilon])(\mathcal{H})$, we demand $0 \leq \epsilon < a$, and we want this projected vector to be nonzero; but we have no *natural* way of choosing ϵ, especially in the case of continuous spectrum, which is the case in the Kac model of $A^2 = -d^2/dx^2$ on $L^2(\mathbb{R})$.

6. Fractional Derivatives

The model telegraph equation

$$D^2u(t) + 2aDu(t) + A^2u(t) = 0 \tag{6.1}$$

(with $D = d/dt$) can alternatively be replaced by a fractional differential equation of the form

$$(D^\gamma)^2u(t) + 2aD^\gamma u(t) + A^2u(t) = 0, \tag{6.2}$$

where $\frac{1}{2} \leq \gamma \leq 1,   \gamma = 1$ corresponding to (6.1). The motivation comes from self similarity and experimental considerations. This will be studied in a separate article [2].

7. Concluding Remarks

Compared to the heat equation, the telegraph equation seems to be a superior model for describing certain fluid flow problems involving suspensions. For the telegraph equation (or for its fractional derivative analogue), one needs to specify two pieces of initial data; the heat equation only requires one. If the telegraph equation is to describe an experiment, the experiment must be able to give enough data to produce two initial conditions. One of these conditions may need to be linked strongly to the experimental method and its limitations, which accordingly would not be natural in a mathematical sense. If or when such an arrangement to set a value of ϵ cannot be provided, one may be forced to use a first order (in time) equation as a model.

References

[1] A. Acrivos, Bingham Award Lecture, 1994: Shear induced particle diffusion in concentrated suspensions of non-colloidal particles, *J. Rheol.* 39(1995), 813- 826.
[2] R. Cascaval, C.L. Frota and J.A. Goldstein, in preparation.
[3] E.C. Eckstein, J.A. Goldstein and M. Leggas, in preparation.
[4] R. Fürth, *Zeits für Physik* 2(1920),244.
[5] J.A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford U. Press, 1985.
[6] J.A. Goldstein, Autocorrelation, equipartition of energy and random evolutions in *Probability Measures on Groups*, H. Heyer (ed.), Springer Lecture Notes in Math. No. 928, Springer (1982), 176-182.
[7] S. Goldstein, On diffusion by discontinuous movements and on the telegraph equation, *Quart. J. Math. Mech.* 4(1951), 129-156.
[8] Griego, R., and R. Hersh, Theory of random evolutions with applications to partial differential equations, *Trans. Amer. Math. Soc.* 156(1971), 405-418.

[9] R. Hersh, Stochastic solutions of hyperbolic problems. In: *Partial Differential Equations and Related Topics* (ed. by J.A. Goldstein), Lecture Notes in Math. No. 446, Springer (1975), 283-399.

[10] R. Hersh, Random evolutions: A survey of results and problems, *Rocky Mtn. J. Math.* 4(1974), 443-477.

[11] M. Kac, A stochastic model related to the telegrapher's equation. In: Some stochastic problems in physics and mathematics. Magnolia Petroleum Co., Lectures in Pure and Applied Science, No. 2, 1956. Reprinted in: *Rocky Mtn. J. Math.* 4(1974), 497-504.

[12] M. Leggas, Thesis, Biomedical Engineering, University of Tennessee Health Sciences Center, Memphis, 1999.

[13] M. Leggas and E.C. Eckstein, in preparation.

[14] L.S. Ornstein, *Versl. Acad. Amst.* 26(1917), 1005 (=*Proc. Acad. Amst.* 21(1919), 96).

[15] M. Pinsky, Differential equations with a small parameter and the central limit theorem for functions defined on a finite Markov chain, *Z. Warsch. Verw. Geb.* 9(1968), 101-111.

[16] G.I. Taylor, Diffusion by continuous movements, *Proc. London Math. Soc.* 20(1922), 196-212.

[17] G.E. Uhlenbeck and C.S. Ornstein, On the theory of Brownian motion, *Phys. Rev.* 36(1930), 823-841 (reprinted in [Wa]).

[18] C. Van den Broeck, Taylor dispersion revisited, *Physica A* 168(1990), 677-696.

[19] N. Wax (editor), *Selected Papers on Noise and Stochastic Processes,* Dover, 1954.

[20] G.H. Weiss, *Aspects and Applications of the Random Walk,* North Holland, 1994.

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