Differential Equations and Computational Simulations III
J. Graef, R. Shivaji, B. Soni J. & Zhu (Editors)
Electronic Journal of Differential Equations, Conference 01, 1997, pp. 149–160.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp 147.26.103.110 or 129.120.3.113 (login: ftp)

On the Existence of Steady Flow in a Channel with one Porous Wall or Two Accelerating Walls *

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Abstract

This paper presents a rigorous proof of the existence of steady flows in a channel either with no-slip at one wall and constant uniform suction or injection through another wall, or with two accelerating walls. The flows are governed by the fourth order nonlinear differential equation $F^{iv} + R(FF''' - F'F'') = 0$. In the former case, the flow is subject to the boundary conditions $F(-1) = F'(-1) = F'(1) = 0$, $F(1) = -1$. In the latter case, the boundary conditions are $F(-1) = F(1) = 0$, $F'(-1) = -1$, $F'(1) = 1$.

1 Introduction

A similarity solution of the Navier-Stokes equation that describes the steady two-dimensional flow of a viscous incompressible fluid through a channel with porous walls was first introduced by Berman in 1953. Since the flows have numerous applications such as the control of boundary layer separation with suction of injection, membrane separation processes, filtration, and biological transport processes, etc., the Berman problem has attracted many researchers' attention and Berman's results have been extended since then. A special case that the channel has one porous wall and one impermeable wall was studied by Cox in [3], in which he classified all possible types of solutions for the flows not only in a channel with one porous and one impermeable, but also in a channel where both walls are impermeable, but accelerating. This paper presents a rigorous proof of existence of those solutions. It is very interesting to study the special flow in a channel with one wall impermeable because [3]

> in laboratory experiment such a flow could be visualized easily than in the case which has previously been considered where there is suction through both walls, and experimental apparatus would simply obscure the view.

* *1991 Mathematics Subject Classifications:* 34B15, 76D05.
Key words and phrases: laminar flow, similarity solutions, Navier-Stokes equations.

Published November 12, 1998.

Assume that the fluid is viscous and incompressible, and that the channel is sufficiently long, and therefore, can be described as $-\infty < x < \infty$, $-1 < y < 1$, where $y = -1$ denotes the lower wall and $y = 1$ the upper wall. Let u be the velocity component along the channel, and v the velocity component across the channel. Then, a stream function $\Phi(x, y)$ can be introduced which satisfies

$$u = \frac{\partial \Phi(x,y)}{\partial y}, \quad v = -\frac{\partial \Phi(x,y)}{\partial x}.$$

There is a similarity solution of the form $\Phi(x, y) = xF(y)$, and the Navier-Stokes equations can be reduced to the Berman equation

$$F^{iv} + R(FF''' - F'F'') = 0 \tag{1}$$

where R is the Reynolds number, and $R > 0$ corresponds to suction and $R < 0$ to injection. If we assume the lower wall is impermeable and constant suction is through the upper wall, then the boundary conditions become

$$F(-1) = 0, \quad F'(-1) = 0\,, \tag{2}$$

$$F(1) = -1, \quad F'(1) = 0\,. \tag{3}$$

If the channel has two accelerating walls with equal rates of acceleration, the boundary conditions are

$$F(-1) = 0, \quad F'(-1) = -1\,, \tag{4}$$

$$F(1) = 0, \quad F'(1) = 1\,. \tag{5}$$

When only the upper wall is accelerating. the boundary conditions are

$$F(-1) = 0, \quad F'(-1) = 0\,, \tag{6}$$

$$F(1) = 0, \quad F'(1) = 1\,. \tag{7}$$

The paper proves the existence of solutions for these boundary value problems. For brief, we will call the boundary value problem related to only one porous wall (2)-(3) as the first boundary value problem, or BVP (1), the boundary value problem related to two accelerating walls (4)-(5) as the second boundary value problem, or BVP (2), and the boundary value problem related to the upper accelerating wall (6)-(7) as the third boundary value problem, or BVP (3).

2 Main Results

The main results in the paper are the following three theorems.

Theorem 1 *For each real number R, the equation (1) with the boundary conditions (2) and (3) has at least one solution. If $R < 0$, the solution is unique.*

Theorem 2 *For each real number R, the equation (1) with the boundary conditions (4)-(5) has at least one solution. If R is sufficiently large, then there are at least three solutions.*

Theorem 3 *For each real number R, the equation (1) with the boundary conditions (6)-(7) has at least one solution.*

The proofs of the theorems rely on the following transformation and lemmas.

3 Transformation and lemmas

We first consider BVP (1). Let

$$RF(y) = -\frac{1}{2}bg(\xi), \quad \xi = \frac{1}{2}b(y+1), \tag{8}$$

where $b > 0$ is a positive parameter to be determined. Substituting (8) into (1), we obtain

$$g^{iv} = gg''' - g'g'' \tag{9}$$

And the substitution converts the boundary conditions (2) and (3) to the forms:

$$g(0) = g'(0) = 0\,, \tag{10}$$

$$g(b) = \tfrac{2R}{b}, \quad g'(b) = 0\,. \tag{11}$$

This change of variables transforms the original boundary value problem into an initial value problem that has two parameters R and b. If we consider the initial value problem (9) subject to

$$g(0) = g'(0) = 0, g''(0) = \beta, g'''(0) = \gamma, \tag{12}$$

then any solution of the boundary value problem, (8) with (2) and (3), must correspond to a pair (β, γ). If a solution $g(\xi, \beta, \gamma)$ of the initial value problem, (9) with (12), has a zero of its first derivative at a positive point, then we say that the pair (β, γ) provides a solution to the boundary value problem. The idea of the proof of the theorem 1 is to find all possible pairs (β, γ) such that the corresponding solution of the initial value problem has a positive zero b. It is clear that the values of b and $R = bg(b)/2$ are functions of β and γ. Throughout the paper, we always consider nonzero independent variable ξ.

For BVP (3), following the same change of variables (8), the boundary conditions are converted to

$$g(0) = g'(0) = 0\,, \tag{13}$$

$$g(b) = 0, \quad g'(b) = -4R/b^2\,. \tag{14}$$

As for BVP (2), we only consider the symmetric solutions, i.e., an odd function $F(y)$ satisfying (1) and (4)-(5). Thus, $F(0) = F''(0) = 0$, we may only consider the half channel $[0, 1]$. In this case, we set

$$\xi = by, \quad RF(y) = -bg(\xi).$$

The resulting equation is exactly the same as (9), but the boundary conditions take the form:

$$g(0) = g''(0) = 0\,, \tag{15}$$

$$g(b) = 0, \quad g'(b) = -R/b^2\,. \tag{16}$$

The idea to find a solution of BVP (2) or BVP (3) is the same as that used in solving BVP (1).

In order to cover all the three cases, we may consider more general cases as follows. Let $g(\xi, \alpha, \beta, \gamma)$ denote the solution of (9) with the initial conditions

$$g(0) = 0, \quad g'(0) = \alpha, \quad g''(0) = \beta, \quad g'''(0) = \gamma\,. \tag{17}$$

Thus, in order to get a solution to a boundary value problem above, we need only find a suitable triple (α, β, γ) such that the solution of (9) with (17) has a zero point of g' so that it provides an $R = R(\alpha, \beta, \gamma)$. To find all solutions of the boundary value problems, we must consider all possible such triples. The following two lemmas hold for the initial value problem.

Lemma 1 *If $g(\xi, \alpha, \beta, \gamma)$ is a nonlinear solution of (9) - (17) solving the above three boundary value problems, then $g^{iv}(\xi, \alpha, \beta, \gamma) < 0$ for $\xi > 0$ and as long as the solution exists.*

Proof . For BVP (1) and BVP (3), the initial value is $(0, \beta, \gamma)$, and BVP (3) is related to the initial value $(\alpha, 0, \gamma)$. In either case, we see from (9) that $g^{iv}(0) = 0$. Differentiating (9) once with respect to ξ, we obtain

$$g^v = gg^{iv} - (g'')^2, \tag{18}$$

from which

$$g^{iv} = -\int_0^\xi (g'')^2 e^{\int_s^\xi g\, dt}\, ds. \tag{19}$$

Since g is not linear, $g'' \neq 0$ at least for small $\xi > 0$, hence, $(g'')^2 > 0$ initially, which leads to the lemma. □

Lemma 2 $g(\xi, \beta, \gamma) = \lambda g(\lambda\xi, \alpha\lambda^{-2}, \beta\lambda^{-3}, \gamma\lambda^{-4})$ *for any positive λ and as long as the solution exists.*

Proof. Let $f(\xi) = \lambda g(\lambda\xi, \alpha\lambda^{-2}, \beta\lambda^{-3}, \gamma\lambda^{-4})$. A simple substitution shows that f solves the equation (9) and satisfies the initial condition $f(0) = 0$, $f'(0) = \alpha$, $f''(0) = \beta$, and $f'''(0) = \gamma$. By the uniqueness of the solution to the initial value problem, the lemma follows. □

Remark. This lemma was first introduced by Wang and Hwang in [6].

The following corollary is a consequence of lemma 2.

Corollary 1 *$R(\alpha, \beta, \gamma) = R(\alpha\lambda^{-2}, \beta\lambda^{-3}, \gamma\lambda^{-4})$ for any $\lambda > 0$ as long as one of them exists.*

Lemma 3 *For any triple (α, β, γ) with $\alpha = 0$ or $\beta = 0$, the solution $g(\xi, \alpha, \beta, \gamma)$ of the initial value problem exists for all $\xi \in [0, \infty)$.*

Proof. Suppose not. There would be a triple (α, β, γ) such that the corresponding solution g only exists on the maximal interval $[0, l)$ for a real number l. By lemma 1, g''' is decreasing, and hence must diverge to $-\infty$ as $\xi \to l^-$ for otherwise g''', g'', g', and g are all bounded, hence g^{iv} is bounded in $[0, l)$ and the solution can be extend to the point l. Since g'' is concave down and $g''' \to -\infty$, it follows that $g'' \to -\infty$ as $\xi \to l^-$. Consecutive integration of (9) gives

$$g''' = gg'' - (g')^2 + \alpha^2 + \gamma\,, \tag{20}$$

$$g'' = gg' - 2\int_0^\xi (g')^2 dt + (\alpha^2 + \gamma)\xi + \beta\,. \tag{21}$$

From (21), g' must be unbounded as $\xi \to l^-$. Since also $g' < \alpha + \beta\xi + \frac{1}{2}\gamma\xi^2$, g' must diverge to $-\infty$ as $\xi \to l^-$. Then, for sufficiently large number $N > 0$, there is a number $k = k(N)$ such that $g' < -N, g'' < -N$ and $g''' < -N$, and g keeps one sign in (k, l). Also, $|g'|$, $|g''|$, and $|g'''|$ are bounded by N in $[0, k]$.

If $g < 0$ in (k, l), from (20), $gg'' - (g')^2 < 0$ in (k, l) which implies

$$\frac{g''}{g'} < \frac{g'}{g}\,. \tag{22}$$

Integrating (22) once from k to $\xi \in (k, l)$ shows

$$g'(\xi) \geq cg(\xi), \tag{23}$$

for all $\xi \in (k, l)$ where $c = \frac{g'(k)}{g(k)} > 0$ is a constant , from which $g \to -\infty$ as $\xi \to l^-$. An integration of (23) produces $g(\xi) \geq -|g(k)|e^{c(\xi-k)}$, and $\xi \in (k, l)$. This contradicts the fact that g is unbounded at l.

If $g > 0$ in (k, l), from (21), since $g' < -N$ for $\xi \in (k, l)$ and $|g'| \leq N$ in $[0, k]$, there exists a number $M > 0$ such that $g''(\xi) > -2\int_0^\xi (g')^2 dt > -Mg'(\xi)^2$. Hence

$$\frac{g''}{g'} < -Mg'. \tag{24}$$

Integrating (24) from k to any $\xi \in (k,l)$ leads to $|g'(\xi)| \leq Ce^{-Mg(\xi)}$, where $C = |g'(k)|e^{Mg(k)}$, a constant. Since $g(\xi) > 0$, this contradicts the assumption $g' \to -\infty$. The proof is completed. □

Remark. Lemma 3 generalizes the similar lemma in [4]

Since the proofs of the three theorems are similar, for the sake of briefness, we only provide the detailed proof of theorem 1, outline the proof of theorem 2 and omit the proof of theorem 3.

4 Proof of Theorem 1

In this case, $g'(0) = \alpha = 0$, therefore, to get all solutions of BVP (1), we need to consider all possible pairs (β, γ). Denote $g(\xi, 0, \beta, \gamma) = g(\xi, \beta, \gamma)$, and $R(0, \beta, \gamma) = R(\beta, \gamma)$. It is immediately seen from lemma 1 that $g'(\xi, \beta, \gamma)$ has no zero at all if both β and γ are non-positive. Then, there are only three possible cases that need to be investigated: (1) $\beta > 0$ and $\gamma \leq 0$, (2) $\beta \geq 0$ and $\gamma \geq 0$, and (3) $\beta \leq 0$ and $\gamma \geq 0$. Of course, the case $\beta = \gamma = 0$ is excluded.

Case (1). If $\gamma < 0$, then, since $g^{iv} < 0$ for all $\xi > 0$, we have $g'(\xi) < \beta\xi + \frac{\gamma}{2}\xi^2$ for all $\xi > 0$, by Taylor's theorem. It then follows that there is a point $\xi = r$ at which $g' = 0$. If $\gamma = 0$, then from (18), $g^v(0) = -\beta^2 < 0$, hence $g'''(x_0) \approx -\beta^2\frac{x_0^2}{2} < \gamma_0 < 0$ for sufficiently small $x_0 > 0$ where γ_0 is a constant. Then, $g'(\xi) < \beta(\xi - x_0) + \frac{\gamma_0}{2}(\xi - x_0^2)$ for all $\xi > x_0$. A similar argument can show the existence of r at which $g' = 0$. And this is the only zero point of g', namely, $g' < 0$ for all $\xi > r$. This shows that any pair (β, γ) where $\beta > 0$ and $\gamma < 0$ presents an $R > 0$. Recall that $R = R(\beta, \gamma) = bg(b)/2$ which implies that R is a continuous function of β and γ since the zero of g', in this case, is a simple zero, i.e., $g'' < 0$ whenever $g' = 0$ from (20).

From corollary 1, $R(\beta, \gamma) = R(1, \frac{\gamma}{\beta^{4/3}}) = R(\frac{\beta}{|\gamma|^{3/4}}, -1)$. As $\frac{\gamma}{\beta^{4/3}} \to 0^-, R \to R(1, 0)$; and as $\frac{\gamma}{\beta^{4/3}} \to -\infty, R \to R(0, -1) = 0$. Since $(-\infty, 0]$ is connected and the range of continuous functions with a connected domain is also connected, all the possible R that solutions of this kind contribute is an interval $(0, R_1) = I_1$ where $R_1 = \sup R(1, s)$ for all $s \in (-\infty, 0]$. The continuity of R guarantees $R_1 < \infty$. In addition, $R(1, 0) \leq R_1$.

Case (2). In this case, $g'' > 0$ initially (i.e., for sufficiently small $\xi > 0$), which implies that $g, g' > 0$ initially. From (18) and lemma 1, $g^v < -(g'')^2$ as long as $g > 0$. This means that the graph of g''' is concave down and decreasing as long as $g > 0$. Thus, either $g''' = 0$ or $g = 0$ eventually. If $g = 0$, then there must be a point at which $g' = 0$; if $g''' = 0$ with $g' > 0$, then g'' becomes concave down and decreases, thus, g'' must cross the ξ axis, then g' is concave down and decreasing and must have a zero point. Similar to above, by applying corollary 1, the maximal interval for R, which admits a solution of this kind, is $(\inf R(1, b), \sup R(1, b)) = I_2$ for $b \in (0, \infty)$. Since $R(1, 0) \in I_1$ and is a limit

point of I_2, we see that $I_1 \cup I_2$ is connected. In addition, there is only one zero of g' in this case. To see this, let r be the first zero point of g'. Since $g'' > 0$ initially and r is the first zero of g', it must be that $g''(r) \leq 0$ and $g' > 0$ in $(0, r)$. Then, there is a point, say s, in $(0, r)$ such that $g'''(s) < 0$. By lemma 1, $g''' < 0$ and hence $g'', g' < 0$ for all $\xi > r$. Note that I_2 must be an open interval and $R(1, 0)$ is either $\inf R(1, b)$ or $\sup R(1, b)$ for $b \in (0, \infty)$.
Case (3). Unlike the two cases above, some β and γ may not provide an R at all. The discussion of this case consists of the following three lemmas.

Lemma 4 *If $\beta < 0$ and $\gamma > 0$ with $\beta \leq -\sqrt{2}\gamma^{3/4}$, then $g'(\xi, \beta, \gamma)$ has no zero for all $\xi > 0$.*

Proof. By corollary 1, we may only consider $g(\xi, l, 1)$ where $l = \frac{\beta}{\gamma^{3/4}}$. Then, equation (20) takes the form

$$g''' = gg'' - (g')^2 + 1 \tag{25}$$

subject to

$$g(0) = g'(0) = 0, \quad g''(0) = l. \tag{26}$$

From the condition of the lemma, $l \leq -\sqrt{2}$. Since $g^{iv} < 0$ for all $\xi > 0$, we see $g''' < 1$, then $g'' \leq \xi + l$ and $g' \leq l\xi + \frac{\xi^2}{2}$ for all $\xi > 0$. Thus, $g'' < 0$ for $\xi < |l|$ and $g'(|l|) \leq -1$. Hence, $g' < 0$ for $0 < \xi \leq |l|$. Next, we show that $g' < 0$ for all $\xi > |l|$. To see this, it suffices to show that $g'' < 0$ for all $\xi > 0$. Suppose, for contradiction, that there is a first zero of g'', say x_0. Then, $x_0 > |l|$, $g'(x_0) < -1$, and $g'''(x_0) \geq 0$ from (9). However, from (25), $g'''(x_0) < 0$, a contradiction. □

Lemma 5 *There exists a real number $l_0 < 0$ such that if $0 > \beta > l_0\gamma^{\frac{3}{4}}$ then, $g'(\xi, \beta, \gamma)$ has exactly two zeros $r_1 < r_2$ with $g(r_1) < 0$ and $g(r_2) > 0$. Also, $g'', g''' > 0$ at r_1 and $g'', g''' < 0$ at r_2.*

Proof. Consider the initial value problem (25)-(26) with $l = 0$. Then, $g'''(0) = 1$, which implies $g', g'' > 0$ initially. Similar to the case (2) above, there is a unique zero point r_0 of $g'(\xi, 1, 0)$. Consider a compact set $[0, 2r_0]$. From the proof of case (2), $g'(2r_0), g''(2r_0) < 0$. It follows that for given $\epsilon > 0$, there is a $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0), |g^{(k)}(\xi, 0, 1) - g^{(k)}(\xi, -\delta, 1)| < \epsilon$ on $[0, 2r_0]$ for $k = 0, 1, 2, 3$. Choosing ϵ small enough, we can make $g(\frac{r_0}{2}, -\delta, 1) > 0$, $g'(\frac{r_0}{2}, -\delta, 1) > 0$, $g'(2r_0, -\delta, 1) < 0$ and $g''(2r_0, -\delta, 1) < 0$. Noting that for this function g, the initial condition $g''(0) < 0$, hence g, $g', g'' < 0$ initially. Thus, g' must have a first zero point $r_1 \in (0, \frac{r_0}{2})$ with $g(r_1) < 0$ and a second zero point $r_2 \in (\frac{r_0}{2}, 2r_0)$. Since the first zero point of g' is the first relative minimum and the second zero point of g' the relative maximum, $g''(r_1) \geq 0$ and $g''(r_2) \leq 0$. Following the argument similar to above, there is no more zero point of g' for $\xi > r_2$. It is obvious that $g(r_1) < 0$. Since g''' decreases and changes signs in (r_1, r_2), we see $g''' < 0$ for all $\dot{\xi} \geq r_2$. It can be seen that $g(r_2) > 0$, for otherwise, $g'''(r_2) \geq 1$ from (25), a contradiction. □

Now we show, by contradiction, that for any $\delta > 0$, $g'(\xi, -\delta, 1)$ has either no zero or two zeros. If not, then let r_1 denote the first zero of g'. Thus, $g(r_1) < 0$. Since $g, g', g'' < 0$ initially, we first see $g''(r_1) \geq 0$. Next we claim the equal sign cannot hold, otherwise, from (25), that $g' = g'' = 0$ implies $g'''(r_1) = 1$, which is impossible because of lemma 1. With $g(r_1) < 0, g'(r_1) = 0$, and $g''(r_1) > 0$, one can see that after r_1 the function g begins increasing. Now the question is: can the function g keep increasing and negative for all ξ? If so, then there would be a zero point of g'' with $g < 0$ and $g''' \leq 0$, which implies $g^{iv} \geq 0$. But, this is impossible by lemma 1.

Define

$$A = \{b < 0 \mid g'(\xi, \beta, 1) \text{ has two zeros for all } \ \beta \in (b, 0)\}.$$

It is seen from lemma 5 that the set A is open and not empty. It follows from lemma 4 that the set A is bounded below. Let

$$\beta_1 = \inf A.$$

Then, $\beta_1 < 0$, $\beta_1 \notin A$, and so $g'(\xi, \beta_1, 1)$ has no zero. Also, $g''(\xi, \beta_1, 1) < 0$ for all $\xi > 0$, otherwise, if $g'' = 0$ at an x, then $g'''(x) = g^{iv}(x)/g(x) > 0$. It then follows that $g'', g''' > 0$ as long as $g' < 0$ for otherwise, at the first zero point of g'' after x, if any, $g''' < 0$, which made $g^{iv} > 0$, a contradiction. It turns out that g' becomes zero because of the concavity. From the above argument, g' would have a second zero. This contradicts the definition of β_1.

Lemma 6 *Suppose that $\beta \in A$ and that $\beta \to \beta_1$. Then* $\lim g(r_1) = -\infty$*, and* $\liminf r_2 g(r_2) = \infty$*.*

Proof. Since $g''(\xi, \beta_1, 1) < 0$ for all $\xi > 0$, we see $g(\xi, \beta_1, 1) \to -\infty$ as $\xi \to \infty$ and $g'(\xi, \beta_1, 1) < -\sigma$ for some $\sigma > 0$ and for all $\xi > x_0$ where x_0 is a constant. Then, for any $M_0 > 0$, there is an interval $(\beta_1, b_0) \subset A$ such that for $\beta \in (\beta_1, b_0)$, $g'(M_0, \beta, 1) < 0$ and $g(M_0, \beta, 1) < -\frac{\sigma}{2}M_0$. This implies that for $\beta \in (\beta_1, b_0)$, $r_1 > M_0$ where r_1 is the first zero of g', and $g(r_1) < -\frac{\sigma}{2}M_0$. This proves the first conclusion of the lemma and $r_1 \to \infty$ as $\beta \to \beta_1^+$. To prove the second conclusion, we only need to prove that $\liminf g(r_2) > 0$. Suppose not. Then, for any small $\epsilon > 0$ there would be infinitely many $\beta \in A$ such that the corresponding $g(r_2) < \epsilon$. Consider the second zero point x_2 of g'' at which $g''' < 0$ and $g > 0$ (it is impossible to have $g'' = 0, g''' < 0$, and $g \leq 0$ simultaneously). From (25), $g'(x_2) > 1$. Let x_2 be the point at which $g' = 1$. Then, $\epsilon > g(r_2) > g(x_1)$. Since $g'' < 0$ after x_2 and $g''(x_1) = 1$,we have $g(\xi) > \xi - x_2 + g(x_2)$ for $\xi \in (x_2, x_1)$ since $g' \geq 1$ over that interval. This implies $x_1 - x_2 < \epsilon$. Hence, by the mean value theorem and lemma 1,

$g'''(x_1) \le \frac{0-g''(x_1)}{x_2-x_1} < \frac{g''(x_1)}{\epsilon}$. On the other hand, from (9), $g''' = gg''$ at x_1, and so $g(x_1)g''(x_1) < \frac{g''(x_1)}{\epsilon}$, which implies $g(x_1) > \frac{1}{\epsilon}$, a contradiction. □

Lemma 6, in fact, implies that the solutions of BVP (1) exist for sufficiently large $|R|$. Furthermore, we can show that as $\beta \to 0^-$, $g(r_1) \to 0$ and $g(r_2) \to y_0 = g(r, 0, 1)$. It then follows, from the fact that continuous functions map connected set into connected sets, that the solution of BVP (1) exists for all $R \le 0$, and for all $R > R_3$ where $R_3 = \inf R(\beta, 1) \le R(0, 1)$.

Now the picture is clear, the solutions exist for all $R \le 0$. As R increases, the solutions of case (1) appear till R_1 (defined as above) while the solutions of case (2) appear for $R \in I$ where I is given in the proof of case (2). Then, the solutions of case (3) appear for all $R > R_3$. It can be seen that $R_3, R_1 \in \overline{I_2}$, the closure of I_2, since

$$R_1 = \sup_{s \le 0} R(1, s) \ge R(1, 0) \ge \inf_{s > 0} R(1, s)$$

which implies $R_1 \in \overline{I_2}$. By the corollary,

$$\sup_{s \in (0,\infty)} R(1, s) = \sup_{t \in (0,\infty)} R(t, 1)$$

which implies $R_{3 \in \overline{I_2}}$ since $\inf R(\beta, 1) < \sup R(\beta, 1)$. This shows that there exists at least one solution for each $R \in (-\infty, \infty)$.

Next we prove the uniqueness of the solution of BVP (1) for $R < 0$, which is a consequence of the following lemma.

Lemma 7 *Suppose that $r_1 = r_1(\beta)$ denotes the first zero point of $g'(\xi, \beta, 1)$ for $\beta < 0$. Then both r_1 and $g(r_1)$ are increasing function of β.*

Proof. Let $\beta_1 < \beta_2 < 0$ be given such that the corresponding solutions g_1 and g_2 have zero points of their first derivatives. Since $g_1'''(0) = g_2'''(0) = 1$, $(g_2 - g_1)'' > 0$, and $(g_2 - g_1)' > 0$ for sufficiently small $\xi > 0$. This shows that $g_2^2 < g_1^2$, and $g_2 > g_1$ initially. Using the integral form of g^{iv} given by (19), we obtain that $g_2^{iv} > g_1^{iv}$ initially. Hence, $g_2''' > g_1'''$ initially. We claim that $g_2^{(k)} > g_1^{(k)}$ for $k = 0, 1, 2, 3$ as long as $g_1'' \le 0$ and $g_2' \le 0$, which implies that g_2'' becomes zero before g_1'' does. To see this, we subtract the equation $g_1^{iv} = g_1 g_1''' - g_1' g_1''$ from $g_2^{iv} = g_2 g_2''' - g_2' g_2''$ and obtain

$$(g_2 - g_1)^{iv} = g_2(g_2 - g_1)''' + (g_2 - g_1)g_1''' - g_2'(g_2 - g_1)'' - g_1''(g_2 - g_1)'\,.$$

Let $\psi = g_2 - g_1$. The last equation can be written in the form

$$\psi^{iv} = g_2\psi''' - g_2'\psi'' - g_1''\psi' + g_1'''\psi\,. \tag{27}$$

Also, all of ψ, ψ', ψ'', ψ''', and ψ^{iv} are positive in $(0, \delta)$ where δ is sufficiently small. Note that $g_1''' > 0$ for $\xi < r_1(\beta_1)$. We then see from (27) that $\psi''' > 0$ as long as $g_1'' \leq 0$ and $g_2' \leq 0$ since at the first zero x of ψ''', if any, $\psi^{iv}(x) > 0$, a contradiction. This shows that $g_2'' = 0$ before g_1'' does.

Now there are two possibilities: (1). $g_2' = 0$ before or on $g_1'' = 0$, which implies the lemma, and (2). $g_2' = 0$ at a point with $g_1'' > 0$. In the latter case, we begin with the point x_1 where $g_2'' = 0$. Of course, $g_1''(x_1) < 0$, and $\psi^{(k)}(x_1) > 0$ for $k = 0, 1, 2, 3$. Since $g_2'' > 0$ for $\xi > x_1$, ψ'', hence ψ' and ψ are positive as long as $g_1'' \leq 0$. Let x_2 be the first zero point of g_1''. Then, at $\xi = x_2$, ψ, ψ', and ψ'' are positive, and $g_2''(x_2) > 0$, $g_1'(x_2) < 0$, and $g_1'(x_2) < 0$. Consider the third order equation of (27)

$$\psi''' = g_2\psi'' - (g_2' + g_1')\psi' + g_1''\psi \,,$$

with the initial condition $\psi^{(k)}(x_2) > 0$ for $k = 0, 1, 2$. At the first zero x_3 of ψ, if any, $\psi''' > 0$ from the last equation.. This shows that after x_2, $0 > g_2' > g_1'$ before one of them reaches the zero point. Consequently, g_2' becomes zero before g_1' does. Therefore, $r_2 < r_1$, and $0 > g_2(r_1(\beta_2)) > g_1(r_1(\beta))$. The proof of theorem 1 is completed. □

Remark. Lemma 7 can be generalized and then applied to prove the uniqueness of the solution for the original Berman problem with injection. This will be presented in another paper.

5 Proof of theorem 2

BVP (2) is related the initial value problem with the initial conditions

$$g(0) = g''(0) = 0, g'(0) = \alpha, g'''(0) = \gamma \,.$$

Any symmetric solution of BVP (2) corresponds to a solution $g(\xi, \alpha, \gamma)$ of the initial value problem which satisfies

$$g(b) = 0, g'(b) = -\frac{R}{b^2} \,.$$

Then, there are only three possible cases to be considered: (I) $\alpha > 0$ and $\gamma \leq 0$, (II) $\alpha \geq 0$ and $\gamma \geq 0$ with $(\alpha, \gamma) \neq (0, 0)$, and (III) $\alpha < 0$ and $\gamma > 0$.

Case (I) Applying lemma 1, one sees that $g' = 0$ at an $r > 0$ because the concavity and monotonicity of g', and then g must cross the ξ-axis at a $b > 0$ with $g'(b) < 0$ and $g''(b) < 0$. This is the only zero of g in $(0,\infty)$.

Case (II) We apply the equation (18) and see that $g^v < 0$ as long as $g > 0$, which implies g''' is concave down and decreasing in $(0, \infty)$. Thus, g''' must crass ξ-axis, hence g'' becomes concave down and decreasing. Continuing the similar argument shows that there exists a unique b at which $g(b) = 0$ with $g'(b)$, $g''(b) < 0$.

Case (III) We can prove that g has either none or two positive zero points. Suppose that there is a zero point of g. Let b_1 be the first such point. Then, g' and g'' must be of the same sign by lemma 1. It turns out that $g'(b_1), g''(b_1) > 0$. Since the equation is autonomous, an argument similar to that of case (II) shows the existence of the second positive zero point of g. We now define

$$B = \{\alpha_0 < 0 |\ g(\xi, \alpha, 1) \ \text{ has two positive zeros for all } \alpha \in (\alpha_0, 0)\}.$$

It can be easily seen that the set B nonempty sine B consists of at least some $\alpha < 0$ with small absolute values. B is also an open set because of the transversality of the function when it crosses the ξ-axis. By the definition, the set B is connected. Let $\inf B = \tilde{\alpha}$. Then, for any $\alpha \in (\tilde{\alpha}, 0)$, $g(\xi, \alpha, 1)$ has exact two zero pints in (0,∞). Note that the first zero b_1 of g provides a negative R while the second zero b_2 of g gives a positive R.

Next, we determine the feasible set for R considering all possible pairs (α, γ). In the case (I), the solution is concave down. As $\alpha \to 0^+$, $R(\alpha, -1) \to 0$ since both b and $g(b)$ approach zero. If $\alpha \to \infty$, we apply corollary 1, $R(\alpha, -1) = R(1, -\frac{1}{\alpha^2})$. Since $g(\xi, 1, 0)$ is the linear function $g = \xi$, we can choose α sufficiently large such that the positive zero point $b > M$ for any give $M > 0$. To show that $R \to \infty$, as above, we need to prove $g'(b)$ is bounded above by a nonzero constant. An integration of (9), with the initial condition $g'(0) = \alpha$, $g''(0) = 0$, $g'''(0) = -1$ where $\alpha \gg 1$, produces

$$g''' = gg'' - (g')^2 + \alpha^2 - 1. \tag{28}$$

At $\xi = b$, $g''' < 0$. This implies $g'(b) > \sqrt{\alpha^2 - 1}$, which shows that $g'(b) \to \infty$. This shows that the type I solutions are valid for all $R \in (0, \infty)$.

In the case (II), similar to the case (I), $R \to \infty$ as $\alpha \to \infty$. However, as $\alpha \to 0^+$, $R \to R(0, 1) > 0$. let $R_2 = \inf R(\alpha, 1)$ for $\alpha \in (0, \infty)$. The type II solutions exist for all $R > R_2$.

In the case (III), we consider the processes $\alpha \to 0$ and $\alpha \to \tilde{\alpha}$ which is give above. For positive R, we see the feasible set is $(\inf b_2^2|g'(b_2)|, \sup b_2^2|g'(b_2)|) = (R_3, \infty)$. Here, we applied corollary 1 again, $R(\alpha, 1) = R(-1, \frac{1}{\alpha^2})$. The argument for having $\sup b_2^2|g'(b_2)| = \infty$ is b_2 can be sufficiently large and $g'(b)$ is bonded below as $\alpha \to \tilde{\alpha}$. Why is b_2 can be sufficiently large? Since either $\tilde{\alpha} = -\infty$ or $g(\xi, \tilde{\alpha}, 1)$ has no zero point at all. In the former case, we consider $g(\xi, -1, 0) = -\xi$; in the latter case, $g(\xi, \tilde{\alpha}, 1)$ is a decreasing function defined on $(0, \infty)$. In either case, we can choose and α such that the corresponding b_2 is sufficiently large (so is b_1). For the first zero b_1, we see that as $\alpha \to 0^-$, $g(\xi, \alpha, 1)$ and $g(\xi, 0, 1)$ can be arbitrarily close on a closed interval $[0, T]$ where T is a positive constant. This shows as $\alpha \to 0^-$, $R \to 0^-$. As $\alpha \to \tilde{\alpha}$, $b_1 \to \infty$, and $g'(b)$ must be bounded below, otherwise, if $g'(b_1) \to 0$ (or a sequence), then $g'''(b_1) \to \tilde{\alpha}^2 + 1 > 1$, which is impossible, since g''' is a decreasing function. The proof of theorem 2 is complete.

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