15th Annual Conference of Applied Mathematics, Univ. of Central Oklahoma,
Electronic Journal of Differential Equations, Conference 02, 1999, pp. 115–124.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Mathematical model for the basilar membrane as a two dimensional plate *

H. Y. Alkahby, M. A. Mahrous, & B. Mamo

Abstract

In this paper we present two mathematical models for the basilar membrane. In the first model the membrane is represented as an annular region. In the second model the basilar membrane is treated as a rectangular region. Comparison of the two models allows us to study the effect of the curvature of the basilar membrane on the range of the frequencies of hearing. The differential equation of both models is a fourth order partial differential equation derived from the classical plate theory. Boundary conditions are defined as a region with four sides. The conditions are different on each side and together form an interesting physiological combination, relative to standard engineering problems. Eigenvalues of the differential equations of the two models are obtained numerically. A comparison of the eigenvalues of the two models clearly shows that the range of the hearing frequencies of the first model is larger than that of the second model. The results indicate strongly that the curvature of the basilar membrane plays an important role in the hearing process. Curvature and measurement of curvature should be allowed in future models and experiments of the inner ear.

1 Introduction

Before the mathematical models of the basilar membrane are presented, it is necessary to briefly describe the components of the inner ear. This gives a better understanding of the role of the basilar membrane's curvature in the hearing process. The inner ear is the location in the auditory system where mechanical and electrophysiological mechanisms are combined. The cochlea of the inner ear is a small bony structure with a small coiled tube in its interior. The walls of this tube are composed of special hard bone (the hardest in the body).

* *1991 Mathematics Subject Classifications:* 92C05, 92610, 35G15, 34B10.
Key words and phrases: Basilar membrane, eigenvalue, hearing frequencies.

Published January 21, 2000.

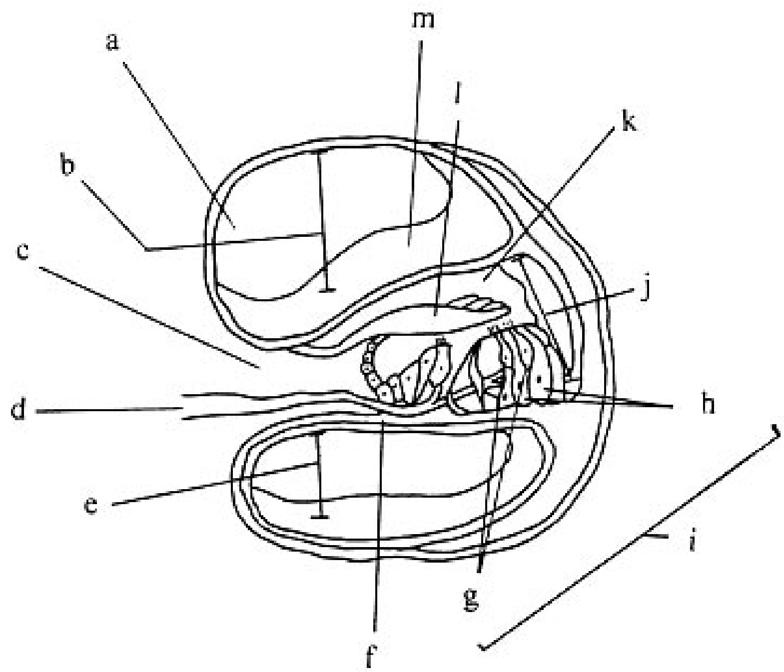

Figure 1: Schematic diagram of the cross section of the cochlear duct. Perylimph space (a), scala vestibuli (b), modiolus (c), cochlear nerve (d), scala tympani (e), basilar memebrane (f), hair celss (g), supporting cell (h), organ of Corti (i), cochlear duct (j), endolymph (within membrane) (k), tectotial membrane (l), vestibular (Reissner's) membrane (m)

In a cross section of the coiled tube (Fig. 1), there are three distinct chambers, namely, the scala vestibuli, the scala media and the scala tympani. The scala media is bounded by the Reissner's membrane and the basilar membrane. Vibrations of the oval window are transferred to the perilymph in the scala vestibuli, which transfers them to the basilar membrane, triggering the electrical impulses in the orgin of Corti, where the terminals of the acoustic nerve reside. The organ of Corti rests on top of the basilar membrane. Therefore, the bsilar membrane is considered to be of primary importance in the stimulation of the hair cells and the transmission of signals to the brain. The basilar membrane is a three-dimensional structure. It forms the helical spiral ramp. The edges, described as being from the base to the apex, from a diminishing spiral with radii of curvature becoming increasingly shorter. Interestingly, the basilar membrane is coiled, with no exception, in any species. The length of the basilar membrane varies from a short as 7 mm in laboratory mice, 20 mm in cats, 32 - 35 mm in humans and sheep to 60 mm in elephants (see [2] for references and Fig.2). The number of coils or "turns" ranges from 2 to 4.25

spiral turns. The number of coils in man is 2.25, 3 coils in cats and dogs, and 4 coils in the guinea-pig. The width of the basilar membrane (in man) is 0.1 mm at the basal end and increases to 0.5 mm at the apical end. From the size and location of the basilar membrane, direct experimental procedures are almost impossible. Nevertheless, Bekesy [1] pioneered an important experimental work on the inner ear for which he received a Nobel Laureat in 1961.

Presently, there are several theories of hearing and all assume a place principle; that is, different frequencies are triggered at different locations on the membrane. In general, this principle assumes that hearing frequency is a function of the mechanical properties of the basilar membrane and also a functioin of the location on the basilar membrane. In the most recent experiments, with the aid of advanced technological tools, better information is available about the basilar membrane [5]. Yet, data for an important parameter "curvature at the edges" is still missing. The work of Bekesy [2] supports a place principle where the basilar membrane is more sensitive to successively low frequencies progressively toward the apical end and to successively higher frequencies toward the basal end. It is pointed out that "the place principle predicts that apical or mid-apical regions of the cochlea are the first to mature and that basal regions are last, and just the opposite results are consistently found [4]." Thus, a more reliable place principle is needed.

From the above discussion, it is clear that the curvature of the edges of the basilar membrane must play some role in the hearing mechanism, in addition to the accepted evolution's explanation as a space saving feature. Notice that curvature information embodies information about the height of the cochlea, its base diameter, the number of coils and the diameter of each coil, and also the diminishing rate of the spiral. Thus, any attempt to model the mechanism of the basilar membrane in the hearing process must allow for curvature. In this paper two models are presented to illustrate the effect of the curvature on the vibration response of the membrane. In the following sections we present results that do, indeed, indicate that the curvature of the basilar membrane is an important property that cannot be ignored.

In the first model, the basilar membrane is considered as an incomplete annular region (see Fig. 3). The radii of the curvature of the inner and outer circles of the annular region simulate the curvature of the edges of the basilar membrane. The mathematical model is derived from the classical plate theory. It is a linearized fourth order partial differential equation. Eigenvalues for this differential equation are obtained numerically and they are dependent on the radii of the curvature. We believe that the dependence of the eigenvalues on the radii of the curvature has an important influence on the hearing process. To emphasize the importance of this conclusion the above model is compared with a second model. In the second model the membrane on a rectangular region is considered. The rectangular and the annular models have the same properties, and the same boundary conditions. We also have chosen the regions that have equal areas. Two sets of eigenvalues, for both models, are compared. Comparison clearly indicates the importance of the curvature and its effect on hearing frequencies.

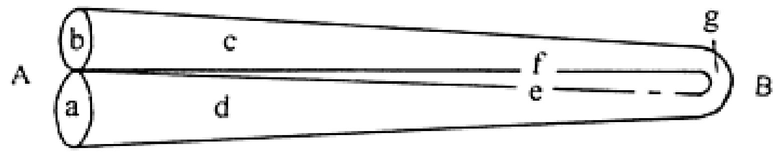

Figure 2: Schematic diagram of uncoiled cochlea and basilar membrane. Round window (a), Oval window connected to the stapes of the middle eat (b), Scala vestibuli (c), Scal tympani (d), Basilar membrane (f), Helicotrema (apical end) (g)

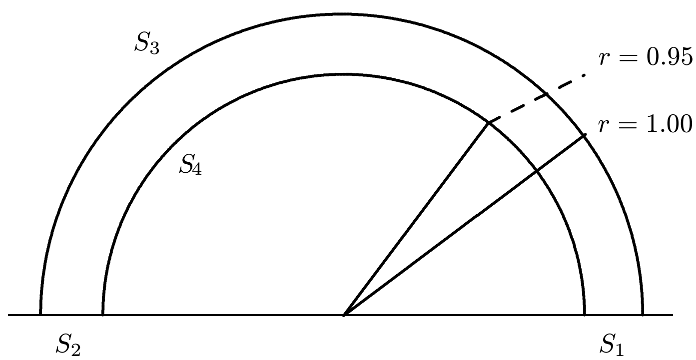

Figure 3: Semiannular plate configuration

Finally, in this work the membrane is represented by two dimensional models, where most of the previous "membrane specific" models are one dimensions. For example, [3] the basilar membrane was modeled as a one dimensional beam, which implies that the vibrations are dependent only on the longitudinal direction along the membrane. On the other extreme, the fibers of the basilar membrane are examined in the radial direction [5].

2 Basic equations and boundary conditions

The basilar membrane is considered as a plate according to classical plate theory. The equation of motion for the traverse displacement, u, is given as:

$$\boldsymbol{D}\nabla^4\boldsymbol{u} + 2c\frac{\partial u}{\partial t} + \zeta\boldsymbol{h}\frac{\partial^2 u}{\partial t^2} = 0 \;, \tag{1}$$

where D is flexural rigidity and defined by

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

h is plate thickness, E is Young's modules of plate material, v is Poisson's ratio for plate material, ζ is density of plate material, c is damping coefficient, and ∇ is the gradient operator.

Thickness, density and flexural rigidity are considered constant. The basilar membrane is assumed to be a region with the following four sides:

"S_1" corresponding to the basal end of the membrane,

"S_2" corresponding to the apical end of the membrane (at the helicotrema),

"S_3" corresponding to the outer wall of the cochlea, and

"S_4" corresponding to the inner wall (see Fig. 3).

Since the side "S_1" is clamped we have:

$$\boldsymbol{u}|_{t=0} = 0 \ , \qquad \frac{\partial u}{\partial n} \ , \tag{2}$$

where n is the normal direction to S_1.

The basilar membrane is not attached to anything at the helicotrema, so the condition on S_2 is:

$$u_{ss} + v u_{nn} = 0 \ , \tag{3}$$

$$u_{sss} - 2(1-v)u_{snn} = 0 \ . \tag{4}$$

The side "S_3" is attached to the outside cochlear wall and simply supported. On this side, usually called the Spiral Ligament (SL),the fibers present little resistance to moment. The condition on S_3 is:

$$u_{ss} + u_{nn} = 0 \ . \tag{5}$$

At the primary osseous spiral lamina (SPL), which is represented by S_4, the filaments are held between an upper and lower bony layers and enter the support with zero slope. Therefore, this side is clamped, and the condition on S_4 is:

$$\boldsymbol{u}|_{t=0} = 0 \ , \qquad \frac{\partial u}{\partial n} = 0 \ . \tag{6}$$

In this section the basilar membrane is modeled as an annular plate. Moreover, a is the radius of the outer circle and b is the radius of the inner circle.

3 Solution of the model as an annular region

In this section the basilar membrane is modeled as an annular plate. The radii of curvature are constants. In spite of the absence of information about the real curvature of the membrane edges, we still get results for the radial waves as well as the longitudinal waves. Equation (1) is considered in polar coordinates where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \ . \tag{7}$$

Let $u = W(r,\theta)f(t)$ in (1), and assume that

$$f(t) = \exp(-ct/\omega h)\sin(i\omega) \ , \tag{8}$$

and $f(0) = 0$, then equation (1) becomes

$$\nabla^4 W - k^4 W = 0 \ , \tag{9}$$

where

$$\begin{aligned} \omega^2 &= \frac{\zeta h D k^4 - c^2}{\zeta^2 h^2} \ , \quad \text{or} \\ k^4 &= \frac{\zeta^2 h^2 \omega^2 + c^2}{\zeta h D} \ . \end{aligned} \tag{10}$$

The boundary conditions (2-6) are now as follows: On $S_1 : \theta = 0$ and

$$W(r,0) = 0 \ , \qquad \frac{\partial W}{\partial r}(r,0) = 0 \ . \tag{11}$$

On $S_2 : \theta = \pi$ and

$$\begin{aligned} W_{\theta\theta}(r,\pi) = 0 \quad &, \qquad W_{rr}(r,\pi) = 0 \ , \\ W_{\theta\theta\theta}(r,\pi) = 0 \quad &, \qquad W_{\theta\pi\pi}(r,\pi) = 0 \ . \end{aligned} \tag{12}$$

On $S_3 : r = a$ and

$$W_{rr}(a,\theta) = 0 \ , \qquad W_\theta(a,\theta) = 0 \ . \tag{13}$$

On $S_4 : r = b$ and

$$W(b,\theta) = 0 \ , \qquad W_\theta(b,\theta) = 0 \ . \tag{14}$$

The differential equation (9) has a solution of the following form:

$$W(r,\theta) = \sum_{m=1}^{\infty} g_m(r)\cos m\theta + \sum_{m=1}^{\infty} \bar{g}_m(r)\sin m\theta \ , \tag{15}$$

where

$$g_m(r) = A_m J_m(kr) + B_m Y_m(kr) + C_m I_m(kr) + D_m K_m(kr) \ , \tag{16}$$

and

$$\bar{g}_m(r) = \bar{A}_m J_m(kr) + \bar{B}_m Y_m(kr) + \bar{C}_m I_m(kr) + \bar{D}_m K_m(kr) \ . \tag{17}$$

The coefficients A_m, B_m, C_m,$\bar{A}_m$, $\bar{B}_m$, $\bar{C}_m$, $\bar{D}_m$ are constant to be evaluated using the boundary conditions, and J_m, Y_m, I_m, K_m- Bessel functions and modified Bessel functions of the first and second kind. Using the boundary conditions (11 - 14), and after some simplification, we obtain the following frequency determinant T. A nontrivial solution is obtained with $T = 0$.

$$T = \begin{vmatrix} J_m(\lambda_1) & Y_m(\lambda_1) & I_m(\lambda_1) & K_m(\lambda_1) \\ T_{21} & T_{22} & T_{23} & T_{24} \\ J_m(\lambda_2) & Y_m(\lambda_2) & I_m(\lambda_2) & K_m(\lambda_2) \\ J_{m+1}(\lambda_2) & Y_{m+1}(\lambda_2) & -I_{m+1}(\lambda_2) & K_{m+1}(\lambda_2) \end{vmatrix} \tag{18}$$

where

$$\begin{aligned} T_{21} &= (1-v)J_{m+1}(\lambda_1) - 2\lambda_1 J_m(\lambda_1) \ , \\ T_{22} &= (1-v)Y_{m+1}(\lambda_1) - 2\lambda_1 Y_m(\lambda_1) \ , \\ T_{23} &= -(1-v)I_{m+1}(\lambda_1) \ , \\ T_{24} &= (1-v)K_{m+1}(\lambda_1) \ , \\ \lambda_1 &= ka, \qquad \lambda_2 = kb \ . \end{aligned} \tag{19}$$

Moreover, a is the radius of the outer circle, b is the radius of the inner circle and k is defined in (10).

4 Solution of the model as a rectangular plate

The above model of the basilar membrane as an annular plate is closer to the real model than a rectangular plate model. Our interest in a rectangular plate in this section is to compare the eigenvalues from both models for plates with the same characteristics and essentially the same area. The literature is wealthy in results for rectangular plates but results for the specific boundary condition we selected for the basilar membrane are scarce. Therefore two solutions that lead to some estimates of eigenvalues are given here.

The boundary conditions are as follows:

On $S_1 : x = 0$ and

$$W(0, y) = W_x(0, y) = 0 \ . \tag{20}$$

On $S_2 : x = a$ and

$$\begin{aligned} W_{xx}(a, y) + (2-v)W_{yyx}(a, y) &= 0 \ , \\ W_{xx}(a,) + vW_{yy}(a, y) &= 0 \ . \end{aligned} \tag{21}$$

On $S_3 : y = 0$ and

$$W(x, 0) = W_{yy}(x, 0) = 0 \ . \tag{22}$$

On $S_4 : y = b$ and

$$W_{yy}(x, b) + vW_{xx}(x, b) = 0 \ . \tag{23}$$

The operator ∇^2 has the standard form:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \ .$$

Let

$$W = \sum_{m=1}^{\infty} \Big[\sum_{n=1}^{4} (A_{mn} \sin cy \quad + \quad B_{mn} \cos cy$$
$$C_{mn} \sinh dy \quad + \quad D_{mn} \cosh dy) F_n(x) \Big] \qquad (24)$$

where A_{mn}, B_{mn}, C_{mn}, D_{mn} are constants and

$$F_1(x) = \sin ax, F_2(x) = \cos ax, F_3(x) = \sinh ax . F_4(x) = \cosh ax \ , \qquad (25)$$

where

$$c = \sqrt{k^2 - a^2} \ , \quad d = \sqrt{k^2 + a^2} \ . \qquad (26)$$

Using the boundary conditions (20-23) we obtain

$$\tan cb = \frac{c}{d} \tan db \ , \quad \text{and} \quad \tan aa = \tan aa \ . \qquad (27)$$

Table 1. Eigenvalues for annular plate and rectangular plate models

a/b	Harmonics	Annular	Plate	Rectangular	Plate
		λ_1	λ_2	λ_1	λ_2
0.8	1	15.5611	12.4489		
0.8	2	31.5640	25.2512		
0.8	3	47.0748	37.6598		
0.8	4	62.906	50.3252		
0.8	5	78.5104	62.8083		
0.8	6	94.2975	75.438		
0.8	7	109.9347	87.9478		
0.8	8	125.7010	100.5608		
0.8	9	141.3555	13.0844		
0.8	10	157.1095	125.6876		
0.8	11	172.7740	138.2190		
0.9	1	31.1005	27.990		
0.9	2	62.8981	56.6083		
0.9	3	94.143	84.7280		
0.9	4	125.697	13.1270		
0.9	5	175.017	41.3153		
0.95	1	62.4469	59.3245	44.8387	42.5968
0.95	2	125.695	119.4100	93.8786	89.1847

5 Computations and conclusions

The eigenvalues of both models are computed numerically. It is shown that if the curvature of the basilar membrane is taken into account the range of the hearing frequencies is significantly wider. On the other hand, for the rectangular model, the hearing frequencies appeared to be significant only when the rectangular region tends to a secured region. This, of course, contradicts the real shape of the basilar membrane. As a result, any experimental or theoretical work should take the curvature of the basilar membrane into account.

Finally, these numerical computations for the eigenvalues were obtained on a VAX-8600 system with IMSL standard mathematical routines.

References

[1] Bekesy, G. von. "The variation of phase along the basilar membrane with sinusoidal vibrations," *J. Acous. Soc. Amer.* **19**(1947) , 452–460.

[2] Bekesy, G. von. *Experiments in hearing*, McGraw-Hill, (1960).

[3] Cooper, N. P., and Rhodes, W. S. "Basilar membrane mechanics in the hook region of dat and guinea-pig cochlea: sharp tuning and non linearity in the absence of baseline position shifts," *Hear. Res.* **63**, (1/2), 163–190.

[4] Diependaal, R. J. and Viergever, M. A. (1989) "Nonlinear and active two-dimensional cochlear models: Time-down in solution," *J. Acoust. Soc. Am.* **85**(2)(1992), 803–812.

[5] Inselberg, A. and Forester, H. V. "A mathematical model of the basilar membrane,"*Mathematical Biosciences* **7**(1970) , 341–363.

[6] Khanna, S. M. and Leonard, D. G. B. "Measurement of basilar membrane vibrations and evaluation of the cochlear condition," *Hear. Res*, **23**(1986), 37–54.

[7] Khanna, S. M. and Leonard, D. G. B. "Relationship between basilar membrane tuning and hair cell condition," *Hear. Res*, **23**(1986), 55–70.

[8] Leonard, D. G. B. and Khanna, S. M. "Histological evaluation of damage in cat cochleas used for measurement of basilar membrane mechanics," *J. Acoust. Soc. Amer.* **75** (1984), 515–527.

[9] Muller, M., Ott, H. and Bruns, V. "Frequency representation and spiral ganglion cell density in the cochlea of the gerbil," *Pachyuromys duprasi.* Hear. Res., **56**(1991), (1/2), 191–196.

[10] Nuttall, A. L., Dolan, D. F. and Avinash, G. "Laser doppler velocimetry of basilar membrane vibration," *Hear. Res.* **51**(1991), (2) 203–213.

[11] Patuzzi, R., Sellick, P. M. and Johnstone, B. M. "The modulation of the sensitivity of the mammalian cochlea by low frequency tones. III basilar membrane motion. *Hear. Res.* **13**(1984), 19–27.

[12] Robles, L. Ruggero, M.A. and Rich, N. C. "Basilar membrane mechanics at the base of the chinchilla cochlea. I. Input-output functions, tuning curves and response phases, *J. Acouost. Soc. Am.* **80**(1989), 1364–1374.

[13] Robles, L. Ruggero, M.A. and Rich, N. C. "Two-tone distortion in the basilar membrane of the cochlea," *Nature* Lond. **349**(1991), 413–414.

[14] Ruggero, M.A. "Response to sound of the basilar membrane of the mammalian cochlea," *Curr. Opin. Neurobiol.* **2**(4)(1992) , 449–456.

[15] Ruggero, M.A. and Rich, N. C. "Application of a commercially-manufactured doppler-shift laser velocimeter to the measurement of basilar-membrane vibration," *Hear. Res.* **51** (1991), 215–230.

[16] Ruggero, M. A., Robles, L., Rich, N. C. and Recio, A. "Basilar membrane response to two-tone and broadband stimuli," *Philos. Trans. R. Soc. Lond. Biol. Sci.* **336**(1992), 307–314.

[17] Sellick, P. M., Paluzzi, R. and Johnstone, B. M. "Measurement of basilar membrane motion iin the guinea-pig using Mossbauer techniques," *J. Acoust. Soc. Am.* **72**(1982), 131–141.

[18] Sokolowski, B. H. A., Sachs, M. B. and Goldstein J. L. "Auditory nerve rate-level functions for two-tone stimuli: possible relation to basilar membrane nonlinearity," *Hear Res.* **41**(1989), 115–124.

[19] West, C. D. "The relationship of the spiral turns of the cochlea and the length of the basilar membrane to the range of audible frequencies in ground dwelling mammals," *J. Acoust. Soc. Am.* **77** (3)(1985), 1091–1101.

H. Y. Alkahby & B. Mamo
Division of the Natural Sciences, Dillard University
New Orleans, LA 70122 USA
Telephone: 504-286-4731 e-mail: halkahby@aol.com

M. A. Mahrous
Department of Mathematics, University of New Orleans
New Orleans, LA 70148, USA