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ON OSCILLATORY SOLUTIONS OF THIRD ORDER DIFFERENTIAL EQUATION WITH QUASIDERIVATIVES

Miroslav Bartušek

Abstract. This paper gives sufficient conditions under which all oscillatory solutions of a third order nonlinear differential equation with quasiderivatives vanish at infinity. Applications to third order differentials equation with a middle term are also given.

I. Introduction

Consider the differential equation

$$y^{[3]} = \left(\frac{1}{a_2}\left(\frac{1}{a_1}y'\right)'\right)' = r(t)\, f(y) \tag{1}$$

where $J = [0,T)$, $T \leqslant \infty$, $r \in C^\circ(J)$, $f \in C^\circ(R)$, $R = (-\infty,\infty)$, $a_i \in C^1(J)$, $i = 1,2$, a_i are positive on J,

$$r(t) > 0 \text{ on } J, \qquad f(x)x > 0 \quad \text{for} \quad x \neq 0\,, \tag{H1}$$

and $y^{[i]}$, $i = 0,1,2,3$, is the i-th quasiderivative of y defined by

$$y^{[0]} = y\,, \quad y^{[i]} = \frac{1}{a_i(t)}\left(y^{[i-1]}\right)'\,, \quad i = 1,2, \quad y^{[3]} = \left(y^{[2]}\right)'\,. \tag{2}$$

Let a function $y : I \to R$ have the continuous quasiderivatives up to the order 3 on I and let (1) hold on I. Then y is called a solution of (1). A solution y is called oscillatory if it is defined on J, $\sup_{\tau \leqslant t < T} |y(t)| > 0$ for an arbitrary $\tau \in J$ and if there exists a sequence of its zeros tending to T. Denote by $\mathcal{O}$ the set of all oscillatory solutions of (1).

Due to the methods used, we will study two cases:

$$\left(\frac{a_2(t)}{a_1(t)}\right)' \leqslant 0\,, \quad t \in J\,, \tag{H2}$$

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and

$$\left(\frac{a_2(t)}{a_1(t)}\right)' \geqslant 0\,, \quad t \in J\,. \tag{H3}$$

A great effort has been exerted to the study of the asymptotic behaviour of oscillatory solutions of (1) and its special cases, see e.g. [1–6, 8, 10, 12].

If $a_2 \equiv a_1 \equiv 1$ and $T = \infty$, sufficient conditions are given in [1,10] for every oscillatory solution y of (1) to vanish at infinity, i.e.

$$\lim_{t\to\infty} y(t) = 0\,. \tag{3}$$

Theorem A ([10])**.** *Let $T = \infty$,* (H1) *hold, $a_1 \equiv a_2 \equiv 1$, and $0 < M \leqslant r(t)$ on $\mathbb{R}_+$. If $y \in \mathcal{O}$, then* (3) *holds.*

The same problem is solved for (1) in [6].

Theorem B. *Let* (H1) *and* (H3) *hold and let*

$$0 < M \leqslant r(t)\,, \quad a_1(t)\, a_2(t) \leqslant M_1 < \infty \quad \text{on} \quad J\,. \tag{4}$$

If y be an oscillatory solution of (1)*, then $\lim\limits_{t\to T_-} y(t) = 0$.*

Proof. The assertion is proved in [6] if $T = \infty$ and (4) holds on $\mathbb{R}_+$. But in the proof, the fact that J is infinite is not used; thus the statement holds for $T < \infty$ as well. □

The following example shows that (3) can not be valid.

Example 1. The differential equation

$$\left(\left(\mathrm{e}^{-t}\, y'\right)'\right)' = 2\,\mathrm{e}^{-t}\, y\,, \qquad t \in \mathbb{R}_+$$

has an oscillatory solution $y = \sin t$ and (3) does not hold. Note that (H1) and (H2) are valid.

Besides (3), other asymptotic behaviour of oscillatory solutions of (1) with $T = \infty$ are often investigated. In [3, 4] we give sufficient conditions under which the sequences of the absolute values of all local extrema of $y^{[i]}$, $i \in \{0, 1, 2\}$, in a neighbourhood of ∞ are monotone for an oscillatory solution y of (1) in case $r(t) \leqslant 0$.

In this paper, the above mentioned results are extended to (1) under the hypothesis (H1). In the last paragraph, applications to the third order differential equation with a middle term are given.

We do not discuss the problem of the existence of oscillatory solutions of (1). It is solved in [8, 12], and for the case of usual derivatives (i.e., for $a_1 \equiv a_2 \equiv 1$), in the monographes [1] and [10] (for $T = \infty$).

The following lemma is a simple consequence of the definition of quasiderivatives and of (H1).

Lemma 1. *Let* (H1) *hold and let y be a solution of* (1) *defined on $I = [t_1, t_2] \subset J$, $t_1 < t_2$. Let $y^{[-1]} \equiv y^{[2]}$. If $i \in \{0, 1, 2\}$ and $y^{[i]}(t) > 0$ (< 0) on I, then $y^{[i-1]}$ is increasing (decreasing) on I.*

Remark 1. Note that $<$ and increasing ($>$ and decreasing) can be replaced by $\leqslant$ and nondecreasing ($\geqslant$ and non-increasing).

The following lemma describes the structure of oscillatory solutions of (1).

Lemma 2 ([2]). *Let $y \in \mathcal{O}$. Then sequences $\{t_k^i\}$, $i = 0,1,2$, $k = 1,2,\dots$ exist such that $\lim_{k\to\infty} t_k^0 = T$,*

$$t_k^0 < t_k^1 < t_k^2 < t_{k+1}^0\,, \quad y^{[i]}(t_k^i) = 0\,, \quad i = 0,1,2\,,$$

$$\begin{aligned}(-1)^{j+1} y^{[j]}(t)\, y(t) &> 0 \quad on \quad (t_k^0, t_k^j)\,,\\ &< 0 \quad on \quad (t_k^j, t_{k+1}^0)\,, \quad j = 1,2;\ k = 1,2,\dots\end{aligned}$$

Remark 2. Note that according to Lemmas 1 and 2, the sequences $\{|y(t_k^1)|\}_1^\infty$, $\{|y^{[1]}(t_k^2)|\}_1^\infty$ and $\{|y^{[2]}(t_k^0)|\}_1^\infty$ are the sequences of the absolute values of all local extrema of y, $y^{[1]}$ and $y^{[3]}$ on $[t_0^0, T)$, respectively.

Sometimes it is useful to express (1) in an equivalent form.

Lemma 3. *Let $a_0 \in C^\circ(J)$ be positive. Then the transformation*

$$x(t) = \int_0^t a_0(s)\, ds\,,\ Y(x) = y(t)\,,\ t \in J,\ x \in [0, x^*)\,,\ x^* = x(T)$$

transforms (1) into

$$\left(\frac{1}{A_2}\left(\frac{1}{A_1}\overset{\bullet}{Y}\right)^{\bullet}\right)^{\bullet} = R(x)\, f(Y) \tag{5}$$

where $A_i(x) = \frac{a_i(t(x))}{a_0(t(x))}$, $i = 1,2$, $R(x) = \frac{r(t(x))}{a_0(t(x))}$, $\frac{d}{dx} = {}^\bullet$ and $t(x)$ is the inverse function to $x(t)$. At the same time,

$$Y^{\{i\}}(x) = y^{[i]}(t)\,, \quad i = 0,1,2,3\,, \tag{6}$$

where

$$Y^{\{0\}} = Y\,, \quad Y^{\{j\}} = \frac{1}{A_j(x)}\left(Y^{\{j-1\}}\right)^{\bullet}\,,\ j = 1,2\,, \quad Y^{\{3\}} = \left(Y^{\{2\}}\right)^{\bullet}\,.$$

Proof. Use a direct computation or see [4]. □

2. Case (H2)

Some results will be used that are obtained for (1) under a different assumptions than (H1). Consider

$$\left(\frac{1}{b(\sigma)} Z''\right)' + \bar r(\sigma)\, f(Z) = 0 \tag{7}$$

where $I \subset \mathbb{R}_+$, $b \in C^1(I)$, $\bar r \in C^1(I)$, $f \in C^\circ(I)$, $f(x)x > 0$ for $x \neq 0$,

$$b(\sigma) > 0\,, \quad \bar r(\sigma) \geqslant 0 \text{ on } I\,, \quad f'(x) \geqslant 0 \text{ on } R\,.$$

The quasiderivatives are given by

$$Z^{[0]} = Z\,,\ Z^{[1]} = Z'\,,\ Z^{[2]} = \frac{Z''}{b(\sigma)}\,.$$

Note that the sign of $\bar r$ is opposite to the one of r.

Lemma 4. *Let $b' \geqslant 0$ and $\bar{r}' \geqslant 0$ on I. Let Z be a solution of* (7) *the second quasiderivatives $Z^{[2]}$ of which has three consecutive zeros σ_0, σ_1 and $\sigma_2 \in I$, $\sigma_0 < \sigma_1 < \sigma_2$. Then*

$$\sqrt{2}|Z'(\sigma_2)| < |Z'(\sigma_1)|\,.$$

Proof. The assertion is proved for $T = \infty$ and for an oscillatory solution in [4] (see Lemma 2.4 and Definition 2.1). But it follows from the proof that only information on $[\sigma_1, \sigma_2]$ and the existence of the zero σ_0 were used. Thus, the statement is valid under our assumptions as well. □

The following theorem investigates the asymptotic behaviour of the first and the second quasiderivatives of an oscillatory solution of (1).

Theorem 1. *Let* (H1) *and* (H2) *hold. Let $y \in \mathcal{O}$ and $\{t_k^i\}$, $i = 0, 1, 2$, $k = 1, 2, \dots$, be given by Lemma 2.*

(i) *Then the sequence $\{|y^{[1]}(t_k^2)|\}_1^\infty$ of the absolute values of all local extrema of $y^{[1]}$ on $[t_1^0, T)$ is decreasing.*

(ii) *Let $r \in C^1(J)$, $f \in C^1(R)$, $f' \geqslant 0$ on R and $\left(\frac{r(t)}{a_1(t)}\right)' \leqslant 0$ on J.*
Then $\lim\limits_{t \to T} y^{[1]}(t) = 0$ and

$$\left|y^{[1]}(t_k^2)\right| \leqslant 2^{\frac{1-k}{2}} \left|y^{[1]}(t_1^2)\right|\,, \quad k = 1, 2, \dots \tag{8}$$

(iii) *Let $r \in C^1(J)$, $f \in C^1(R)$, $f' \geqslant 0$ on R and $\left(\frac{r(t)}{a_2(t)}\right)' \leqslant 0$ on J.*
Then the sequence $\left\{|y^{[2]}(t_k^0)|\right\}_1^\infty$ of the absolute values of all local extrema of $y^{[2]}$ on $[t_1^0, T)$ is decreasing.

Proof. Note that according to Remark 2, the sequences $\left\{|y^{[1]}(t_k^2)|\right\}_1^\infty$ and $\left\{|y^{[2]}(t_k^0)|\right\}_1^\infty$ are the sequences of the absolute values of all local extrema of $y^{[1]}$ and $y^{[2]}$, respectively.

(i) Let $k \in \{2, 3, \dots\}$ and suppose, without loss of generality, that

$$y(t) > 0 \quad \text{on} \quad (t_k^0, t_{k+1}^0)\,.$$

Thus, according to Lemmas 1 and 2 there exists t_k^* such that

$$\begin{gathered} t_k^* \in (t_k^0, t_k^1)\,,\ y(t_k^*) = y(t_k^2)\,, \\ y \text{ is increasing (decreasing) on } [t_k^*, t_k^1] \text{ (on } [t_k^1, t_k^2])\,, \\ y^{[1]}(t) > 0\ (< 0) \text{ on } [t_k^*, t_k^1) \text{ (on } t_k^1, t_k^2)\,, \\ y^{[1]}(t_k^1) = 0\,, \quad y^{[1]}(t_{k-1}^2) > y^{[1]}(t_k^*) > 0\,, \\ y^{[2]}(t) < 0 \text{ and } |y^{[2]}| \text{ is decreasing on } [t_k^*, t_k^2)\,, y^{[2]}(t_k^2) = 0\,. \end{gathered} \tag{9}$$

Let φ and ψ be the inverse functions to y:

$$\begin{gathered} t_k^* \leqslant \varphi(v) \leqslant t_k^1\,, \quad y(\varphi(v)) = v\,, \\ t_k^1 \leqslant \psi(v) \leqslant t_k^2\,, \quad y(\psi(v)) = v\,, \\ v \in I = [y(t_k^*), y(t_k^1)]\,. \end{gathered}$$

We prove by an indirect proof that

$$y^{[1]}(\varphi(v)) \geqslant \left|y^{[1]}(\psi(v))\right|, \quad v \in I. \tag{10}$$

Observe that (9) yields $y^{[1]}(\varphi(v)) > 0$ and $y^{[1]}(\psi(v)) < 0$ for $v \in I_1 = \big[y(t_k^*), y(t_k^1)\big)$. Define

$$S(v) = y^{[1]}(\varphi(v)) - |y^{[1]}(\psi(v))|, \quad v \in I.$$

Suppose, contrarily, that there exists $\bar{v} \in I_1$ such that

$$S(\bar{v}) < 0. \tag{11}$$

Then using (2), (9) and (H2), we have

$$\begin{aligned}\frac{d}{dv}S(v) &= \frac{y^{[2]}(\varphi(v))a_2(\varphi(v))}{y'(\varphi(v))} + \frac{y^{[2]}(\psi(v))a_2(\psi(v))}{y'(\psi(v))} \\ &= \frac{y^{[2]}(\varphi(v))}{y^{[1]}(\varphi(v))}\,\frac{a_2(\varphi(v))}{a_1(\varphi(v))} + \frac{y^{[2]}(\psi(v))}{y^{[1]}(\psi(v))}\,\frac{a_2(\psi(v))}{a_1(\psi(v))} \\ &\leqslant y^{[2]}(\psi(v))\frac{a_2(\psi(v))}{a_1(\psi(v))}\left[\frac{1}{y^{[1]}(\varphi(v))} + \frac{1}{y^{[1]}(\psi(v))}\right], \quad v \in I_1.\end{aligned}$$

Thus

$$v \in I_1\,,\ S(v) < 0\ \mathbb{R}_+ightarrow \frac{d}{dv}S(v) < 0\,.$$

¿From this and from (11), it is clear that

$$S(v) < 0 \text{ on } [\bar{v}, y(t_k^1)]\,,$$

and this contradicts $S(y(t_k^1)) = 0$. Thus, (10) holds and using $v = y(t_k^*)$ in (10) and (9), $y^{[1]}(t_{k-1}^2) > |y^{[1]}(t_k^2)|$.

(ii) Let $t_0 < t_1 < t_2$, $t_0^1 \leqslant t_0$ be consecutive zeros of $y^{[2]}$. Let us transform (1) into (5) according to Lemma 3 with $a_0 \equiv a_1$. Then x_i, $x_i = x(t_i)$, $i = 0, 1, 2$, are the consecutive zeros of $Y^{\{2\}}$, $x_0 < x_1 < x_3$.

The next transformation

$$\sigma = x_2 - x\,,\ Y(x) = Z(\sigma)\,,\ x \in [x_0, x_2]\,,\ \sigma \in [0, x_2 - x_0]\,, \tag{12}$$

transforms (5) into (7) where

$$b(\sigma) = \frac{a_2(t(x_2-\sigma))}{a_1(t(x_2-\sigma))}\,,\ \bar{r}(\sigma) = \frac{r(t(x_2-\sigma))}{a_1(t(x_2-\sigma))}$$

and according to (H2) and $\frac{d}{dt}\left(\frac{r(t)}{a_1(t)}\right) \leqslant 0$, we have

$$b'(\sigma) \geqslant 0 \text{ and } \bar{r}'(\sigma) \geqslant 0 \text{ on } [0, x_2 - x_0]\,, \frac{d}{d\sigma} = {}'\,.$$

As $\sigma_0 = 0$, $\sigma_1 = x_2 - x_1$, and $\sigma_2 = x_2 - x_0$ are consecutive zeros of $Z^{[2]}$, Lemma 4 yields

$$\sqrt{2}|Z'(x_2 - x_0)| < |Z'(x_2 - x_1)|\,. \tag{13}$$

Using (12) and (6) we have

$$|y^{[1]}(t_0)| = |Y^{\{1\}}(x_0)| = |\overset{\bullet}{Y}(x_0)| = |Z'(x_2 - x_0)|\,,$$
$$|y^{[1]}(t_1)| = |Y^{\{1\}}(x_1)| = |\overset{\bullet}{Y}(x_1)| = |Z'(x_2 - x_1)|$$

and thus (13) yields $\sqrt{2}|y^{[1]}(t_1)| < |y^{[1]}(t_0)|$.

¿From this the inequality (8) holds and $\lim\limits_{t\to T_-} y^{[1]}(t) = 0$.

(iii) We prove the third statement for (5) with $a_0 \equiv a_2$

$$\left(\left(\frac{1}{A_1}\overset{\bullet}{Y}\right)^{\bullet}\right)^{\bullet} = R(x)\,f(Y)\,,$$
$$A_1(x) = \frac{a_1(t(x))}{a_2(t(x))}\,,\ R(x) = \frac{r(t(x))}{a_2(t(x))}\,,\ Y^{\{1\}} = \frac{1}{A_1(x)}Y^{\bullet}\,,\ Y^{\{2\}} = (Y^{\{1\}})^{\bullet}\,;$$

then according to (6), it will hold for (1) too.

Applying Lemma 2 to (5), sequences $\{x_k^i\}$, $k = 1, 2, \dots$, $i = 0, 1, 2$ exist such that

$$\begin{aligned} & x_k^0 < x_k^1 < x_k^2 < x_{k+1}^0\,,\ k = 1, 2, \dots,\ \lim_{k\to\infty} x_k^0 = x(T)\,,\\ & Y^{\{i\}}(x_k^i) = 0\,,\ (-1)^{j+1}Y^{\{j\}}(x)Y(x)\ \begin{array}{l} > 0 \text{ on } (x_k^0, x_k^j)\,,\\ < 0 \text{ on } (x_k^j, x_{k+1}^0)\,, \end{array}\\ & k = 1, 2, \dots\,; \quad j = 1, 2\,. \end{aligned} \tag{14}$$

Let $k \in \{1, 2, \dots\}$. Put $\tau_0 = x_k^1$, $\tau_1 = x_k^2$, $\tau_2 = x_{k+1}^0$, $\Delta_1 = [\tau_0, \tau_1]$, $\Delta_2 = [\tau_1, \tau_2]$, $\delta_1 = \tau_1 - \tau_0$, $\delta_2 = \tau_2 - \tau_1$ and suppose, for simplicity, that $Y^{\{1\}}(x) \leqslant 0$ on Δ_1. Then (14) and Lemma 1 yield

$$\begin{aligned} & Y(x) > 0\,,\ Y^{\{1\}}(x) < 0\,,\ Y^{\{2\}}(x) < 0\,,\ Y \text{ and } |Y^{\{2\}}| \text{ are decreasing}\\ & \text{and } |Y^{\{1\}}| \text{ is increasing on } (\tau_0, \tau_1)\,;\\ & Y(x) > 0\,,\ Y^{\{1\}}(x) < 0\,,\ Y^{\{2\}}(x) > 0\,,\ Y \text{ and } |Y^{\{1\}}| \text{ are decreasing}\\ & \text{and } Y^{\{2\}} \text{ is increasing on } (\tau_1, \tau_2)\,. \end{aligned} \tag{15}$$

The statement will be valid if we prove that

$$|Y^{\{2\}}(x_k^0)| > |Y^{\{2\}}(\tau_0)| > Y^{\{2\}}(\tau_2)\,.$$

As the first inequality follows from (14) and Lemma 1, the second one only must be proved. Thus, suppose that

$$|Y^{\{2\}}(\tau_0)| \leqslant Y^{\{2\}}(\tau_2)\,. \tag{16}$$

According to (15) and the assumptions of the theorem, the function $Y^{\{2\}}$ is concave on $[\tau_0, \tau_2]$:

$$\begin{aligned}\left(Y^{\{2\}}(x)\right)^{\bullet\bullet} &= \left(Y^{\{3\}}(x)\right)^{\bullet} = \left[\frac{r(t(x))}{a_2(t(x))}\, f(Y(x))\right]^{\bullet} = \\ &= \left(\frac{r(t(x))}{a_2(t(x))}\right)^{\bullet} f(Y(x)) + \frac{r(t(x))}{a_2(t(x))}\, f'(Y(x))Y^{\{1\}}(x)\frac{a_1(t(x))}{a_2(t(x))} \leqslant 0\,, \\ &\qquad x \in \Delta_1 \cup \Delta_2\,.\end{aligned} \tag{17}$$

Thus, $Y^{\{2\}}$ is above the secant line on $\Delta_1 \cup \Delta_2$, and using (14) and (15), we have

$$|Y^{\{1\}}(\tau_1)| = \int_{\Delta_1} |(Y^{\{1\}}(x))^{\bullet}|\, dx = \int_{\Delta_1} |Y^{\{2\}}(x)|\, dx \leqslant |Y^{\{2\}}(\tau_0)|\, \frac{\delta_1}{2}\,,$$
$$|Y^{\{1\}}(\tau_1)| \geqslant Y^{\{1\}}(\tau_2) - Y^{\{1\}}(\tau_1) = \int_{\Delta_2} Y^{\{2\}}(x)\, dx \geqslant Y^{\{2\}}(\tau_2)\, \frac{\delta_2}{2}\,.$$

¿From this and (16),

$$\delta_1 \geqslant \delta_2\,. \tag{18}$$

Furthermore, according to (1), (15) and (17), $Y^{\{3\}} \geqslant 0$ is decreasing on $\Delta_1 \cup \Delta_2$. From this it follows that

$$|Y^{\{2\}}(\tau_0)| = \int_{\Delta_1} Y^{\{3\}}(x)\, dx > Y^{\{3\}}(\tau_1)\, \delta_1\,,$$
$$Y^{\{2\}}(\tau_2) = \int_{\Delta_2} Y^{\{3\}}(x)\, dx < Y^{\{3\}}(\tau_1)\, \delta_2\,.$$

Thus, with respect to (16), $\delta_1 < \delta_2$ and this contradicts (18). □

The following theorem states a sufficient condition under which oscillatory solutions tend to zero as $t \to T$.

Theorem 2. *Let* (H1) *and* (H2) *hold,* $r \in C^1(J)$, $f \in C^1(R)$, $f' \geqslant 0$ *on* R,

$$\left(\frac{r(t)}{a_1(t)}\right)' \leqslant 0\,, \tag{19}$$

and let one of the following assumptions hold:

(i) $\left(\frac{r(t)}{a_2(t)}\right)' \leqslant 0, \quad 0 < M \leqslant \frac{r(t)}{a_1(t)}$ *for* $t \in J$*;*

(ii) $\frac{a_2(t)}{a_1^2(t)}\, r(t) \geqslant M > 0$ *for* $t \in J$*;*

(iii) $\int_0^T a_1(s)\, ds < \infty$.

If $y \in \mathcal{O}$, *then* $\lim\limits_{t\to\infty} y^{(j)}(t) = 0$ *for* $j = 0, 1$.

Proof. Let $y \in \mathcal{O}$. According to Lemma 3 with $a_0 \equiv a_1$, it is sufficient to prove the results for(5) only:

$$\left(\frac{1}{A_2(x)} Y^{\bullet\bullet}\right)^{\bullet} = R(x)\, f(Y)\,,\ \frac{d}{dx} = {}^{\bullet}\,, \tag{20}$$

$$A_1 \equiv 1\,,\ A_2(x) = \frac{a_2(t(x))}{a_1(t(x))}\,,\ R(x) = \frac{r(t(x))}{a_1(t(x))}\,,\ x \in I = [0, x^*)\,,\ x^* = x(T)\,,$$

$$Y^{\{1\}} = Y^{\bullet}\,,\ Y^{\{2\}} = \frac{1}{A_2(x)} Y^{\bullet\bullet}\,. \tag{21}$$

Denote by $\{x_k^i\}$, $i = 0, 1, 2$, $k = 1, 2, \ldots$, the sequences given by Lemma 2 for (20) (i.e. $x_k^i = t_k^i$) and put

$$\Delta_k = [x_k^0, x_k^1]\,.$$

Then, according to Lemmas 1 and 2,

$$\begin{gathered} Y^{\{1\}}(x)\, Y(x) \geqslant 0\,,\ Y^{\{2\}}(x)\, Y(x) \leqslant 0 \text{ for } x \in \Delta_k\,, \\ |Y^{\{1\}}| \text{ and } |Y^{\{2\}}| \text{ are decreasing on } \Delta_k\,. \end{gathered} \tag{22}$$

Furthermore, using (19),

$$\left(\frac{R(x)}{A_1(x)}\right)^{\bullet} = R^{\bullet}(x) = \left(\frac{r(t)}{a_1(t)}\right)' t^{\bullet}(x) \leqslant 0 \quad \text{on } I\,,$$

the assumptions of Th. 1 (ii), applied to (20), are fulfilled. Thus, $\lim\limits_{x \to x^*} Y^{\{1\}}(x) = 0$ and

$$|Y^{\{1\}}(x_k^0)| \leqslant |Y^{\{1\}}(x_{k-1}^2)| \leqslant 2^{\frac{2-k}{2}} |Y^{\{1\}}(x_1^2)|\,, \quad k \geqslant 2\,; \tag{23}$$

note that the first inequality follows from Lemmas 1 and 2.

We prove indirectly that

$$\lim_{t \to T} Y(t) = 0\,. \tag{24}$$

Thus suppose, without loss of generality, that

$$|Y(x_k^1)| \geqslant M_1 > 0\,, \quad k = 1, 2, \ldots$$

Then, according to Lemmas 1 and 2, there exists a sequence $\bar{x}_k \in (x_k^0, x_k^1)$ such that

$$|Y(\bar{x}_k)| = \frac{M_1}{2}\,, \frac{M_1}{2} \leqslant |Y(x)| \leqslant M_1 \text{ on } \bar{\Delta}_k = [\bar{x}_k, x_k^1]\,. \tag{25}$$

Let $\delta_k = x_k^1 - \bar{x}_k$. Using (22) and (23), we have

$$\begin{aligned} \frac{M_1}{2} &\leqslant |Y(x_k^1) - Y(\bar{x}_k)| = \int_{\bar{\Delta}_k} |Y^{\{1\}}(x)|\, dx \\ &\leqslant |Y^{\{1\}}(x_k^0)|\, \delta_k \leqslant\leqslant 2^{\frac{2-k}{2}} \delta_k |Y^{\{1\}}(x_1^2)| \end{aligned}$$

and thus

$$\lim_{k \to \infty} \delta_k = \infty\,. \tag{26}$$

(i) According to (19) and (22),

$$|Y^{\{2\}}(x_k^0)| \geqslant \left[Y^{\{2\}}(x_k^1) - Y^{\{2\}}(\bar{x}_k)\right] \operatorname{sgn} Y(x_k^1) = \int_{\bar{\Delta}_k} Y^{\{3\}}(x) \operatorname{sgn} Y(x)\, dx$$
$$= \int_{\bar{\Delta}_k} R(x)\, f(Y(x)) \operatorname{sgn} Y(x)\, dx \geqslant M\delta_k \min_{\frac{M_1}{2} \leqslant s \leqslant M_1} |f(s)| > 0$$

and thus (26) yields $\lim_{k\to\infty} Y^{\{2\}}(x_k^0) = \infty$ which contradicts Theorem 1 (iii).

(ii) Using (22), (H2) and the assumptions, we have for $x \in \bar{\Delta}_k$:

$$A_2(x)|Y^{\{2\}}(x)| \geqslant A_2(x)\left[Y^{\{2\}}(x_k^1) - Y^{\{2\}}(x)\right] \operatorname{sgn} Y(x_k^1) =$$
$$= A_2(x) \int_x^{x_k^1} |Y^{\{3\}}(s)|\, ds \geqslant \int_x^{x_k^1} R(s)\, A_2(s)|f(Y(s))|\, ds \geqslant$$
$$\geqslant MM_2(x_k^1 - x)\,, \qquad M_2 = \min_{\frac{M_1}{2} \leqslant s \leqslant M_1} |f(s)| > 0\,.$$

¿From this and from (21),

$$Y^{\{1\}}(\bar{x}_k) = \int_{\bar{\Delta}_k} A_2(x)|Y^{\{2\}}(x)|\, dx \geqslant MM_2 \int_{\bar{\Delta}_k} (x_k^1 - x)\, dx = \frac{MM_2}{2}\, \delta_k^2\,.$$

Since $\lim_{x\to x^*} Y^{\{1\}}(x) = 0$, $Y^{\{1\}}(\bar{x}_k)$ is bounded, say

$$|Y^{\{1\}}(\bar{x}_k)| \leqslant M_3\,, \quad k = 1, 2, \dots\,,$$

and we can conclude that δ_k is bounded as well. This contradiction to (26) proves the statement.

(iii) In this case, $x^* < \infty$ and I is bounded which contradicts (26). □

Remark 3. (i) Note that $\left(\frac{r(t)}{a_1(t)}\right)' \leqslant 0$ follows from (H2) and the fact that $\left(\frac{r(t)}{a_2(t)}\right)' \leqslant 0$:

$$\left(\frac{r}{a_1}\right)' = \left(\frac{r}{a_2}\,\frac{a_2}{a_1}\right)' = \left(\frac{r}{a_2}\right)' \frac{a_2}{a_1} + \frac{r}{a_2}\left(\frac{a_2}{a_1}\right)' \leqslant 0\,.$$

(ii) The differential equation in Ex. 1 fulfills all assumptions of Th. 2 (i) with the exception of $0 < M \leqslant \frac{r(t)}{a_1(t)}$.

3. Case (H3)

In this section (1) will be studied under the assumption (H3).

Theorem B gives us a sufficient condition for every oscillatory solution to vanish at T. We generalize this result as follows.

Theorem 3. *Let* (H1) *and* (H3) *hold and let*

$$M \in (0,\infty), \quad a_1(t)\, a_2(t) \leqslant M r^2(t), \quad t \in J\,.$$

Then for every oscillatory solution y *of* (1), $\lim\limits_{t\to T^-} y(t) = 0$.

Proof. Using Lemma 3 with $a_0 = r$, the statement follows from Theorem B applied to (5). □

Remark 4. (i) It is proved in [6] that if (H3) holds, then $\big\{\sqrt{\frac{a_1}{a_2}}|y^{[1]}|\big|_{t=t_k^2}\big\}_{k=1}^{\infty}$ is a decreasing sequence.

(ii) Note that Theorem B is the special case of Theorem 3. Furthermore, if

$$a_1(t) = a_2(t) = r(t) = e^{-t}, \quad J = \mathbb{R}_+\,,$$

then the assumptions of Theorem 3 are fulfilled and the ones of Theorem 3 not. Thus Theorem 3 is a generalization of Theorem B.

4. Applications

We apply the previous results to the equation

$$y''' + q(t)y' = s(t)\, f(y) \tag{27}$$

where $q \in C^\circ(\mathbb{R}_+)$, $s \in C^\circ(\mathbb{R}_+)$, $f \in C(R)$,

$$s(t) > 0 \text{ on } \mathbb{R}_+\,, \;\; f(x)x > 0 \text{ for } x \neq 0\,. \tag{H4}$$

A solution y of (27) is called oscillatory if it is defined on $\mathbb{R}_+$, $\sup\limits_{\tau \leqslant t < \infty} |y(t)| > 0$ for every $\tau \in \mathbb{R}_+$ and there exists a sequence of zeros of y tending to ∞.

Let h be a positive solution on $[\tau,\infty)$, $\tau \in \mathbb{R}_+$, of the equation

$$h'' + q(t)h = 0\,. \tag{28}$$

Then (27) is equivalent to (1) (see [5] or make a direct computation) on $J = [\tau,\infty)$, where $T = \infty$,

$$\begin{gathered} a_1(t) = h(t)\,, \quad a_2(t) = \frac{1}{h^2(t)}\,, \quad r(t) = s(t)\, h(t)\,, \\ y^{[1]} = \frac{y'}{h}\,, \quad y^{[2]} = h^2 (y^{[1]})'\,. \end{gathered} \tag{29}$$

Thus (H1) is satisfied, (H2) holds if h is increasing, and (H3) holds if h is decreasing.

Theorem 4. *Let* (H4) *hold,*

$$q(t) \leqslant 0\,, s(t) \geqslant M > 0 \quad \text{for} \quad t \in [M_1,\infty)$$

and $\int\limits_0^\infty t|q(t)|\, dt < \infty$ *where* M *and* M_1 *are positive constants. Then every oscillatory solution of* (27) *tends to zero as* $t \to \infty$.

Proof. If follows from [11] and from $\int\limits_0^\infty t|q(t)|\, dt < \infty$ that (28) is non-oscillatory and there exists a positive solution h of (27) that is decreasing for large t and $\lim\limits_{t\to\infty} h(t) = h_0 \in (0,\infty)$. Thus, the conclusion follows from Theorem 3. □

Theorem 5. *Let* (H4) *hold,* $s \in C^1(\mathbb{R}_+)$*,* $f \in C^1(R)$*,* $f' \geqslant 0$ *on* R*,*

$$q(t) \geqslant 0\,, \quad 0 < M \leqslant s(t)\,, \quad s'(t) \leqslant 0 \quad \textit{for } t \in [M_1, \infty)\,,$$

and $\int\limits_0^\infty tq(t)\,dt < \infty$ *where* M *and* M_1 *are positive constants. Then every oscillatory solution of* (27) *tends to zero as* $t \to \infty$ *along with its first derivative.*

Proof. It follows from [7] and from $\int_0^t tq(t)\,dt < \infty$ that (28) is nonoscillatory and there exists a positive solution h of (28) that is increasing for large t and $\lim\limits_{t\to\infty} h(t) = h_0 \in (0, \infty)$. Then (27) is equivalent (1) and (29). Thus, the statement follows from Theorem 2 (ii) and the fact that $\lim\limits_{t\to\infty} y'(t) = \lim\limits_{t\to\infty} y^{[1]}(t)\,h(t) = 0$ (see Theorem 1 (ii)). □

Remark 5. Theorems 4 and 5 expand the results obtained in [9].

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Miroslav Bartušek
Department of Mathematics, Masaryk University
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
E-mail address: `bartusek@math.muni.cz`