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On Variational Inequalities Associated with the Navier-Stokes Equation: Some Bifurcation Problems *

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1 Introduction

This paper is devoted to a study of bifurcation problems for the steady states of Navier-Stokes problems where several types of constraints are imposed. In an abstract setting this leads to the study of bifurcation problems for variational inequalities. We show how the tools developed in [4] may be employed to analyze these problems. Some of the problems considered here, have already been analyzed in [4] for nonlinear versions of the Stokes equation (see [4, pp. 72-77]); but we expand considerably upon the development there. We first give the statement of the problem and then provide several constraint situations where bifurcation results may be obtained.

The results discussed here were first presented in a lecture given during May 1997 by the second author at the third *Mississippi State Conference on Differential Equations and Computational Simulations* at Mississippi State University.

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary. Let

$$V = \{v \in [H_0^1(\Omega)]^3 : \operatorname{div} v = 0 \text{ a.e. in } \Omega\}.$$

Then V is a subspace of the Hilbert space $[H_0^1(\Omega)]^3$ with the restricted norm and scalar product. For $u \in V$, we denote by Du the 3×3 matrix of distributional derivatives,

$$Du = [\partial_i u_j]_{1 \le i,j \le 3}\,.$$

For $u, v \in V$, let

$$Du : Dv = \sum_{i,j=1}^{3} \partial_i u_j \partial_i v_j\,.$$

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Let $b : V \times V \times V \to \mathbb{R}$ be the trilinear form

$$b(u,v,w) = \int_\Omega \sum_{i,j=1}^{3} u_i(\partial_i v_j) w_j = \int_\Omega u^T (Du) w\,,$$

(for all $u, v, w \in [H_0^1(\Omega)]^3$). Notice that $b(u,v,w) = -b(v,u,w) = -b(u,w,v)$, for all u, v, w. Let $g : \Omega \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$, with

$$(x, u, \lambda) \mapsto g(x, u, \lambda)$$

being a mapping that satisfies the Carathéodory conditions for each component g_i, $i = 1, 2, 3$. Furthermore, assume that g is differentiable with respect to u, and that $g, D_u g$ satisfy the usual growth conditions:

$$\begin{aligned} |g(x,u,\lambda)| &\le A(\lambda) + B(\lambda)|u|^{s-1} \\ |D_u g(x,u,\lambda)| &\le A(\lambda) + B(\lambda)|u|^{s-2}, \end{aligned} \tag{1}$$

for a.e. $x \in \Omega$, all $u, \lambda \in \mathbb{R}$, with $A, B \in L^\infty_{loc}(\mathbb{R})$, $1 < s < 6 = 2^*$ (the Sobolev conjugate of 2). (Note that the above correct several misprints on page 76 of [4].) We shall assume further that

$$g(x, 0, \lambda) = 0 \ \text{ for a.e. } x \in \Omega,\ \forall \lambda \in \mathbb{R}.$$

Also assume that

$$j : V \to [0, \infty],\ j(0) = 0,$$

is a convex, lower semicontinuous functional.

We consider the variational inequality below, where $\nu > 0$ is the so-called viscosity constant.

$$\nu \int_\Omega Du : D(v-u) + b(u,u,v-u) + j(v) - j(u) \ge \int_\Omega g(x,u,\lambda) \cdot (v-u), \tag{2}$$

for all $v \in V, u \in V$. i.e., we seek $u \in V$ such that the inequality in (2) holds for all $v \in V$. We note that, by hypotheses, $u = 0$ is a solution. We shall establish conditions, for various choices of j, in order that (2) have nontrivial solutions for certain values of λ. More specifically, we establish the existence of connected sets $\mathcal{C} \subset V$ of solutions of (2) such that for $(u, \lambda) \in \mathcal{C}$, $u \neq 0$ and $\overline{\mathcal{C}} \cap \{0\} \times \mathbb{R} \neq \emptyset$, i.e., $\mathcal{C}$ bifurcates from the trivial solution set $\{0\} \times \mathbb{R}$.

Many interesting cases (we shall present several of these) are covered by choosing j to be the indicator function of some closed convex set $K \ni 0$, i.e.,

$$j(u) = I_K(u) = \begin{cases} 0, & u \in K \\ \infty, & u \notin K, \end{cases}$$

in which case (2) is equivalent to the variational inequality

$$\nu \int_\Omega Du : D(v-u) + b(u,u,v-u) - \int_\Omega g(\cdot, u, \lambda) \cdot (v-u) \ge 0 \tag{3}$$

for all $v \in K$, $u \in K$. Also, in the case

$$j : V \to [0, \infty)$$

is C^1, by choosing $v = u + tw$, $t > 0$ in (2), dividing by t and letting $t \to 0^+$, we obtain the inequality

$$\nu \int_\Omega Du : Dw + b(u,u,w) + \langle j'(u), w\rangle \geq \int_\Omega g(x,u,\lambda)\cdot w,\ \forall w \in V,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing of V^* and V. Hence (since w may be replaced by $-w$) we obtain the equation

$$\nu \int_\Omega Du : Dw + b(u,u,w) + \langle j'(u), w\rangle = \int_\Omega g(x,u,\lambda)\cdot w\,, \tag{4}$$

for all $w \in V$, $u \in V$. Which, when $j = 0$, is the usual variational form of the Navier-Stokes equation (cf. [5], [9], or [10]). (Note that, if $j : V \to [0, \infty)$ is C^1 and convex, then (2) and (4) are equivalent problems.) Now, we define the operators $A : V \to V^*$ and $B : V \times \mathbb{R} \to V^*$ as follows

$$\langle A(u), v\rangle = \nu \textstyle\int_\Omega Du : Dv\,,$$
$$\langle B(u,\lambda), v\rangle = \textstyle\int_\Omega g(x,u,\lambda)\cdot v - b(u,u,v),\ u, v \in V\,.$$

Then (2) may be rewritten as

$$\langle A(u) - B(u,\lambda), v - u\rangle + j(v) - j(u) \geq 0\,, \tag{5}$$

for all $v \in V$, $u \in V$.

In order to obtain information about possible bifurcations from the trivial solution of (5), we need to establish some properties of the operator B and study (5) in a neighborhood of the trivial solutions. This is carried out in the next section.

2 Preliminaries

Lemma 1 *The operator A is linear, continuous and coercive, in the sense that there exists a constant $\alpha > 0$ such that*

$$\langle A(v), v\rangle \geq \alpha \|v\|^2,\ \forall v \in V.$$

Proof. This statement follows easily from the definition of A, the Poincaré's inequality for $H_0^1(\Omega)$, and the conventional norm of $[H_0^1(\Omega)]^3$ as derived from the norm of $H_0^1(\Omega)$.

Now we define the operator $f : V \times \mathbb{R} \to V^*$, by

$$\begin{aligned} \langle f(u,\lambda), v\rangle &= \int_\Omega \sum_{i,j=1}^3 D_{u_i} g_j(x,0,\lambda) u_i(x) v_j(x) \\ &= \int_\Omega [v(x)]^T D_u g(x,0,\lambda) u(x)\,. \end{aligned} \tag{6}$$

Lemma 2 *The operators B and f are completely continuous, and f is the linearization of B at 0 (with $p = 2$ in the sense of (A7), Chapter 6, [4]).*

Proof. It is clear that if f is the linearization of B in the sense of (A7), [4]. Then f is the (partial) Fréchet derivative of B. Hence, once we have shown that B is completely continuous, it will follow that f is completely continuous by a result of Krasnosel'skii ([3]).

Let $B_0 : V \times \mathbb{R} \to V^*$ be given by

$$\langle B_0(u,\lambda), v\rangle = \int_\Omega g(\cdot, u, \lambda)\cdot v, \ \forall u, v \in V, \ \lambda \in \mathbb{R}.$$

Then the continuity of B_0 follows ¿From the continuity of the Niemitskii operator and the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$, for $q < 6 = 2^*$. Now, we check that f is the linearization of B_0 at 0 in the sense of (A7) of [4]. In fact, let $\{u_n\}$ be an arbitrary sequence in V converging weakly to a function u, $u_n \rightharpoonup u$, and let $\{\sigma_n\} \subset \mathbb{R}$ with $\sigma_n > 0$, for all n, $\sigma_n \to 0$, and $\{\lambda_n\} \subset \mathbb{R}$, $\lambda_n \to \lambda$.

We can estimate $\left|\langle \frac{1}{\sigma_n} B_0(\sigma_n u_n, \lambda_n) - f(u,\lambda), v\rangle\right|$ (for $v \in V$, $\|v\| = 1$), using Hölder's inequality to obtain

$$\begin{aligned} &\left\|\frac{1}{\sigma_n} B_0(\sigma_n u_n, \lambda_n) - f(u,\lambda)\right\|_* \\ &\leq \ C\left[\int_\Omega \left|\frac{1}{\sigma_n} g(x, \sigma_n u_n, \lambda_n) - D_u g(x, 0, u, \lambda)u\right|^{s/(s-1)}\right]^{(s-1)/s}. \end{aligned}$$

Since $u_n \rightharpoonup u$ in $[H_0^1(\Omega)]^3$, $u_n \to u$ in $[L^s(\Omega)]^3$, $1 \leq s \leq 6$, and hence, by passing to a subsequence if needed,

$$u_n \to u \ \text{ a.e. in } \ \Omega \quad \text{and} \quad |u_n| \leq h\,,$$

with $h \in L^s(\Omega)$ (cf. [1]). Hence,

$$\frac{1}{\sigma_n} g(x, \sigma_n u_n, \lambda_n) \to D_u g(x, 0, u, \lambda)u \ \text{ a.e. in } \ \Omega,$$

because of the Carathéodory conditions and differentiability assumptions on g. Further, from the mean value theorem,

$$\begin{aligned} |g(x, \sigma_n u_n, \lambda_n)| &= |g(x, \sigma_n u_n, \lambda_n) - g(x, 0, \lambda_n)| \\ &\leq \sup_{|v| \leq |u_n|} |D_u g(x, v, \lambda_n)|\,|\sigma_n u_n| \\ &\leq \sup_{|v| \leq |u_n|} [A(\lambda) + B(\lambda)|v|^{s-2}]\sigma_n |u_n| \\ &\leq [A(\lambda) + B(\lambda)|u_n|^{s-2}]\sigma_n |u_n| \\ &\leq \sigma_n [A(\lambda) + B(\lambda)|h|^{s-2}]|h|\,. \end{aligned}$$

Hence,

$$\left|\frac{1}{\sigma_n}g(x,\sigma_n u_n,\lambda_n)\right| \le [A(\lambda)+B(\lambda)|h|^{s-2}]|h| (\in L^{s/(s-1)}(\Omega)).$$

By the dominated convergence theorem,

$$\int_\Omega \left|\frac{1}{\sigma_n}g(x,\sigma_n u_n,\lambda_n) - D_u g(x,0,u,\lambda)u\right|^{s/(s-1)} \to 0,$$

which implies

$$\frac{1}{\sigma_n}B_0(\sigma_n u_n,\lambda_n) \to f(u,\lambda) \text{ in } V^*.$$

Let the mapping $Q : V \to V^*$ be defined by

$$\langle Q(u), v\rangle = b(u,u,v),\ \forall u,v \in V.$$

Then, since the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ is compact, the mapping Q will be completely continuous (see e.g. Chapter 72, Lemma 72.5, [10]). Hence $B(u,\lambda) = B_0(u,\lambda) - Q(u)$ is completely continuous. If now $u_n \rightharpoonup u$ in V and $\sigma_n \to 0^+$, then

$$\langle\frac{1}{\sigma_n}Q(\sigma_n u_n), v\rangle = \frac{1}{\sigma_n}b(\sigma_n u_n,\sigma_n u_n,v) = \sigma_n b(u_n,u_n,v),$$

or

$$\frac{1}{\sigma_n}Q(\sigma_n u_n) = \sigma_n Q(u_n)\,.$$

Since $Q(u_n) \to Q(u)$ in V^*, it follows that $\frac{1}{\sigma_n}Q(\sigma_n u_n) \to 0$ in V^*. Hence it follows that

$$\frac{1}{\sigma_n}B(\sigma_n u_n,\lambda_n) \to f(u,\lambda) \text{ in } V^*,$$

whenever $u_n \rightharpoonup u$ in V, $\sigma_n \to 0^+$, $\lambda_n \to \lambda$. This shows that f is the linearization of B at 0. ◇

We next assume that there exists a convex, lower semicontinuous functional $J : V \to [0,\infty]$ having the property below. (see (A8), pp. 117-120, [4])

If $v_n \rightharpoonup v$ in V and $\sigma_n \to 0^+$, then

$$J(v) \le \liminf \frac{1}{\sigma_n^2}j(\sigma_n v_n),$$

and if $v \in V$ and $\sigma_n \to 0^+$, then there exists a sequence $\{v_n\} \subset V$ such that

$$v_n \to v \text{ and } \frac{1}{\sigma_n^2}j(\sigma_n v_n) \to J(v)\,.$$

We note that J, if it exists, is uniquely determined (see [4]).

To (5) we assign the variational inequality

$$\langle A(u) - f(u,\lambda), v-u\rangle + J(v) - J(u) \ge 0,\ \forall v \in V\,,\ u \in V\,. \tag{7}$$

The proof of the following lemma is straightforward.

Lemma 3 *(i) For each* $t > 0$ *and* $u \in V$*,*

$$f(tu, \lambda) = tf(u, \lambda), \ J(tu) = t^2 J(u).$$

(ii) If u *is a solution of (7), then so is* tu *for all* $t > 0$.

As an immediate corollary, we obtain the following necessary conditions for bifurcation from the trivial solution (see [4]).

Corollary 1 *Assume that* $(0, \lambda_0)$ *is a bifurcation point from the trivial solution for (5). Then there exists* $u_0 \neq 0$ *such that for all* $t > 0$*,* tu_0 *solves (7) with* $\lambda = \lambda_0$.

3 A general result in global bifurcation

¿From classical results ([4], [5]), it follows that for each $f \in V^*$, there exists a unique solution, $u = P_{A,j}(f)$, to the variational inequality

$$\langle A(u) - f, v - u \rangle + j(v) - j(u) \geq 0, \ \forall v \in V, \ u \in V\,,$$

and a unique solution, $u = P_{A,J}(f)$, to the variational inequality

$$\langle A(u) - f, v - u \rangle + J(v) - J(u) \geq 0, \ \forall v \in V \quad \in V\,.$$

Furthermore, the mappings $P_{A,j}, P_{A,J} : V^* \to V$ are continuous and problems (5), respectively (7), are equivalent to the fixed point equations

$$u - P_{A,j} B(u, \lambda) = 0, \tag{8}$$

respectively,

$$u - P_{A,J} f(u, \lambda) = 0. \tag{9}$$

Of course, both equations have the trivial solution. A necessary condition for $(0, \lambda_0)$ to be a bifurcation point of (8) is that (9), for $\lambda = \lambda_0$, have a nontrivial solution, hence a ray of such.

We have the following bifurcation theorem (see[4]).

Theorem 1 *Assume that* a_1 *and* a_2 $(a_1 < a_2)$ *are such that (9) has only the trivial solution for* $\lambda = a_1$ *and* $\lambda = a_2$*. Further assume that*

$$\deg(I - P_{A,J}[f(\cdot, a_1)], B_r(0), 0) \neq \deg(I - P_{A,J}[f(\cdot, a_2)], B_r(0), 0),$$

where $\deg(\cdot, \cdot, \cdot)$ *denotes the Leray-Schauder degree. Then if*

$$\mathcal{S} = \overline{\{(u, \lambda) : (u, \lambda) \text{ is a solution of (8) with } u \neq 0\}} \cup (\{0\} \times [a_1, a_2])\,,$$

and $\mathcal{C}$ *is the connected component of* S *containing* $\{0\} \times [a_1, a_2]$*, it follows that either*

(i) $\mathcal{C}$ *is unbounded in* $V \times \mathbb{R}$*, or*

(ii) $\mathcal{C} \cap (\{0\} \times (\mathbb{R} \setminus [a_1, a_2])) \neq \emptyset$.

In other words, we have the global Krasnosel'skii-Rabinowitz bifurcation alternative ([8]) for bifurcation from the trivial solution segment $\{0\} \times [a_1, a_2]$.

We shall apply this theorem in several situations, and note that the application will be most straightforward if the problem (7) yields an equation, e.g. that J is a C^1 functional or $J = I_K$, where K is a closed subspace of V.

4 Examples

Velocity constraints

We first consider some situations where there are imposed constraints on the velocity u, i.e., there exists a closed convex set K in V such that $j = I_K$. Several choices of K will be considered.

Let us suppose that K is given by

$$K = \{w \in V : |w(x)| \leq c \ \text{ for a.e } \ x \in \Omega\}. \tag{10}$$

In order to apply Theorem 1, we must compute the functional J. To this end, we note that if $w \in V \cap [C_0^\infty(\Omega)]^3$, then $tw \in K$, for $t > 0$, sufficiently small. Hence, as $V \cap [C_0^\infty(\Omega)]^3$ is dense in V, we obtain that

$$V = \overline{\bigcup_{t>0} tK} = K_0,$$

which is the so-called support cone of K, and we deduce that $J = I_V$, i.e., $J = 0$. Thus problem (7) becomes

$$\nu \int_\Omega Du : Dv - \int_\Omega v^T D_u g(\cdot, 0, \lambda) u = 0, \ \forall v \in V \ \ u \in V\,. \tag{11}$$

If it is the case that

$$D_u g(x, 0, \lambda) = \lambda k(x), \tag{12}$$

where $k = [k_{ij}]_{i,j=1,2,3}$ is a matrix in $[L^\infty(\omega)]^9$, then (11) is the usual eigenvalue problem for the Stokes equation

$$\nu \int_\Omega Du : Dv - \lambda \int_\Omega v^T k u = 0, \ \forall v \in V\,, \ u \in V\,. \tag{13}$$

It follows that all eigenvalues of odd multiplicity of (13) yield global bifurcation branches of (11).

If it is the case that the flow is restricted for some components of the velocity field on a sub-domain $\Omega_0 \subset \Omega$, e.g.

$$K = \{w \in V : w_1(x) \geq -c, \ w_2(x) \geq -d \text{ for a.e. } x \in \Omega_0\},$$

then $J = I_{K_0}$, where the support cone

$$K_0 = \{w \in V : w_1 \geq 0, w_2 \geq 0 \text{ a.e. in } \Omega_0\}$$

then (7) becomes

$$\nu \int_\Omega Du : D(v-u) - \lambda \int_\Omega (v-u)^T ku \geq 0, \ \forall v \in K_0\,, \ u \in K_0, \qquad (14)$$

and the bifurcation values are contained in the "spectrum" of (14), i.e., the set of those $\lambda \in \mathbb{R}$, for which (14) has a nontrivial solution.

Other interesting cases where the support cone K_0 is the whole space V (and hence $J = 0$) are given by

$$K = \{u \in V : |(\nabla \times u)(x)| \leq c \text{ for a.e. } x \in \Omega\},$$

where $c > 0$ is given, or a constraint of a nonlocal nature, e.g. if S be a compact oriented smooth surface in Ω and a limitation is imposed on the magnitude of the flux of the flow across S, e.g.,

$$K = \big\{u \in V : \big| \int_S u \cdot \nu dS \big| \leq c \big\},$$

where ν is the unit normal vector field to S and c is a nonnegative constant.

Visco-plastic Bingham fluids

We consider here the variational inequality modeling the equilibrium of a steady state rigid visco-plastic Bingham fluid. This viscous-rigid fluid is a generalization of the usual Newtonian fluid, whose equilibrium is represented by the Navier-Stokes equations.

Here we consider the convex functional $j : V \to [0, \infty)$ given by:

$$j(u) = \int_\Omega \mu(x)|Du| = \int_\Omega \mu(x) \left[\sum (\partial_i u_j)^2\right]^{1/2}, \qquad (15)$$

or

$$j(u) = \int_\Omega \mu(x) \left[\sum \epsilon_{ij}^2(u)\right]^{1/2}, \qquad (16)$$

where $\epsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. Here, $\mu \in L^\infty(\Omega)$, $\mu \geq 0$, a.e. on Ω, $\not\equiv 0$, represents the yield limit between the rigidity and viscosity of the fluid flow (cf. [2], [6]). We shall consider the case that j is given by (15), the other case in (16) being similar in nature. Let $\Omega_0 = \{x \in \Omega : \mu(x) = 0\}$ and

$$W = \{u \in V : Du = 0 \ \text{ a.e. on } \ \Omega \backslash \Omega_0\}. \qquad (17)$$

It is clear from the definition of j that it is a nonnegative convex and lower semicontinuous functional on V and $j(0) = 0$. We have the following lemma:

Lemma 4 *Given j as above, the functional J exists and is given by*

$$J = I_W,$$

where W is given by (17).

Proof. Let $\{u_n\}$ be a sequence with $u_n \rightharpoonup u$ and let $\{\sigma_n\} \subset \mathbb{R}^+$ a sequence with $\sigma_n \to 0^+$. We first show that

$$\liminf \frac{j(\sigma_n u_n)}{\sigma_n^2} \geq I_W(u). \tag{18}$$

If $u \in W$ then (18) obvious holds, since $j \geq 0$. If $u \notin W$, then $\mu(x)|Du| > 0$ on a subset of positive measure, hence,

$$j(u) = \int_\Omega \mu(x)|Du|\,dx > 0.$$

We have

$$\frac{1}{\sigma_n} j(\sigma_n u_n) = \int_\Omega \mu(x)|Du_n|,$$

since j is homogeneous of degree 1. Since j is weakly lower semicontinuous, $\liminf j(u_n) \geq j(u) > 0$. Hence

$$\begin{aligned}\liminf \frac{j(\sigma_n u_n)}{\sigma_n^2} &= \liminf \frac{j(u_n)}{\sigma_n} \\ &\geq \lim \frac{1}{\sigma_n} \cdot \liminf j(u_n) \;=\; \infty \;=\; I_W(u).\end{aligned}$$

Next, let $u \in V$, $\sigma_n \to 0^+$ and choose $u_n = u$, $\forall n$, then

$$\begin{aligned}\lim \frac{j(\sigma_n u_n)}{\sigma_n^2} &= \begin{cases} 0, & \text{if } \; j(u) = 0 \\ \infty, & \text{if } \; j(u) > 0 \end{cases} \\ &= \begin{cases} 0, & \text{if } \; u \in W \\ \infty, & \text{if } \; u \notin W \end{cases} \\ &= I_W(u),\end{aligned}$$

hence $J = I_W$, by earlier remarks.

Since W is a subspace, inequality (7) becomes

$$\nu \int_\Omega Du : Dv - \lambda \int_\Omega v^T D_u g(\cdot, 0, \lambda) u = 0, \; \forall v \in W\,,\; u \in W. \tag{19}$$

By Theorem 1, we may conclude that eigenvalues of odd multiplicity of (19) will yield bifurcation points for global bifurcation of (2).

An extension of the above is the case that j is given by

$$j(u) = \int_\Omega \mu(x)|Du|^\gamma\,dx \tag{20}$$

with $\gamma \geq 1$. Again $j : V \to [0, \infty]$ is a convex and lower semicontinuous functional with the effective domain $D(j)$ (which always is a vector subspace of V)

$$D(j) = V, \; 1 \leq \gamma \leq 2,$$

and for $\gamma > 2$,

$$\begin{aligned} D(j) &= \{u \in V : \mu|Du|^{\gamma} \in L^1(\Omega)\} \\ &\supset V \cap [W^{1,\gamma}(\Omega)]^3. \end{aligned}$$

Also, we may compute the functional J and obtain

$$D(J) = V,\ 1 \leq \gamma < 2$$

In case $\gamma = 2$, j is homogeneous of degree 2 and hence clearly $J = j$. In fact j is differentiable with

$$\langle j'(u), v\rangle = \int_{\Omega} \mu(x) Du : Dv,\ \forall u, v \in V, \tag{21}$$

hence in this case inequality (14) becomes (see also (4))

$$\int_{\Omega} [1 + \mu(x)] Du : Dv - \lambda \int_{\Omega} v^T k u = 0,\ \forall v \in V\ ,\ u \in V\,. \tag{22}$$

Hence we may conclude that eigenvalues of odd multiplicity of (22) yield global bifurcation points for (2).

We finally consider the case where $\gamma > 2$. As noted, $D(j)$ is a vector subspace of V, which however, will not be closed in general. However, we may conclude

Lemma 5 *Let j be given by (20). Then, for $\gamma > 2$, the functional J is given by*

$$J = I_V = 0.$$

Proof. Let $u_n \rightharpoonup u$, $\sigma_n \to 0^+$. Since $j \geq 0$, we have that

$$\liminf \frac{1}{\sigma_n^2} j(\sigma_n u_n) \geq 0 = I_V(u).$$

This shows (A8) (a) of [4]. Let now $v \in V$ and $\sigma_n \to 0^+$, we shall choose a sequence $\{v_n\} \subset V$ such that

$$v_n \to v \ \text{ in } V, \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{\sigma_n^2} j(\sigma_n v_n) = 0\,. \tag{23}$$

Since

$$V \cap [C_0^{\infty}(\Omega)]^3 = \{u \in [C_0^{\infty}(\Omega)]^3 : \operatorname{div} u = 0\}$$

is dense in V, we can find a sequence $\{u_n\} \subset V \cap [C_0^{\infty}(\Omega)]^3$ such that $u_n \to v$ in V. But

$$j(u_n) = \int_{\Omega} \mu(x) |Du_n|^{\gamma} < \infty,$$

hence, since $\sigma_n^{\gamma-2} \to 0$ as $n \to \infty$ (since $\gamma > 2$), we may, for each k, find $n_k \in \mathbb{N}$ sufficiently large such that $n_k > n_{k-1}$ and

$$\left(\int_{\Omega} \mu |Du_k|^{\gamma}\right) \sigma_j^{\gamma-2} < \frac{1}{k},\ \forall j \geq n_k.$$

Now, define the sequence $\{v_j\}$ as follows:

For each $j \in \mathbb{N}$, there exists a unique $k = k(j)$ such that

$$n_k \leq j < n_{k+1}, \tag{24}$$

(since the sequence $\{n_k\}$ is strictly increasing). Define

$$v_j = u_k = u_{k(j)}.$$

If j is large, n_k is also large. Since $\lim_{k\to\infty} u_{n_k} = v$, $\lim_{j\to\infty} v_j = v$.

On the other hand

$$\begin{aligned} j_{\sigma_j}(v_j) &= \left(\int_\Omega \mu|\sigma_j D v_j|^\gamma\right)\sigma_j^{-2} \\ &= \left(\int_\Omega \mu|D v_j|^\gamma\right)\sigma_j^{\gamma-2} \\ &= \left(\int_\Omega \mu|D u_k|^\gamma\right)\sigma_j^{\gamma-2} \\ &< \frac{1}{k} = \frac{1}{k(j)}, \end{aligned}$$

since $j \geq n_k$. As j is sufficiently large, we have, by (24), that n_{k+1} and, then, k is also large. Hence, $\frac{1}{k(j)} \to 0$ as $j \to \infty$. This shows that $\lim_{j\to\infty} j_{\sigma_j}(v_j) = 0$.

We hence obtain again the Stokes equation (11) or (13) as a limiting problem.

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