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A Global Solution Curve for a Class of Semilinear Equations *

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Abstract

We use bifurcation theory to give a simple proof of existence and uniqueness of a positive solution for the problem

$$\Delta u - \lambda u + u^p = 0 \quad \text{for } |x| < 1, \quad u = 0 \quad \text{on } |x| = 1,$$

where $x \in \mathbb{R}^n$, for any integer $n \geq 1$, and real $1 < p < (n+2)/(n-2)$, $\lambda \geq 0$. Moreover, we show that all solutions lie on a unique smooth curve of solutions, and all solutions are non-singular. In the process we prove the following assertion, which appears to be of independent interest: the Morse index of the positive solution of

$$\Delta u + u^p = 0 \quad \text{for } |x| < 1, \quad u = 0 \quad \text{on } |x| = 1$$

is one, for any $1 < p < (n+2)/(n-2)$.

1 Introduction

Following the important work of B. Gidas, W.-M. Ni and L. Nirenberg [4] there has been a lot of interest in the radially symmetric solutions of the problem

$$\Delta u + f(u) = 0 \quad \text{for } |x| < 1, \quad u = 0 \quad \text{on } |x| = 1\,. \tag{1}$$

Indeed they showed (under mild regularity assumption on $f(u)$) that all positive solutions of (1) are radially symmetric. (And the positive solutions are the only ones with a chance of being stable, a physically significant property). Soon after publication of [4] it was recognized that uniqueness of radial solutions is very hard to prove, even for relatively simple equations, see W.-M. Ni [13]. One such equation is the focus of the present work:

$$\Delta u - \lambda u + u^p = 0 \quad \text{for } |x| < 1, \quad u = 0 \quad \text{on } |x| = 1\,. \tag{2}$$

Here λ is a non-negative parameter, $p > 1$ a real constant. M.K. Kwong [10] has shown that for p subcritical, that is $p < (n+2)/(n-2)$ for $n > 2$, the problem

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(2) has a unique positive solution for any $\lambda \geq 0$, and in M.K. Kwong and Y. Li [11], uniqueness is shown for any real λ, and for more general equations; see also L. Zhang [17] and P.N. Srikanth [16]. (In [10] one has $\lambda = 1$, but the ball has an arbitrary radius. By rescaling both u and x, we see that the problems are equivalent.) M.K. Kwong's result [10] also covers the cases of domain being an annulus and of whole of $\mathbb{R}^n$. However, the proofs are rather involved (as the author himself mentions in the introduction). In this paper we use techniques from bifurcation theory to give a short and simple proof of existence and uniqueness for a ball. In addition to simplicity, our proof provides some extra information on the solution set: all solutions lie on one curve (which is significant for numerical computations), $u(0,\lambda)$ is increasing in λ, and it tends to infinity with λ. (We denote the solution of (2) by $u(r,\lambda)$.)

Next we recall a bifurcation theorem of M.G. Crandall and P.H. Rabinowitz, which will be used to continue the solution curve, and to describe the structure of the solution set near any possible turning point.

Theorem 1.1 *[1] Let X and Y be Banach spaces. Let $(\overline{\lambda}, \overline{x}) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighborhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space $N(F_x(\overline{\lambda}, \overline{x})) =$ span x_0 be one-dimensional and codim $R(F_x(\overline{\lambda}, \overline{x})) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R(F_x(\overline{\lambda}, \overline{x}))$. If Z is a complement of span x_0 in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve $(\lambda(s), x(s)) = (\overline{\lambda} + \tau(s), \overline{x} + s x_0 + z(s))$, where $s \to (\tau(s), z(s)) \in \mathbb{R} \times Z$ is a continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.*

2 A global solution curve

We study positive solutions of the Dirichlet problem for the semilinear elliptic equation on the unit ball

$$\Delta u + \lambda f(u) = 0 \quad \text{for } |x| < 1, \quad u = 0 \quad \text{on } |x| = 1, \tag{1}$$

i.e, a Dirichlet problem on a ball in $\mathbb{R}^n$, depending on a positive parameter λ. By the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg [4] positive solutions of (2) are radially symmetric, which reduces (1) to

$$u'' + \frac{n-1}{r} u' + \lambda f(u) = 0 \quad \text{for } 0 < r < 1, \quad u'(0) = u(1) = 0\,. \tag{2}$$

We shall also need the corresponding linearized equation

$$w'' + \frac{n-1}{r} w' + \lambda f'(u) w = 0 \quad \text{for } 0 < r < 1, \quad w'(0) = w(1) = 0\,. \tag{3}$$

For multiplicity and uniqueness results it is important to know if $w(r)$ is of one sign on $(0,1)$. This is part of the strategy we developed jointly with Y. Li and T. Ouyang; see [9] which also has references to our earlier papers. However, positivity of $w(r)$ is usually very hard to prove. Recently we noticed in [8] that it sometimes suffices to prove that $w(r)$ cannot change sign exactly once. This observation is rooted in the important work of M.K. Kwong and L. Zhang [12], and it will be crucial in the present paper. We begin with some some preliminary results.

The following lemma was proved in [6]. We present its proof for completeness.

Lemma 2.1 *Assume that the function* $f(u) \in C^2(\bar{\mathbb{R}}_+)$*, and the problem* (3) *has a nontrivial solution* w *at some* λ*. Then*

$$\int_0^1 f(u) w r^{n-1}\, dr = \frac{1}{2\lambda} u'(1) w'(1). \tag{4}$$

Proof. The function $v = r u_r - u_r(1)$ satisfies

$$\begin{gathered} \Delta v + \lambda f'(u) v = -2\lambda f(u) - \lambda f'(u) u'(1) \quad \text{for } |x| < 1, \\ v = 0 \quad \text{for } |x| = 1. \end{gathered} \tag{5}$$

By the Fredholm alternative the right hand side of (5) is orthogonal to w, i.e.

$$\int_0^1 f(u) w r^{n-1}\, dr = -\frac{1}{2} u'(1) \int_0^1 f'(u) w r^{n-1}\, dr\,. \tag{6}$$

Integrating the equation (3), we obtain

$$\lambda \int_0^1 f'(u) w r^{n-1}\, dr = -w'(1).$$

Using this in (6), we conclude the lemma. ♢

We recall that solution of (2) is called singular provided the corresponding linearized problem (3) has a nontrivial solution. The following lemma follows immediately from the equations (2) and (3). We assume that $f(u) \in C^2(\mathbb{R}_+)$ for the rest of the paper.

Lemma 2.2 *Let* (λ, u) *be a singular solution of* (2)*. Then*

$$\int_0^1 \left(f(u) - f'(u)u\right)\, w r^{n-1}\, dr = 0\,. \tag{7}$$

The following lemma is a consequence of the first two.

Lemma 2.3 *Let* (λ, u) *be a singular solution of* (2)*. Then for any real* γ

$$\int_0^1 \left(\gamma f(u) - f'(u)u\right)\, w r^{n-1}\, dr = \frac{\gamma - 1}{2\lambda} u'(1) w'(1). \tag{8}$$

Proof. Multiplying (4) by $\gamma - 1$, and adding (7), we obtain (8).

The following lemma is known, see e.g. E.N. Dancer [2]. We present its proof for completeness.

Lemma 2.4 *Positive solutions of the problem* (2) *are globally parameterized by their maximum values* $u(0,\lambda)$. *I.e., for every* $p > 0$ *there is at most one* $\lambda > 0$, *for which* $u(0,\lambda) = p$.

Proof. If $u(r,\lambda)$ is a solution of (2) with $u(0,\lambda) = p$, then $v \equiv u(\frac{1}{\sqrt{\lambda}} r)$ solves

$$v'' + \frac{n-1}{r} v' + f(v) = 0, \quad v(0) = p,\, v'(0) = 0. \tag{9}$$

If $u(0,\mu) = p$ for some $\mu \neq \lambda$, then $u(\frac{1}{\sqrt{\mu}} r)$ is another solution of the same problem. This is a contradiction in view of the uniqueness of solutions for initial value problems of the type (9), see [14]. ♢

We shall need the eigenvalue problem corresponding to an arbitrary solution of (2)

$$\begin{gathered} \phi'' + \tfrac{n-1}{r}\phi' + \lambda f'(u)\phi + \mu\phi = 0 \quad \text{for } 0 < r < 1, \\ \phi'(0) = \phi(1) = 0. \end{gathered} \tag{10}$$

We define Morse index of any solution of (2) to be the number of negative eigenvalues of (10).

We shall work with a standard test function $v(r) = r u_r + \mu u$, where μ is a positive constant, to be specified. One verifies that v satisfies

$$v'' + \frac{n-1}{r} v' + \lambda f'(u) v = \lambda \left[\mu u f_u - (\mu + 2) f \right] \quad \text{for } 0 < r < 1\,. \tag{11}$$

We assume that there is a number $u_0 > 0$, such that $f(u) < 0$ on $(0, u_0)$, while $f(u) > 0$ on (u_0, ∞). We define a function $G(u) \equiv \frac{u f'(u)}{f(u)}$. The following lemma follows the idea of M.K. Kwong and L. Zhang [12, Lemma 8].

Lemma 2.5 *Assume that on* (u_0, ∞) *the function* $G(u)$ *is strictly decreasing and* $\lim_{u\to\infty} G(u) > 1$, *while* $G(u) < 1$ *on* $(0, u_0)$. *Then any non-trivial solution of* (3) $w(r)$ *cannot have exactly one zero on* $(0,1)$.

Proof. Since $w(0) \neq 0$, see [14] for the appropriate uniqueness result (if $w(0) = w'(0) = 0$ then $w \equiv 0$), we may assume that $w(0) > 0$. Assume that on the contrary $w(r)$ has exactly one root at some $r = r_0$, i.e.

$$w(r) > 0 \quad \text{on } (0, r_0), \quad w(r) < 0 \quad \text{on } (r_0, 1). \tag{12}$$

We claim that $f(u(r_0)) > 0$, so that $G(u(r_0)) > 1$ by our conditions. Indeed, assuming otherwise, we would have that

$$f(u) < 0 \quad \text{on } (r_0, 1). \tag{13}$$

We now set $\mu = 0$ in (11), then multiply (11) by w, subtract from that equation (3) multiplied by v, and then integrate over $(r_0, 1)$. Obtain

$$r_0^{n-1} v(r_0) w'(r_0) = -2\lambda \int_{r_0}^1 f w r^{n-1} \, dr \, . \tag{14}$$

By (13) the right hand side of (14) is negative, while the left side is positive, a contradiction, proving the claim.

Setting $\gamma = G(u(r_0))$, we see that the horizontal line $y = \gamma > 1$ intersects the graph of $y = G(u)$ exactly once, and

$$\gamma f(u) - u f'(u) \begin{cases} < 0 & \text{for all} \quad u < u(r_0) \\ > 0 & \text{for all} \quad u > u(r_0). \end{cases} \tag{15}$$

Since $\gamma > 1$, we obtain by Lemma 2.3 (notice that $w'(1) > 0$ by (12))

$$\int_0^1 \left[\gamma f(u) - u f'(u)\right] w(r) r^{n-1} \, dr < 0 \, . \tag{16}$$

In view of (12) and (15) the quantity on the left is positive, and we have a contradiction in (16). ♢

We now restrict our attention to the problem

$$u'' + \frac{n-1}{r} u' - \lambda u + u^p = 0 \quad \text{for } 0 < r < 1, \quad u'(0) = u(1) = 0 \, , \tag{17}$$

where $p > 1$ is a constant. One checks that the Lemma 2.5 applies for $f(u) = -\lambda u + u^p$ at any $\lambda \geq 0$. Indeed, $G(u) = 1 + (p-1)\dfrac{u^p}{-\lambda u + u^p} < 1$, where $f(u) < 0$, and $G'(u) < 0$ for all $u > 0$.

We now consider the problem

$$u'' + \frac{n-1}{r} u' + u^p = 0 \quad \text{for } 0 < r < 1, \quad u'(0) = u(1) = 0 \, . \tag{18}$$

For $1 < p < (n+2)/(n-2)$ the problem is known to have a unique positive solution, see [4].

Lemma 2.6 *Assume that $1 < p \leq n/(n-2)$ in case $n > 2$. (We allow any $p > 1$ for $n = 1, 2$.) Then the Morse index of the positive solution of* (18) *is one. Moreover, this solution is non-singular (i.e. the second eigenvalue of* (19) *is positive).*

Proof. We claim that the Morse index cannot be zero. Indeed, assuming that Morse index is zero, let $\mu \geq 0$ and $\phi(r) > 0$ be solutions of (10) with $f(u) = u^p$. We now multiply (10) by u, and subtract the equation (2) multiplied by ϕ, then integrate over $(0, 1)$. Obtain

$$\int_0^1 \left[(p-1)u^p + \mu u\right] \phi \, dr = 0,$$

which is impossible, since the quantity on the left is positive.

Assume now that Morse index is at least two. Then the second eigenvalue of the corresponding linearized operator is negative. That is, we can find a $\mu < 0$ and a function $\phi(r)$, which changes sign exactly once on $(0,1)$, such that ($\phi(r)$ is of course the second eigenfunction)

$$\phi'' + \tfrac{n-1}{r}\phi' + pu^{p-1}\phi + \mu\phi = 0 \quad \text{for } 0 < r < 1, \qquad (19)$$
$$\phi'(0) = \phi(1) = 0\,.$$

We shall use a test function of M. Ramaswamy and P.N. Srikanth [15], $z = ru' + \frac{2}{p-1}u$. From (11) we notice that $z(r)$ satisfies

$$z'' + \frac{n-1}{r}z' + pu^{p-1}z = 0 \quad \text{for } 0 < r < 1, \quad z(1) = u'(1). \qquad (20)$$

We claim that $z(r)$ is decreasing on $(0,1)$. Indeed,

$$z' = ru'' + \frac{p+1}{p-1}u' \le ru'' + (n-1)u' = -ru^p < 0\,,$$

since our condition $p \le \frac{n}{n-2}$ is equivalent to $\frac{p+1}{p-1} \ge n-1$. Since $z(0) > 0$, while $z(1) < 0$, it follows that $z(r)$ changes sign exactly once on $(0,1)$, say at some $\alpha \in (0,1)$. Assume for definiteness that $\phi(0) > 0$, and denote by ξ the point where $\phi(r)$ changes sign. From the equations (19) and (20) we conclude

$$\left[r^{n-1}\left(\phi' z - \phi z'\right)\right]' + \mu\phi z r^{n-1} = 0. \qquad (21)$$

We consider two possibilities.

Case 1. Assume that $\xi \le \alpha$. Integrating (21) over $(0,\xi)$, we have

$$\xi^{n-1}\phi'(\xi)z(\xi) + \mu\int_0^\xi \phi z r^{n-1}\,dr = 0\,.$$

Since the first term on the left is non-positive, while the second one is negative, we have a contradiction.

Case 2. Assume that $\alpha < \xi$. Since z changes sign exactly once, it follows that $z(r) < 0$ on $(\xi, 1]$. Integrating (21) over $(\xi, 1)$, we obtain

$$\phi'(1)z(1) - \xi^{n-1}\phi'(\xi)z(\xi) + \mu\int_\xi^1 \phi z r^{n-1}\,dr = 0\,.$$

Since all terms on the left are negative, we again have a contradiction, completing the proof. ◊

That solution is non-singular was proved first in [15]. This also follows from the above argument. Indeed, if solution is singular then $\mu = 0$ is an eigenvalue of (19), and it has to be the second eigenvalue, since we already proved that the Morse index is one. Then we get the same contradiction as above, except if $\xi = \alpha$. But then $\phi(r)$ and $z(r)$ are solutions of the same linear equation, having the same root. Hence they have to be constant multiples of each other, which is impossible since only one of them vanishes at $r = 1$.

Theorem 2.1 *Assume that* $1 < p < (n+2)/(n-2)$ *in case* $n > 2$. *(We allow any* $p > 1$ *for* $n = 1, 2$.*) Then the Morse index of the positive solution of* (18) *is one.*

Proof. Examining the proof of Lemma 2.6, where a special case of this theorem was proved, we see that the same proof works, provided the function $z(r)$ changes sign exactly once on $(0,1)$. This was shown to be true for $p \le n/(n-2)$. Let us now vary p from $n/(n-2)$ to $(n+2)/(n-2)$. Since by M. Ramaswamy and P.N. Srikanth [15] the positive solution of (18) is non-degenerate, it follows by the implicit function theorem that positive solution of (18) varies continuously with p in C^1 norm, and hence $z(r)$ varies continuously with p. But this implies that $z(r)$ cannot have more than one zero on $(0,1)$. Indeed, assuming otherwise, we can find a $p \in (\frac{n}{n-2}, \frac{n+2}{n-2})$ and a $\xi \in (0,1)$ such that $z(\xi) = z'(\xi) = 0$. By the uniqueness theorem for ODE, $z(r) \equiv 0$, which contradicts $z(0) > 0$. ♢

We are ready to state our main result.

Theorem 2.2 *Assume that* $1 < p < (n+2)/(n-2)$ *in case* $n > 2$, *and* $p > 1$ *in dimensions one and two. Then for any* $\lambda \ge 0$ *there exists a unique positive solution of* (17)*, which is moreover non-singular. In addition,* $u(0,\lambda)$ *is increasing in* λ, *and* $\lim_{\lambda\to\infty} u(0,\lambda) = \infty$.

Proof. We begin with the unique positive solution at $\lambda = 0$. Since this solution is non-singular, we can continue it for small $\lambda > 0$ using the implicit function theorem. By scaling $u = \lambda^{1/(p-1)}v$, we can put the problem (17) in the form (2) with $f(u) = -u + u^p$. Hence we can use Lemma 2.1 to continue the solution curve globally in λ (i.e. at each point either implicit function theorem or the Crandall-Rabinowitz theorem 1.1 applies, see e.g. [9] for more details). It is easy to check that rescaling does not affect the Morse index of solution. We claim that solutions on this curve stay non-singular. Indeed, by Theorem 2.1 at $\lambda = 0$ zero lies between the first and second eigenvalues of the corresponding linearized problem. If a singular solution appears on the curve, it means that zero is now an eigenvalue, and it is either the first or the second one. It cannot be the second one, since then the corresponding eigenfunction satisfies the linearized equation (3), and it vanishes exactly once on $(0,1)$, which is excluded by Lemma 2.5 . If on the other hand, zero is the first or principal eigenvalue, then the nontrivial solution of (2) is of one sign, and we may assume that in fact $w(r) > 0$ on $(0,1)$. But this leads to a contradiction by Lemma 2.2, since $f(u) - f'(u)u < 0$ for all $r \in (0,1)$.

Since solutions on this curve cannot go to infinity at a finite λ by the result of B. Gidas and J. Spruck [5], it follows that this curve continues without turns for all $\lambda \ge 0$. We claim that there are no other solution curves. Indeed, if there were other solutions, not lying on the above curve, they would have to lie on another curve of solutions, for which there would be no place to go for decreasing λ. Hence solution curve is unique, and so for each $\lambda \ge 0$ there exists a unique solution.

Finally, we turn to the properties of $u(0,\lambda)$. By Lemma 2.4 we know that $u(0,\lambda)$ is either strictly increasing or decreasing. Assume contrary to what we want to prove that it is decreasing. Then it tends to a non-negative limit as $\lambda \to \infty$. If the limit is positive, we see from (17) that $u''(0)$ would have to become positive for large λ, which is impossible by [4]. If the limit is zero, then multiplying the equation (17) and integrating, we see that the quantity on the left is negative for large λ, a contradiction. Hence $u(0,\lambda)$ is increasing in λ. If $u(0,\lambda)$ failed to go to infinity, it would have to tend to a finite limit, which was already shown to be impossible. $\diamondsuit$

Remark. It follows from the work of P.N. Srikanth [16] that our solution curve continues for negative λ. In fact, combining both results, we see that a curve of positive non-singular solutions of (17) bifurcates off $\lambda = -\lambda_1$, and continues for all $\lambda > -\lambda_1$. Here λ_1 is the principal eigenvalue of the $-\Delta$ on the unit ball.

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