

Harmonic Parameterization of Geodesic Quadrangles on Surfaces of Constant Curvature and 2-D Quasi-Isometric Grids *

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Abstract

A method for the generation of quasi-isometric boundary-fitted curvilinear coordinates for arbitrary domains is developed on the basis of the quasi-isometric mappings theory and conformal representation of spherical and hyperbolic geometries. A one-parameter family of Riemannian metrics with some attractive invariant properties is analytically described. We construct the quasi-isometric mapping between the regular computation domain \mathcal{R} and a given physical domain \mathcal{D} that is conformal with respect to the unique metric from the proposed one-parameter class. The identification process of the unknown parameter takes into account the high parametric sensitivity of metrics to the parameter. For this purpose we use a new technique for finding the geodesic quadrangle with given angles and a conformal module on the surface of constant curvature, which makes the method more robust. The method allows more direct control of the grid cells size and angle over the field as the grid is refined. Illustrations of this technique are presented for the case of one-element airfoil and several test domains.

1 Introduction

1.1 Quasi-isometric grids

The generation of a 2-D quasi-isometric grids in a given physical region \mathcal{D} (a curvilinear quadrangle with interior angles β_i , $0 < \beta_i < \pi$, $i = 1, \dots, 4$, and a conformal modulus \mathcal{M}) may be considered as a problem of construction of the mapping

$$X = X(\xi, \eta), \quad Y = Y(\xi, \eta) \quad (1.1)$$

between points (ξ, η) of the computational region

$$\mathcal{R} = \{(\xi, \eta) : 0 \leq \xi \leq 1, 0 \leq \eta \leq 1\}$$

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and points $(X, Y) \in \mathcal{D}$ such that (1.1) is the unique solution of the following boundary value problem (BVP): given a quasi-isometric mapping between $\partial\mathcal{R}$ and $\partial\mathcal{D}$ to extend the mapping inside \mathcal{R} as a quasi-isometric solution of the appropriate Beltrami system

$$\sqrt{g_{11}(\xi, \eta, r)g_{22}(\xi, \eta, r) - g_{12}^2(\xi, \eta, r)} X_\xi = -g_{12}(\xi, \eta, r) Y_\xi + g_{11}(\xi, \eta, r) Y_\eta, \quad (1.2)$$

$$\sqrt{g_{11}(\xi, \eta, r)g_{22}(\xi, \eta, r) - g_{12}^2(\xi, \eta, r)} X_\eta = -g_{22}(\xi, \eta, r) Y_\xi + g_{12}(\xi, \eta, r) Y_\eta.$$

The coefficients of the system (1.2) contain one unknown parameter r , restricted to the interval (r^{\min}, r^{\max}) , which is to be found in the process of solving the BVP.

In the capacity of the boundary conditions in this BVP we can also choose so-called "free" conditions, under which grid points on the boundary of the physical region \mathcal{D} are not fixed and can move along $\partial\mathcal{D}$.

As it is shown in [3], the linear elliptic system (1.2) can be treated as a condition for conformality of the mapping (1.1) with respect to the following Riemmanian metric defined on \mathcal{R} :

$$ds^2 = g_{11}(\xi, \eta, r)d\xi^2 + 2g_{12}(\xi, \eta, r)d\xi d\eta + g_{22}(\xi, \eta, r)d\eta^2. \quad (1.3)$$

Suppose the functions $g_{ik}(\xi, \eta, r)$ satisfy the inequality

$$g_{11}(\xi, \eta, r) + g_{22}(\xi, \eta, r) \leq Q \sqrt{g_{11}(\xi, \eta, r)g_{22}(\xi, \eta, r) - g_{12}^2(\xi, \eta, r)}, \quad (1.4)$$

and are continuously differentiable in \mathcal{R} . Then the mapping (1.1) has non-vanishing Jacobian in \mathcal{R} and is called *Q-quasiconformal*. That means that the singular values $\nu_1 = \nu_1(\xi, \eta)$ and $\nu_2 = \nu_2(\xi, \eta)$ of the matrix

$$J = \begin{bmatrix} X_\xi & X_\eta \\ Y_\xi & Y_\eta \end{bmatrix},$$

being enumerated in decreasing order, satisfy the condition $\nu_1(\xi, \eta)/\nu_2(\xi, \eta) \leq Q$ for all $(\xi, \eta) \in \mathcal{R}$. The least possible number Q with such property is called *the coefficient of quasi-conformality* of the mapping (1.1) in the domain \mathcal{R} . Clearly, a 1-quasiconformal mapping is conformal.

Note that the angle α between the grid lines is defined by the formula $\cos \alpha = g_{12}(g_{11}g_{22})^{-1/2}$, and the ratio of lengths of cell sides is $(g_{22}/g_{11})^{1/2}$. Thus, since every solution of the Beltrami system (1.2) generates a metric conformally equivalent to (1.3), we are able to control the grid angle and the cell sides ratio by defining the coefficients g_{ik} of the metric (1.3) manually. In order to have better control of the size of cells as the grid is refined, we want to define g_{ik} in such a way that the solution of the corresponding Beltrami system (1.2) is μ -quasi-isometric.

- A mapping (1.1) is called μ -quasi-isometric if there exists a constant $\mu \in [1, \infty)$ such that

$$\mu = \max \left\{ \sup_{(\xi, \eta) \in \mathcal{R}} \nu_1(\xi, \eta), \sup_{(\xi, \eta) \in \mathcal{R}} \frac{1}{\nu_2(\xi, \eta)} \right\}.$$

The constant μ is called *the coefficient of quasi-isometricity* of the mapping (1.1) in the domain \mathcal{R} .

Under a μ -quasi-isometric mapping an infinitesimal square will go over into a parallelogram and the ratio of the lengths of any side of the parallelogram and a corresponding side of the square will be bounded by $1/\mu$ and μ . If $\mu = 1$ then (1.1) is an isometric mapping. It is clear that μ -quasi-isometric mapping will also be μ^2 -quasiconformal. Note that the relation (1.4) and the system (1.2) imply that the mapping (1.1) is quasi-isometric in any sub-region of \mathcal{R} . As a rule a quasiconformal mapping is not quasi-isometric, that is, function $\nu_1(\xi, \eta)$ or $1/\nu_2(\xi, \eta)$ is not bounded.

Nevertheless use of quasiconformal mappings combined with conditions of existence of bounded derivative of a holomorphic function provides the key to the generation of quasi-isometric coordinate systems.

Several attempts were made in that direction. If we put $g_{11} = 1/r$, $g_{12} = 0$, $g_{22} = r$, we get the quasi-conformal grid generation problem posed by Godunov and Prokopov in [11, 13]. The development of the Godunov-Prokopov method is precisely described in [13] and [24]. In a recent paper Khamayseh and Mastin [17] have extended this method to surface grid generation. For various concepts from tensor analysis and differential geometry applicable to the generation of curvilinear coordinate systems we refer to [25], [9] and [23].

The orthogonal mapping technique has been investigated in recent works [16] and [8], [21] from different points of view.

In [1] Belinsky et. al. proposed to consider the special class of quasi-conformal mappings by defining $g_{11} = e^{2q(\xi)}$, $g_{22} = e^{2p(\eta)}$, $g_{12} = \sqrt{g_{11}}\sqrt{g_{22}}\cos[\beta(\eta) - \alpha(\xi)]$. In other words, the proposed metric contained four arbitrary functions of either ξ or η . This metric was obtained as a result of the Chebyshev mapping \mathcal{R} onto a plain curvilinear parallelogram. In a similar way in the paper [12] for the purpose of construction of structured multi-block quasi-conformal grids in complex domains it was proposed to use a certain class of functions g_{ij} depending on ξ , η and an unknown vector of parameters r . The metric was obtained by mapping a computational domain that consisted of several squares onto a figure made of several parallelograms.

In order to obtain the unique quasi-isometric solution to the grid generation problem, a special one-parameter family of metrics was closely studied by one of author in papers [5, 6]. In a later paper by Godunov et. al. [10] it was proposed to study a five-parameter family of metrics.

The main goal of the present work is to describe analytically several types of one-parametric families of metrics for which the posed BVP has a unique quasi-isometric solution. The metric coefficients g_{ik} will be obtained via mapping the

computational region \mathcal{R} onto a special class of geodesic quadrangles on surfaces of constant curvature. We start by investigating some basic properties of one-parameter families of such geodesic quadrangles with given angles.

After the basic concepts are introduced, we give several different quasi-isometric parameterizations of geodesic quadrangles, that is, quasi-isometric mappings

$$x = x(\xi, \eta, \alpha, \mathbf{r}), \quad y = y(\xi, \eta, \alpha, \mathbf{r}) \quad (1.5)$$

between \mathcal{R} and the geodesic quadrangle \mathcal{P} . These parameterizations have two vectors of parameters: interior angles $\alpha = (\alpha_1, \dots, \alpha_4)$ and so-called Euclidean lengths of sides $\mathbf{r} = (r_1, r_2, r_3, r_4)$ in the parametric plane (x, y) . The plane itself which conformally represents spherical or hyperbolic geometries according to whether the angle defect of \mathcal{P}

$$\delta^* = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - 2\pi$$

is positive or negative. The mapping (1.1) generates the metric tensor g_{ij} , coefficients of which are to be used as coefficients of the Beltrami system (1.2).

In [7] the authors proposed several basic parameterizations of a geodesic quadrangle \mathcal{P} , and the method for finding the parameter \mathbf{r} was described. All parameterizations were obtained by means of using geodesic bundles in order to build vertical and horizontal families of grid lines. In this paper we continue with the subject.

Use of geodesic bundles for parameterizations of \mathcal{P} provides a way to construct a mapping (1.1) which maps uniform grid in the unit square on (ξ, η) -plane into a grid in \mathcal{P} that is invariant under rotations. By rotation we mean the motion that places another vertex of the geodesic quadrangle at the origin of the parametric plane, and ‘‘invariant under rotations’’ means that rotations and the grid generation process commute. In this case the grid lines appear to be level lines of some harmonic functions, consequently, the geodesic grids in \mathcal{P} can be treated as a variant of the Winslow grids [26] with an advantage that in our case the grid in \mathcal{P} can be defined explicitly. These kinds of two-dimensional harmonic mappings have non-vanishing Jacobian, as it was shown in [20] and [13]. In particular, we describe a parameterization (1.1) which generates a metric with coefficients $g_{ik}(\xi, \eta, \alpha, \mathbf{r})$ that have the following properties:

$$\frac{g_{11}(\xi, \eta, \alpha, \mathbf{r})}{g_{22}(\xi, \eta, \alpha, \mathbf{r})} = \frac{g_{22}(1 - \eta, \xi, \sigma(\alpha), \sigma(\mathbf{r}))}{g_{11}(1 - \eta, \xi, \sigma(\alpha), \sigma(\mathbf{r}))}, \quad (1.6)$$

$$\begin{aligned} & \frac{g_{12}(\xi, \eta, \alpha, \mathbf{r})}{\sqrt{g_{11}(\xi, \eta, \alpha, \mathbf{r}) g_{22}(\xi, \eta, \alpha, \mathbf{r})}} \\ &= \frac{-g_{12}(1 - \eta, \xi, \sigma(\alpha), \sigma(\mathbf{r}))}{\sqrt{g_{11}(1 - \eta, \xi, \sigma(\alpha), \sigma(\mathbf{r})) g_{22}(1 - \eta, \xi, \sigma(\alpha), \sigma(\mathbf{r}))}}. \end{aligned} \quad (1.7)$$

where $\sigma(1, 2, 3, 4) = (4, 1, 2, 3)$.

From equalities (1.6) and (1.7) it follows that all elements of vectors α and \mathbf{r} are equivalent parameters and consequently any five of these parameters may

be used as characteristic invariants of \mathcal{P} . In particular, we shall fix angles $\alpha_1, \dots, \alpha_4$ and study a set of geodesic quadrangles depending on the parameter \mathcal{M} for the interval $0 < \mathcal{M} < \infty$. In capacity of the parameter \mathcal{M} we pick the conformal module of \mathcal{P} .

We define the conformal module as follows. For every geodesic quadrangle \mathcal{P} with vertices z_i , $i = 1, \dots, 4$, there exists a unique $\mathcal{M} \in (0, \infty)$ such that there exists a conformal mapping of \mathcal{P} onto the rectangle

$$\{(\xi, \eta) : 0 \leq \xi \leq 1, 0 \leq \eta \leq \mathcal{M}\}$$

under which the vertex z_1 is mapped at the origin, and the other vertices of \mathcal{P} go over into vertices of the rectangle.

The additional parameters r_1, \dots, r_4 are then to be determined as monotonic functions $r_i(\mathcal{M})$ of the fundamental parameter \mathcal{M} . Later we are going to use one of r_i as or main parameter for finding the geodesic quadrangle \mathcal{P} with given angles $\alpha_1, \dots, \alpha_4$ and given a conformal module \mathcal{M} . Thus we want to develop the technique for finding such geodesic quadrangle \mathcal{P} that takes into account the high parametric sensitivity of \mathcal{M} to changes of the parameters r_i . For this purpose we shall introduce positive numbers r_i^{\min}, r_i^{\max} such that for all possible values of \mathcal{M} parameters r_i stay within the boundaries r_i^{\min} , and r_i^{\max} for $i = 1, \dots, 4$. We also will need the derivatives $dr_j(\mathcal{M})/dr_i(\mathcal{M})$.

In later sections we describe a procedure of finding the mapping $f(z)$, $z = x + iy$, by which the geodesic quadrangle \mathcal{P} is mapped conformally onto the physical domain \mathcal{D} ; such conformal mapping exists uniquely by virtue of the Riemann Mapping Theorem [15]. It is important to know how the module of the derivative of the conformal mapping behaves on the boundary of \mathcal{P} , because it can affect the grid cell size near the boundary of the domain when the grid is refined.

Suppose we fix a point b on a boundary of \mathcal{D} . Then by virtue of the uniqueness of the conformal mapping of \mathcal{P} onto \mathcal{D} there exists a unique proimage z of the point b on boundary of \mathcal{P} . Consequently, we allow boundary points of \mathcal{P} to move; we call such boundary conditions *free* on \mathcal{P} , and *fixed* on \mathcal{D} . Similarly, if we allow points of \mathcal{D} to move along one of the boundaries, and fix boundary points of \mathcal{P} on the corresponding boundary, we call such boundary conditions *free* on \mathcal{D} and *fixed* on \mathcal{P} .

In order to construct a quasi-isometric grid we should take into account well-known conditions of the existence of the boundary derivative (see [19], [18] and [14]):

- If the boundary $\partial\mathcal{D}$ is twice continuously differentiable in some neighborhood of the point $w_0 = f(\zeta_0)$ and ζ_0 is not a corner point of \mathcal{P} , then the derivative f' can be continuously extended in certain part of the boundary $\partial\mathcal{P}$, containing ζ_0 , in such a way that $f'(\zeta) \neq 0$.
- If ζ_0 and $w_0 = f(\zeta_0)$ are the corner points of \mathcal{P} and \mathcal{D} respectively and two twice continuously differentiable boundary arcs of \mathcal{D} join in the point w_0 at the same angle as do the corresponding arcs of \mathcal{P} in the point ζ_0 ,

then $f'(\zeta)$ does not equal 0 and is bounded in a certain neighborhood of ζ_0 .

Thus, in order to use the proposed method for construction of a quasi-isometric grid in \mathcal{D} , the following conditions should be satisfied:

1. The boundary $\partial\mathcal{D}$ of the domain \mathcal{D} must consist of four twice continuously differentiable curves;
2. The sought geodesic quadrangle \mathcal{P} is to be chosen in such a way that the angles $\alpha_1, \dots, \alpha_4$ of \mathcal{P} should coincide with the angles β_1, \dots, β_4 of the domain \mathcal{D} , and the conformal modules of \mathcal{P} and \mathcal{D} must be the same.

It was proved in [10, 5] and [6], respectively, that under the restriction

$$\delta^* < 2\alpha_i, \quad i = 1, \dots, 4, \quad (1.8)$$

a geodesic quadrangle \mathcal{P} satisfying the second condition does exist uniquely in both cases of the negative and positive angle defect δ^* .

The sought mapping (1.1) is the superposition of two quasi-isometric mappings mentioned above, and it is to be found as the solution of a variational problem of minimizing a functional of Dirichlet type.

To illustrate the effectiveness of the method, numerical examples of quasi-isometric grids with different boundary conditions are presented.

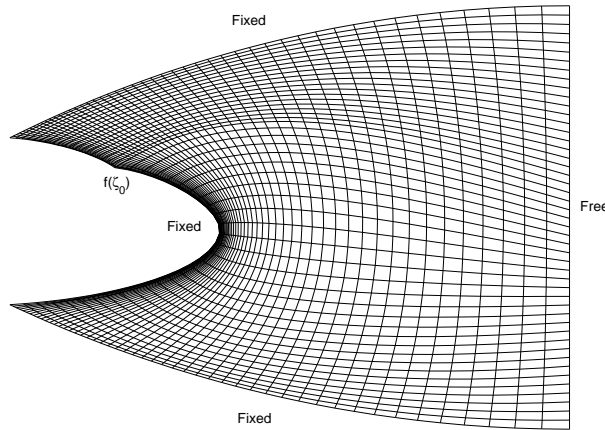


Figure 1a.

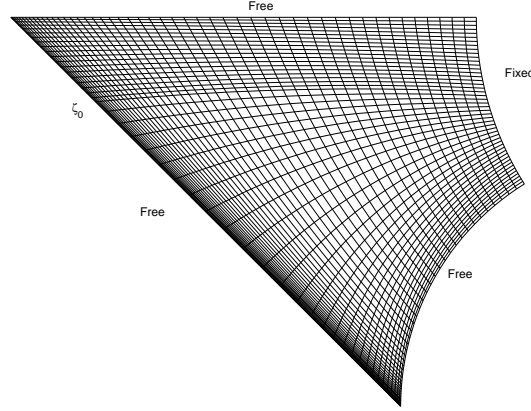


Figure 1b.

In the Figure 1a we show a test domain \mathcal{D} with a quasi-conformal grid in it, and Figure 1b represents the geodesic quadrangle with same angles and conformal module as of the test domain. If boundary points on \mathcal{D} are fixed, then boundary points of \mathcal{P} are free, and vice versa.

Note that the Figure 1 illustrates the case when the conditions 1 and 2 are not satisfied, that is, one side of the physical domain is not a C^2 -curve. We can treat it as a violation of the condition 2. Let the point $f(\zeta_0)$ split the boundary of \mathcal{D} into two C^2 -curves that intersect at the angle $\beta = b \cdot \pi$, $b \in [0, 1]$. Then for the conformal mapping $f : \mathcal{P} \rightarrow \mathcal{D}$ normed such that $\omega_0 = f(\zeta_0) = 0$ in neighborhood of ζ_0 we have

$$f(z) = a(z - \zeta_0)^b + o(|z - \zeta_0|^b), \quad f'(z) = ab(z - \zeta_0)^{b-1} + o(|z - \zeta_0|^{b-1}),$$

where $a \neq 0$. In particular, it follows that at the point ζ_0 the derivative f' becomes infinite, and because of that the grid points next to $f(\zeta_0)$ will not be close to $f(\zeta_0)$ as the grid is refined. Examples of quasi-conformal grids with singularities of such kind can be found in the book [23].

In conclusion we would like to say that from our point of view Riemannian metrics that have properties (1.6) and (1.7), and in particular, the metrics induced by the harmonic parameterization, are more efficient for the generation of grids through the solution of the Beltrami system (1.2) with "free" boundary conditions. We used the harmonic parameterization to generate the distribution of boundary points on the right side of the physical domain in the figure 1.

2 One-parameter families of convex geodesic quadrangles

2.1 Geodesic lines and bundles on surfaces of constant curvature

Consider the surface of constant curvature $K = 4\delta$ as the (x, y) -parametric plane with the following metric:

$$ds^2 = \frac{dx^2 + dy^2}{[1 + \delta(x^2 + y^2)]^2}, \quad (2.1)$$

where δ is a real number [4]. If δ is positive, the metric (2.1) is defined for every x and y including infinite, and we obtain a representation of spherical geometry. If δ is negative, then in the disk $x^2 + y^2 + \delta < 0$ we obtain the Poincare representation of Lobachevsky geometry, or hyperbolic geometry. In the case $\delta = 0$, the metric (2.1) is the Euclidean metric on the plane (x, y) .

Let $q = ax + by + c[1 - \delta(x^2 + y^2)]$. Then geodesics in the metric (2.1) are curves defined by the equation $q = 0$. Note that for every $\delta \neq 0$ circles of the form $q = 0$ are orthogonal to the circle $1 + \delta(x^2 + y^2) = 0$ which is called the absolute.

Each of three types of non-Euclidean spaces of a constant curvature indicated above admits the continuous group of isometric mappings, that is, motions. If we denote $x + iy$ by z , then every motion has the form

$$w(z) = e^{i\omega} \frac{z - \zeta}{1 + \delta\bar{\zeta}z}, \quad (2.2)$$

where $\omega \in R$. The complex number ζ must satisfy the condition $|\zeta| < |\delta|^{-1/2}$ in case $\delta < 0$.

Let s_1 and s_2 be two distinct geodesics defined by equations $q_1 = 0$ and $q_2 = 0$ respectively. The family \mathcal{F} of geodesics orthogonal to s_1 and s_2 is called a geodesic bundle. The geodesic bundle \mathcal{F}^\perp in which s_1 and s_2 can be embedded is called orthogonal to \mathcal{F} .

2.2 Family of geodesic quadrangles with given angles

Let \mathcal{P} be a quadrangle whose sides are geodesics. We assume further that the vertices $z_i = (x_i, y_i)$, $i = 1, \dots, 4$ of the quadrangle \mathcal{P} are enumerated counterclockwise. Let us denote sides of \mathcal{P} as $A_i = z_i z_{i+1}$, angles between A_{i-1} and A_i as α_i , $i = 1, \dots, 4$, (assume $z_5 = z_1$ and $A_0 = A_4$). Let $\varphi_i = \alpha_i - \pi/2$, $i = 1, \dots, 4$.

It is possible to standardize a geodesic quadrangle \mathcal{P} by motions (2.2) in such a way that two of its sides on the (x, y) -parametric plane will be linear segments, as shown on the figure 1. In order to do that it is sufficient to move one vertex of \mathcal{P} (say, z_1) to the origin by an appropriate transformation (2.2). Then by rotation we can place the side A_1 on the x axis so that the vertex z_2

has positive x -coordinate. Denote by r_1 the Euclidean length of the segment A_1 . The invariance of the metric (2.1) under the motions (2.2) implies that we can associate with every A_i its unique Euclidean length r_i .

In order to construct a geodesic quadrangle \mathcal{P} it is sufficient to have five parameters defined. We shall fix angles $\alpha_1, \dots, \alpha_4$ and study one-parameter families of geodesic quadrangles $\mathcal{P}_{\mathcal{M}}, \mathcal{P}_m$ or \mathcal{P}_{r_1} , where in the capacity of varying parameters we can choose the conformal module \mathcal{M} , parameter $m = (r_1 r_3)/(r_2 r_4)$, or a Euclidean side r_1 .

From the results presented in [5, 6] it follows that if we consider a one-parametric family $\mathcal{P}_{\mathcal{M}}$ then invariants $m(\mathcal{M})$ and $r_i(\mathcal{M})$ depend monotonically on the conformal module \mathcal{M} , which ranges from 0 to ∞ . In other words, the following theorem holds.

Theorem 1 *Let $\mathcal{P}_{\mathcal{M}}$ and $\mathcal{P}_{\overline{\mathcal{M}}}$ be two geodesic quadrangles with the same angles and conformal modules \mathcal{M} and $\overline{\mathcal{M}}$, respectively. If $\mathcal{M} = \overline{\mathcal{M}}$ then $\mathcal{P}_{\mathcal{M}} = \mathcal{P}_{\overline{\mathcal{M}}}$; if $\mathcal{M} > \overline{\mathcal{M}}$ then $m(\mathcal{M}) > m(\overline{\mathcal{M}})$, $r_i(\mathcal{M}) > r_i(\overline{\mathcal{M}})$ and $r_{i+1}(\mathcal{M}) < r_{i+1}(\overline{\mathcal{M}})$ for $i = 1, 3$.*

Below we provide some examples of convex geodesic quadrangles $\mathcal{P}_{\mathcal{M}}$ with same angles and different conformal modules.

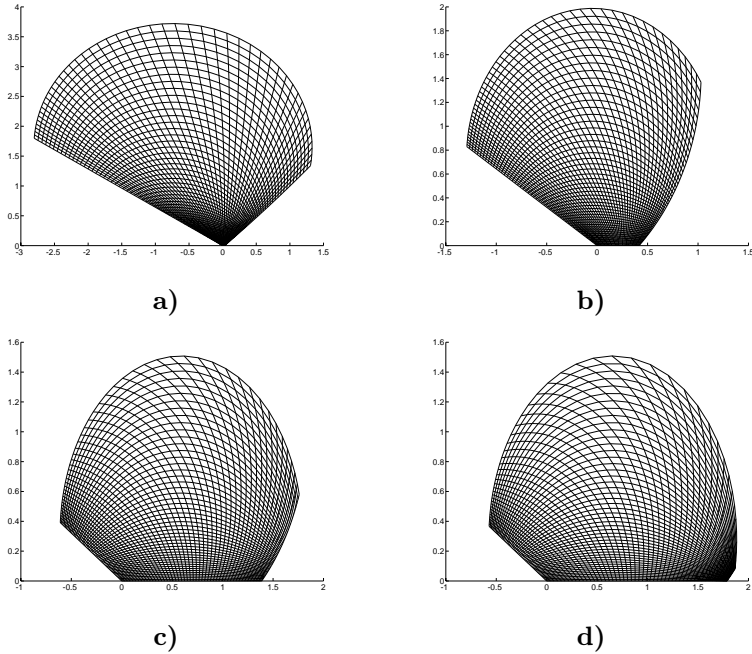


Figure 2. Geodesic quadrangle $\mathcal{P}_{\mathcal{M}}$ with same angles and various conformal module \mathcal{M} .

Note that the parameter $r_1(\mathcal{M})$ varies in the range from $r_1^{min} = r_1(0) = 0$ to $r_1^{max} = r_1(\infty) < \infty$ as \mathcal{M} varies from 0 to ∞ which leads to high parametric sensitivity of conformal module \mathcal{M} to r_1 . When \mathcal{M} is close to zero, practically

we have a geodesic triangle with a polar coordinate system (Figure 2a). It is the topological limit of the geodesic quadrangle $\mathcal{P}_{\mathcal{M}}$ as $\mathcal{M} \rightarrow 0$. In Figure 2d we see that $r_1(\mathcal{M}) \rightarrow r_1^{\max}$ and $r_2(\mathcal{M}) \rightarrow 0$ as $\mathcal{M} \rightarrow \infty$.

2.3 Boundaries for Euclidean lengths

Since we are going to consider Euclidean lengths as our main parameters, one of our tasks is to develop a technique for finding the geodesic quadrangle $\mathcal{P}_{\mathcal{M}}$ by given conformal module \mathcal{M} , which takes into account the high parametric sensitivity of \mathcal{M} to r_i .

First of all for this purpose we shall find positive numbers r_i^{\min} and r_i^{\max} such that $r_i(\mathcal{M})$ belongs to the interval (r_i^{\min}, r_i^{\max}) for all \mathcal{M} , and $r_i^{\min} = r_i(0)$, $r_i^{\max} = r_i(\infty)$ for $i = 1, \dots, 4$. As it was shown in ([7]), we can calculate r_i^{\min} and r_i^{\max} using the special function $B(\psi_1, \psi_2, \psi_3, \psi_4, \delta)$ given by

$$B(\psi_1, \psi_2, \psi_3, \delta) = \sqrt{\frac{\sin \psi \cos(\psi - \psi_3)}{\delta \cos(\psi - \psi_2) \sin(\psi - \psi_2 - \psi_3)}}, \quad (2.3)$$

where $\psi = (\psi_1 + \psi_2 + \psi_3 + \pi/2)/2$. It is convenient for us to fix $\delta = \sin \gamma$, where $\gamma = (\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)/2$.

Let us denote by $l(\mathcal{P})$ the number of sides of \mathcal{P} the sum of whose adjacent angles is not less than π . The value of $l(\mathcal{P})$ can be 0, 1, 2, 3 and 4. Let us consider each case separately. Let Σ_4 be the cyclic subgroup of the permutation group \mathcal{S}_4 , generated by the element

$$\bar{\sigma} \in \mathcal{S}_4, \quad \bar{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},$$

Let $l(\mathcal{P}) = 0$, then for all $\sigma \in \Sigma_4$

$$\begin{aligned} r_{\sigma(1)}^{\min} &= B(\varphi_{\sigma(1)}, \varphi_{\sigma(2)}, -\pi/2, \delta), & r_{\sigma(1)}^{\max} &= B(\varphi_{\sigma(1)}, \pi/2, \varphi_{\sigma(4)}, \delta), \\ r_{\sigma(2)}^{\min} &= B(\varphi_{\sigma(2)}, \varphi_{\sigma(3)}, -\pi/2, \delta), & r_{\sigma(2)}^{\max} &= B(\varphi_{\sigma(2)}, \pi/2, \varphi_{\sigma(1)}, \delta), \\ r_{\sigma(3)}^{\min} &= B(\varphi_{\sigma(3)}, \varphi_{\sigma(4)}, -\pi/2, \delta), & r_{\sigma(3)}^{\max} &= B(\varphi_{\sigma(3)}, \pi/2, \varphi_{\sigma(2)}, \delta), \\ r_{\sigma(4)}^{\min} &= B(\varphi_{\sigma(4)}, \varphi_{\sigma(1)}, -\pi/2, \delta), & r_{\sigma(4)}^{\max} &= B(\varphi_{\sigma(4)}, \pi/2, \varphi_{\sigma(3)}, \delta), \end{aligned}$$

Let $l(\mathcal{P}) = 1$, and $\sigma \in \Sigma_4$ be such that $\varphi_{\sigma(1)} + \varphi_{\sigma(2)} \geq 0$. Then the boundaries will be as follows.

$$\begin{aligned} r_{\sigma(1)}^{\min} &= 0, \\ r_{\sigma(1)}^{\max} &= B(\varphi_{\sigma(1)}, \pi/2, \varphi_{\sigma(4)}, \delta), \\ r_{\sigma(2)}^{\min} &= B(\varphi_{\sigma(3)}, \varphi_{\sigma(2)}, -\pi/2, \delta), \\ r_{\sigma(2)}^{\max} &= B(\varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2, \varphi_{\sigma(3)}, \varphi_{\sigma(4)}, \delta), \\ r_{\sigma(3)}^{\min} &= B(\varphi_{\sigma(3)}, \varphi_{\sigma(4)}, \varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2, \delta), \\ r_{\sigma(3)}^{\max} &= B(\varphi_{\sigma(3)}, -\pi/2, \varphi_{\sigma(2)}, \delta), \\ r_{\sigma(4)}^{\min} &= B(\varphi_{\sigma(1)}, \varphi_{\sigma(4)}, -\pi/2, \delta), \\ r_{\sigma(4)}^{\max} &= B(\varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2, \varphi_{\sigma(4)}, \varphi_{\sigma(3)}, \delta). \end{aligned}$$

Consider now the case $l(\mathcal{P}) \geq 2$. There always exist such $\sigma \in \Sigma_4$ that

$$\varphi_{\sigma(1)} + \varphi_{\sigma(2)} \geq \varphi_{\sigma(3)} + \varphi_{\sigma(4)}, \quad \varphi_{\sigma(1)} + \varphi_{\sigma(4)} \geq \varphi_{\sigma(2)} + \varphi_{\sigma(3)}$$

holds. Then the boundaries for r_j will be as follows:

$$\begin{aligned} r_{\sigma(1)}^{\min} &= 0, \\ r_{\sigma(1)}^{\max} &= B(\varphi_{\sigma(1)} + \varphi_{\sigma(4)} - \pi/2, \varphi_{\sigma(2)}, \varphi_{\sigma(3)}, \delta), \\ r_{\sigma(2)}^{\min} &= B(\varphi_{\sigma(2)}, \varphi_{\sigma(3)}, \varphi_{\sigma(1)} + \varphi_{\sigma(4)} - \pi/2, \delta), \\ r_{\sigma(2)}^{\max} &= B(\varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2, \varphi_{\sigma(3)}, \varphi_{\sigma(4)}, \delta), \\ r_{\sigma(3)}^{\min} &= B(\varphi_{\sigma(3)}, \varphi_{\sigma(4)}, \varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2, \delta), \\ r_{\sigma(3)}^{\max} &= B(\varphi_{\sigma(1)} + \varphi_{\sigma(4)} - \pi/2, \varphi_{\sigma(3)}, \varphi_{\sigma(2)}, \delta), \\ r_{\sigma(4)}^{\min} &= 0, \\ r_{\sigma(4)}^{\max} &= B(\varphi_{\sigma(1)} + \varphi_{\sigma(2)} - \pi/2, \varphi_{\sigma(4)}, \varphi_{\sigma(3)}, \delta). \end{aligned}$$

2.4 Relations between Euclidean lengths

The relations $r_{\sigma(4)}(r_{\sigma(1)})$ and $dr_{\sigma(4)}/d(r_{\sigma(1)})$ for all $\sigma \in \Sigma_4$ can be obtained from the following equation which involves parameters $r_{\sigma(1)}$, $r_{\sigma(4)}$, φ_1 , φ_2 , φ_3 , φ_4 and δ :

$$\begin{aligned} C_{\sigma(3)}S_0 &= \delta C_{\sigma(2)}S_{\sigma(23)}r_{\sigma(1)}^2 + \delta C_{\sigma(4)}S_{\sigma(34)}r_{\sigma(4)}^2 \\ &\quad + 2\delta D_{\sigma}r_{\sigma(1)}r_{\sigma(4)} - \delta^2 C_{\sigma(1)}S_{\sigma(24)}r_{\sigma(1)}^2 r_{\sigma(4)}^2, \end{aligned} \quad (2.4)$$

where

$$\begin{aligned} S_0 &= \sin \gamma, \quad D_{\sigma} = \cos \varphi_{\sigma(2)} \cos \varphi_{\sigma(4)}, \quad C_i = \cos(\gamma - \varphi_i), \\ S_{ij} &= \sin(\gamma - \varphi_i - \varphi_j), \quad \sigma(ij) = (\sigma(i), \sigma(j)), \quad i, j = 1, \dots, 4. \end{aligned}$$

It follows from (2.4) that for $\delta \geq 0$:

$$r_{\sigma(4)}(r_{\sigma(1)}) = \frac{1}{B + \sqrt{B^2 + C}}, \quad (2.5)$$

where

$$B = \frac{r_{\sigma(1)}D_{\sigma}}{C_{\sigma(3)} - r_{\sigma(1)}^2 C_{\sigma(2)}S_{\sigma(23)}}, \quad C = \frac{r_{\sigma(1)}^2 C_{\sigma(1)}S_{\sigma(24)}\delta - C_{\sigma(4)}S_{\sigma(34)}}{C_{\sigma(3)} - r_{\sigma(1)}^2 C_{\sigma(2)}S_{\sigma(23)}}.$$

In the case $\delta < 0$ we have to take as $r_{\sigma(4)}$ the root of (2.4) that belongs to the interval $(r_{\sigma(4)}^{\min}, r_{\sigma(4)}^{\max})$.

Thus given $r_{\sigma(1)}$ and four angles $\alpha_1, \dots, \alpha_4$ we are able by means of (2.4) to determine the function $r_j = r_j(r_i)$, and its derivative with respect to r_i for $i, j = 1, \dots, 4$. In order to do that it is sufficient to find the derivative of $r_{\sigma(4)}(r_{\sigma(1)})$ with respect to $r_{\sigma(1)}$:

$$\frac{dr_{\sigma(4)}(r_{\sigma(1)})}{dr_{\sigma(1)}} = - \frac{C_{\sigma(2)}S_{\sigma(23)}r_{\sigma(1)} + D_{\sigma}r_{\sigma(4)} - \delta C_{\sigma(1)}S_{\sigma(24)}r_{\sigma(1)}r_{\sigma(4)}^2}{C_{\sigma(4)}S_{\sigma(34)}r_{\sigma(4)} + D_{\sigma}r_{\sigma(1)} - \delta C_{\sigma(1)}S_{\sigma(24)}r_{\sigma(1)}^2 r_{\sigma(4)}}. \quad (2.6)$$

3 Quasi-isometric parameterizations of geodesic quadrangles

It should be noted that any five parameters from vectors $\alpha = (\alpha_1, \dots, \alpha_4)$ and $\mathbf{r} = (r_1, r_2, r_3, r_4)$ may be used as characteristic invariants of a geodesic quadrangle \mathcal{P} . We shall now suppose that parameters $\varphi_1, \varphi_2, \varphi_4, r_1$ and r_4 are given.

3.1 The simplest quasi-isometric parameterization

Consider \mathcal{P} as an intersection of two geodesic bundles, which are described by the following algebraic equations

$$\begin{aligned} (\cos \varphi_1 + \xi a_1)x + (\sin \varphi_1 - \xi b_1)y - \xi c_1[1 - \delta(x^2 + y^2)] &= 0, & \xi \in [0, 1] \quad (3.1) \\ \eta a_2 x - (1 + \eta b_2)y + \eta c_2[1 - \delta(x^2 + y^2)] &= 0, & \eta \in [0, 1], \quad (3.2) \end{aligned}$$

where

$$\begin{aligned} a_1 &= \cos \varphi_2(1 - \delta r_1^2) - \cos \varphi_1, & b_1 &= \sin \varphi_2(1 + \delta r_1^2) + \sin \varphi_1, \\ c_1 &= r_1 \cos \varphi_2, & a_2 &= \sin(\varphi_1 + \varphi_4) - \delta r_4^2 \sin(\varphi_1 - \varphi_4), \\ b_2 &= \cos(\varphi_1 + \varphi_4) - \delta r_4^2 \cos(\varphi_1 - \varphi_4) - 1, & c_2 &= r_4 \cos \varphi_4. \end{aligned}$$

Let us denote by

$$\begin{aligned} P(\xi) &= c_2 \cos \varphi_1 + \xi(a_1 c_2 + a_2 c_1), \\ Q(\xi, \eta) &= \xi c_1 - \eta c_2 \sin \varphi_1 + \xi \eta(b_1 c_2 + b_2 c_1), \\ F(\xi, \eta) &= \frac{1}{2} [\cos \varphi_1 + \xi a_1 + \eta(a_2 \sin \varphi_1 + b_2 \cos \varphi_1) + \xi \eta(a_1 b_2 - a_2 b_1)], \\ 2W(\xi, \eta) &= F(\xi, \eta) + \sqrt{F(\xi, \eta)^2 + \delta(Q(\xi, \eta)^2 + \eta^2 P(\eta)^2)}, \end{aligned} \quad (3.3)$$

then by virtue of the relations (3.1)-(3.2) we can obtain the following mapping of the unit square \mathcal{R} onto the geodesic quadrangle \mathcal{P} :

$$x(\xi, \eta) = \frac{Q(\xi, \eta)}{2W(\xi, \eta)}, \quad y(\xi, \eta) = \frac{\eta \cdot P(\xi)}{2W(\xi, \eta)}. \quad (3.4)$$

Note that if the intersection of geodesic bundles (3.2) and (3.1) is a geodesic convex quadrangle then for all $\xi \in [0, 1]$ the inequality $P(\xi) > 0$ holds and the mapping (3.4) is a quasi-isometric one.

From (3.4) we have by differentiation

$$\begin{aligned} x_\xi &= \mathcal{A}[(1 + \eta b_2)W + \delta \eta^2 c_2 P]z, & x_\eta &= P[(b_1 \xi - \sin \varphi_1)W - \delta \xi \eta c_1 P]z, \\ y_\xi &= \eta \cdot \mathcal{A}(a_2 W - \delta c_2 Q)z, & y_\eta &= P[(\cos \varphi_1 + \xi a_1)W + \delta \xi c_1 Q]z, \end{aligned} \quad (3.5)$$

where $P = P(\xi)$, $Q = Q(\xi, \eta)$, $F = F(\xi, \eta)$ and $W = W(\xi, \eta)$ were defined by (3.3) and

$$\begin{aligned} \mathcal{A} &= c_1 \cos \varphi_1 + \eta [(a_1 c_2 + a_2 c_1) \sin \varphi_1 + (b_1 c_2 + b_2 c_1) \cos \varphi_1], \\ 4z &= W^{-2} [F^2 + \delta(Q^2 + \eta^2 P^2)]^{-1/2}. \end{aligned}$$

We should mention that the mapping (3.4) generates a class of conformally equivalent Riemannian metrics. Since any two conformally equivalent metrics lead to the same Beltrami system, we can take in the capacity of a representative the metric with following coefficients

$$\begin{aligned} g_{11} &= \mathcal{A}^2\{[(1 + \eta b_2)W + \delta\eta^2 c_2 P]^2 + \eta^2(a_2 W - \delta c_2 Q)^2\}, \\ g_{22} &= P^2\{[(b_1 \xi - \sin \varphi_1)W - \delta\xi\eta c_1 P]^2 + [(\cos \varphi_1 + \xi a_1)W + \delta\xi c_1 Q]^2\}, \\ g_{12} &= \mathcal{A}P\{[(1 + \eta b_2)W + \delta\eta^2 c_2 P][(b_1 \xi - \sin \varphi_1)W - \delta\xi\eta c_1 P] \\ &\quad + \eta^2(a_2 W - \delta c_2 Q)[(\cos \varphi_1 + \xi a_1)W + \delta\xi c_1 Q]\}. \end{aligned} \quad (3.6)$$

Below we provide the examples where we used the simplest quasi-isometric parameterization in order to construct a grid inside four geodesic quadrangles on the surface of the positive constant curvature with the same angles and conformal module \mathcal{M} , but with different enumerations of the sides and angles.

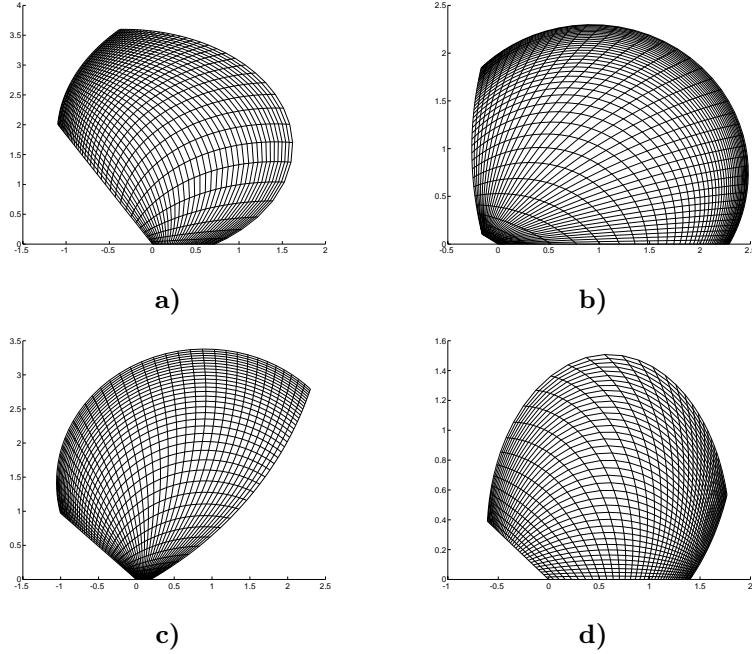


Figure 3. The simplest quasi-isometric parameterization of \mathcal{P}

3.2 “Even” quasi-isometric parameterization

Note that in the transformation (3.4) we can take in the capacity of ξ and η two arbitrary monotonically increasing functions $\xi(\xi_1)$ and $\eta(\eta_1)$ satisfying the conditions $\xi(0) = \eta(0) = 0$ and $\xi(1) = \eta(1) = 1$. In particular, we can choose $\xi = \xi(\xi_1)$ and $\eta = \eta(\eta_1)$ in such a way that under the mapping (3.4) the uniform distribution of points on the lower and the left sides of the unit square holds, i.e., the distribution of points on sides A_1 and A_4 of the geodesic quadrangle \mathcal{P}

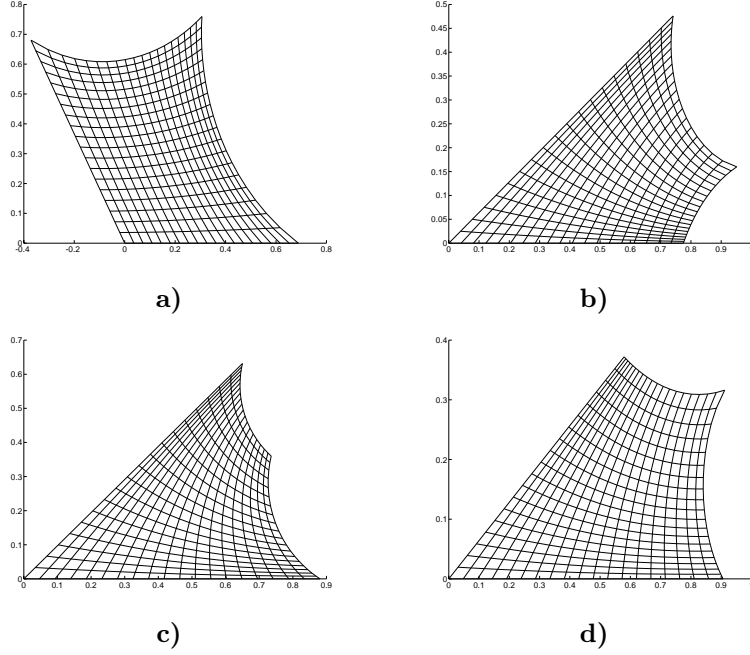


Figure 4. The “even” quasi-isometric parameterization.

is also uniform in a sense of the Euclidean distance. In order to obtain such a mapping it is sufficient to choose $\xi(\xi_1)$ and $\eta(\eta_1)$ as follows:

$$\xi(\xi_1) = \frac{\xi_1 r_1 \cos \varphi_1}{c_1(1 - \delta \xi_1^2 r_1^2) - a_1 \xi_1 r_1},$$

$$\eta(\eta_1) = \frac{\eta_1 r_4 \cos \varphi_1}{c_2(1 - \delta \eta_1^2 r_4^2) - \eta_1 r_4 (a_1 \sin \varphi_1 + b_2 \cos \varphi_1)}.$$

Examples of curvilinear coordinate systems in the geodesic quadrangle on a surface of the constant negative curvature generated by means of “even” quasi-isometric parameterization are given below.

Figure 4 represents four geodesic quadrangles with the same angles and the same “Euclidean” lengths of sides and, consequently, the same conformal module \mathcal{M} . In the Figure 4a is shown the geodesic quadrangle with parameters $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = (0.5, -1.0, -0.8, -1.0)$.

Note that the grids in the geodesic quadrangle \mathcal{P} constructed by the “even” and the simplest parameterizations are not invariant under the cyclic permutation vertices of \mathcal{P} in the origin by means of the motion group. If one wants to obtain the grid that is invariant under the motion group, one can look below in the next two sections where we describe the harmonic parameterization.

3.3 The harmonic parameterization of \mathcal{P} on the Euclidean plane

We now obtain parameterizations which are based upon geodesic bundles and have the invariant properties (1.6)-(1.7). We first consider the case $\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4 = 0$, that is, $\delta = 0$, so that geodesics in the metric (2.1) are straight lines defined by the equation $ax + by + c = 0$. Consider \mathcal{P} as an intersection of two bundles, which are described by the following equations:

$$\begin{aligned} x \cos[\varphi_1 - \xi(\varphi_1 + \varphi_2)] + y \sin[\varphi_1 - \xi(\varphi_1 + \varphi_2)] - r_1 \cos(\varphi_2) \frac{\sin[\xi(\varphi_1 + \varphi_2)]}{\sin(\varphi_1 + \varphi_2)} = 0 \\ 0 \leq \xi \leq 1, \end{aligned} \quad (3.7)$$

$$\begin{aligned} x \sin[\eta(\varphi_1 + \varphi_4)] - y \cos[\eta(\varphi_1 + \varphi_4)] + r_4 \cos \varphi_4 \frac{\sin[\eta(\varphi_1 + \varphi_4)]}{\sin(\varphi_1 + \varphi_4)} = 0 \\ 0 \leq \eta \leq 1 \end{aligned} \quad (3.8)$$

In order to solve the system (3.7)–(3.8) we denote by

$$u(\xi, \eta) = \frac{r_1 \cos \varphi_2}{\cos[\varphi_1 - \xi(\varphi_1 + \varphi_2) - \eta(\varphi_1 + \varphi_4)]} \frac{\sin[\xi(\varphi_1 + \varphi_2)]}{\sin(\varphi_1 + \varphi_2)}, \quad (3.9)$$

$$v(\xi, \eta) = \frac{r_4 \cos \varphi_4}{\cos[\varphi_1 - \xi(\varphi_1 + \varphi_2) - \eta(\varphi_1 + \varphi_4)]} \frac{\sin[\eta(\varphi_1 + \varphi_4)]}{\sin(\varphi_1 + \varphi_4)}, \quad (3.10)$$

then

$$x(\xi, \eta) = u(\xi, \eta) \cos[\eta(\varphi_1 + \varphi_4)] - v(\xi, \eta) \sin[\varphi_1 - \xi(\varphi_1 + \varphi_2)], \quad (3.11)$$

$$y(\xi, \eta) = u(\xi, \eta) \sin[\eta(\varphi_1 + \varphi_4)] + v(\xi, \eta) \cos[\varphi_1 - \xi(\varphi_1 + \varphi_2)], \quad (3.12)$$

is the solution.

From (3.11)–(3.12) we have by differentiation

$$\begin{aligned} x_\xi &= \frac{V(\eta) \cos[\eta(\varphi_1 + \varphi_4)]}{\cos^2[\varphi_1 - \xi(\varphi_1 + \varphi_2) - \eta(\varphi_1 + \varphi_4)]}, \\ x_\eta &= -\frac{U(\xi) \sin[\varphi_1 - \xi(\varphi_1 + \varphi_2)]}{\cos^2[\varphi_1 - \xi(\varphi_1 + \varphi_2) - \eta(\varphi_1 + \varphi_4)]}, \\ y_\xi &= -\frac{V(\eta) \sin[\eta(\varphi_1 + \varphi_4)]}{\cos^2[\varphi_1 - \xi(\varphi_1 + \varphi_2) - \eta(\varphi_1 + \varphi_4)]}, \\ y_\eta &= \frac{U(\xi) \cos[\varphi_1 - \xi(\varphi_1 + \varphi_2)]}{\cos^2[\varphi_1 - \xi(\varphi_1 + \varphi_2) - \eta(\varphi_1 + \varphi_4)]}, \end{aligned}$$

where

$$\begin{aligned} U(\xi) &= r_1(\varphi_1 + \varphi_4) \cos \varphi_2 \frac{\sin[\xi(\varphi_1 + \varphi_2)]}{\sin(\varphi_1 + \varphi_2)} \\ &\quad + r_4 \cos \varphi_4 \cos[\varphi_1 - \xi(\varphi_1 + \varphi_2)] \frac{\varphi_1 + \varphi_4}{\sin(\varphi_1 + \varphi_4)}, \end{aligned} \quad (3.13)$$

$$\begin{aligned} V(\eta) &= r_1 \cos \varphi_2 \cos[\varphi_1 - \eta(\varphi_1 + \varphi_4)] \frac{\varphi_1 + \varphi_2}{\sin(\varphi_1 + \varphi_2)} \\ &\quad + r_4(\varphi_1 + \varphi_2) \cos \varphi_4 \frac{\sin[\eta(\varphi_1 + \varphi_4)]}{\sin(\varphi_1 + \varphi_4)}. \end{aligned} \quad (3.14)$$

It can happen that one obtains an indeterminate expression of the form $\sin[\xi(\varphi_1 + \varphi_2)]/\sin(\varphi_1 + \varphi_2)$ as $\varphi_1 + \varphi \rightarrow 0$ which is evidently equal to ξ .

It is easily seen that for all $(\xi, \eta) \in \mathcal{R}$ the inequalities $U(\xi) > 0$ and $V(\eta) > 0$ hold and the mapping (3.11)–(3.12) is a quasi-isometric transformation \mathcal{R} onto \mathcal{P} . This mapping generates a class conformally equivalent Riemannian metrics. In the capacity of a representative of the class of metrics we can take the metric with the following coefficients:

$$\begin{aligned} g_{11} &= V^2(\eta), & g_{22} &= U^2(\xi), \\ g_{12} &= U(\xi)V(\eta)\sin[-\varphi_1 + \xi(\varphi_1 + \varphi_2) + \eta(\varphi_1 + \varphi_4)]. \end{aligned} \quad (3.15)$$

Let us have a uniform grid in ξ, η - plane, where n, m - are the total numbers of grid points to be used in ξ, η directions. Then the harmonic parameterization has a property that under the cyclic permutation $(m, n) \rightarrow (n, m)$, $(\varphi_1, \varphi_2, \varphi_3, \varphi_4) \rightarrow (\varphi_4, \varphi_1, \varphi_2, \varphi_3)$, $(r_1, r_2, r_3, r_4) \rightarrow (r_4, r_1, r_2, r_3)$, the transformation (3.11)–(3.12) results in the same grid in \mathcal{P} .

The metric (3.15) by virtue of its simplicity can be used for the generation of quasi-conformal grids in the physical domain \mathcal{D} with the angles satisfying the inequality $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \neq 2\pi$ as well. For this purpose it is sufficient to substitute in (3.13)–(3.15)

$$\varphi_i = \alpha_i - \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4}, \quad i = 1, \dots, 4.$$

3.4 The harmonic parameterization of \mathcal{P} on a surface of constant curvature

Let \mathcal{P} be a geodesic quadrangle on a surface of the constant curvature $K = 4\delta$, where $\delta = \sin[(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)/2]$. Let us define for $i = 1, 2$ the angles w_i of the intersection of sides A_i and A_{i+2} of the geodesic quadrangle \mathcal{P} , constants Δ_i and functions $SH(\xi w_i)$, $CH(\xi w_i)$:

$$\begin{aligned} \omega_i &= \begin{cases} \operatorname{arccot}(d_i/b_i), & \text{if } l_i \geq 0, \\ \operatorname{coth}^{-1}(d_i/b_i), & \text{if } l_i < 0, \end{cases} & \Delta_i &= \begin{cases} +1, & \text{if } l_i \geq 0, \\ -1, & \text{if } l_i < 0, \end{cases} \\ SH(\xi w_i) &= \begin{cases} \sin(\xi w_i), & \text{if } l_i \geq 0, \\ \sinh(\xi w_i), & \text{if } l_i < 0, \end{cases} \\ CH(\xi w_i) &= \begin{cases} \cos(\xi w_i), & \text{if } l_i \geq 0, \\ \cosh(\xi w_i), & \text{if } l_i < 0 \end{cases} \end{aligned} \quad (3.16)$$

where $l_i = a_i^2 + 4\delta c_i^2$ and

$$\begin{aligned} a_1 &= -\sin(\varphi_1 + \varphi_2) + \delta r_1^2 \sin(\varphi_1 - \varphi_2), \\ a_2 &= \sin(\varphi_1 + \varphi_4) - \delta r_4^2 \sin(\varphi_1 - \varphi_4), \\ b_1 &= \sqrt{|l_1|} \operatorname{sign} \delta, & b_2 &= \sqrt{|l_2|} \operatorname{sign} \delta, \\ c_1 &= r_1 \cos \varphi_2, & c_2 &= r_4 \cos \varphi_4, \\ d_1 &= \cos(\varphi_1 + \varphi_2) - \delta r_1^2 \cos(\varphi_1 - \varphi_2), \\ d_2 &= \cos(\varphi_1 + \varphi_4) - \delta r_4^2 \cos(\varphi_1 - \varphi_4). \end{aligned}$$

Here we suppose that $\text{sign } \delta = 1$ if $\delta = 0$.

Consider \mathcal{P} as an intersection of two geodesic bundles

$$\begin{aligned} C(b_1, a_1, \xi\omega_1)x - S(b_1, a_1, \xi\omega_1)y + c_1SH(\xi\omega_1)[1 - \delta(x^2 + y^2)] &= 0, \\ 0 \leq \xi \leq 1, \end{aligned} \quad (3.17)$$

$$\begin{aligned} a_2SH(\eta\omega_2)x - b_2CH(\eta\omega_2)y + c_2SH(\eta\omega_2)[1 - \delta(x^2 + y^2)] &= 0, \\ 0 \leq \eta \leq 1, \end{aligned} \quad (3.18)$$

where

$$C(b_1, a_1, \xi\omega_1) = -b_1 \cos \varphi_1 CH(\xi\omega_1) + a_1 \sin \varphi_1 SH(\xi\omega_1), \quad (3.19)$$

$$S(b_1, a_1, \xi\omega_1) = b_1 \sin \varphi_1 CH(\xi\omega_1) + a_1 \cos \varphi_1 SH(\xi\omega_1). \quad (3.20)$$

Introducing functions

$$\begin{aligned} \bar{C}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2) &= b_2 CH(\eta w_2) C(b_1, a_1, \xi w_1) \\ &\quad - a_2 SH(\eta w_2) S(b_1, a_1, \xi w_1), \end{aligned} \quad (3.21)$$

$$\begin{aligned} \bar{S}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2) &= b_2 CH(\eta w_2) S(b_1, a_1, \xi w_1) \\ &\quad + a_2 SH(\eta w_2) C(b_1, a_1, \xi w_1). \end{aligned} \quad (3.22)$$

we can write the desired parameterization of \mathcal{P} in the form

$$x(\xi, \eta) \quad (3.23)$$

$$= \frac{2c_2 SH(\eta w_2) S(b_1, a_1, \xi w_1) - 2c_1 b_2 SH(\xi w_1) CH(\eta w_2)}{\bar{C}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2) + \Delta_0 \sqrt{\bar{C}^2(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2) + 4\delta h(\xi, \eta)}}, \quad (3.24)$$

$$y(\xi, \eta)$$

$$= \frac{2SH(\eta w_2)[c_2 C(b_1, a_1, \xi w_1) - c_1 a_2 SH(\xi w_1)]}{\bar{C}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2) + \Delta_0 \sqrt{\bar{C}^2(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2) + 4\delta h(\xi, \eta)}},$$

where we have set

$$\Delta_0 = \text{sign} \{SH(\eta w_2)[c_2 C(b_1, a_1, \xi w_1) - c_1 a_2 SH(\xi w_1)]\},$$

$$\begin{aligned} h(\xi, \eta) &= c_1^2 SH^2(\xi w_1)[a_2^2 SH^2(\eta w_2) + b_2^2 CH^2(\eta w_2)] \\ &\quad + c_2^2 SH^2(\eta w_2)[a_1^2 SH^2(\xi w_1) + b_1^2 CH^2(\xi w_1)] \\ &\quad - 2c_1 c_2 SH(\xi w_1) SH(\eta w_2) \bar{S}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2). \end{aligned}$$

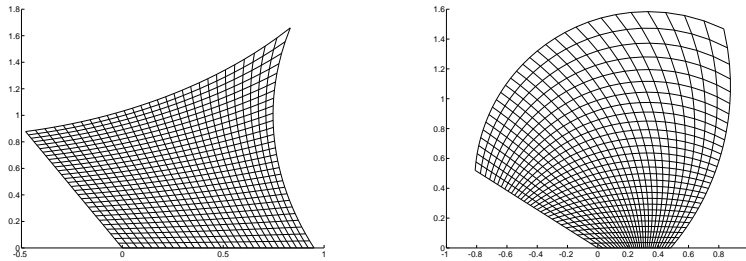


Figure 5. An example of “harmonic” parameterization.

3.5 Riemannian metric of the harmonic parameterization of \mathcal{P}

The calculation of x_ξ and y_ξ is based on the relationships:

$$\begin{aligned}\frac{\partial}{\partial \xi} C(b_1, a_1, \xi w_1) &= w_1 S(a_1, b_1, \Delta_1 \xi w_1), \\ \frac{\partial}{\partial \xi} S(b_1, a_1, \xi w_1) &= -w_1 C(a_1, b_1, \Delta_1 \xi w_1),\end{aligned}\quad (3.25)$$

$$\begin{aligned}\frac{\partial}{\partial \xi} \bar{C}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2) &= w_1 \bar{S}(b_2, a_2, a_1, b_1, \Delta_1 \xi w_1, \eta w_2), \\ \frac{\partial}{\partial \xi} \bar{S}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2) &= -w_1 \bar{C}(b_2, a_2, a_1, b_1, \Delta_1 \xi w_1, \eta w_2), \\ \frac{\partial}{\partial \eta} \bar{C}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2) &= -w_2 \bar{S}(a_2, b_2, b_1, a_1, \xi w_1, \Delta_2 \eta w_2).\end{aligned}\quad (3.26)$$

We use the abbreviations

$$\begin{aligned}\bar{C} &= \bar{C}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2), \quad \bar{S} = \bar{S}(b_2, a_2, b_1, a_1, \xi w_1, \eta w_2), \\ h &= h(\xi, \eta), \quad \bar{C}_1 = \bar{C}(b_2, a_2, a_1, b_1, \Delta_1 \xi w_1, \eta w_2), \\ \bar{C}_2 &= \bar{C}(a_2, b_2, b_1, a_1, \xi w_1, \Delta_2 \eta w_2), \quad z_0 = \Delta_0 \sqrt{C^2 + 4\delta h}, \\ \bar{S}_1 &= \bar{S}(b_2, a_2, a_1, b_1, \Delta_1 \xi w_1, \eta w_2), \quad z_1 = \bar{C} + z_0, \\ \bar{S}_2 &= \bar{S}(a_2, b_2, b_1, a_1, \xi w_1, \Delta_2 \eta w_2),\end{aligned}$$

and find

$$\begin{aligned}\frac{\partial}{\partial \xi} [C(b_1, a_1, \xi w_1)/z_1] &= w_1 [S(a_1, b_1, \Delta_1 \xi w_1) \bar{C} - C(b_1, a_1, \xi w_1) \bar{S}] / (z_1 z_0) \\ &\quad + 2\delta [2w_1 h S(a_1, b_1, \Delta_1 \xi w_1) - h_\xi \bar{C}(b_1, a_1, \xi w_1)] / (z_1^2 z_0), \\ \frac{\partial}{\partial \xi} [SH(\xi w_1)/z_1] &= w_1 [CH(\xi w_1) \bar{C} - SH(\xi w_1) \bar{S}_1] / (z_1 z_0) \\ &\quad + 2\delta [2w_1 h CH(\xi w_1) - h_\xi SH(\xi w_1)] / (z_1^2 z_0), \\ \frac{\partial}{\partial \xi} [S(b_1, a_1, \xi w_1)/z_1] &= -w_1 [C(a_1, b_1, \Delta_1 \xi w_1) \bar{C} + S(b_1, a_1, \xi w_1) \bar{S}_1] / (z_1 z_0) \\ &\quad - 2\delta [2w_1 h C(a_1, b_1, \Delta_1 \xi w_1) + h_\xi S(b_1, a_1, \xi w_1)] / (z_1^2 z_0).\end{aligned}$$

It is easily seen that

$$\begin{aligned}S(a_1, b_1, \Delta_1 \xi w_1) \bar{C} - C(b_1, a_1, \xi w_1) \bar{S} &= -a_1 b_1 a_2 SH(\eta w_1), \\ CH(\xi w_1) \bar{C} - SH(\xi w_1) \bar{S}_1 &= b_1 C(b_2, a_2, -\eta w_2), \\ C(a_1, b_1, \Delta_1 \xi w_1) \bar{C} + S(b_1, a_1, \xi w_1) \bar{S}_1 &= a_1 b_1 b_2 CH(\eta w_2),\end{aligned}$$

and we therefore have

$$\begin{aligned}
x_\xi &= -2b_1b_2w_1CH(\eta w_2)[c_1C(b_2, a_2, -\eta w_2) + c_2a_1SH(\eta w_2)]/(z_1z_0) \\
&+ 4\delta \left\{ 2w_1h[c_1b_2CH(\xi w_1)CH(\eta w_2) + c_2C(a_1, b_1, \Delta_1\xi w_1)SH(\eta w_2)] \right. \\
&\left. + h_\xi[-c_1b_2SH(\xi w_1)CH(\eta w_2) + c_2S(b_1, a_1, \xi w_1)SH(\eta w_2)] \right\}/(z_1^2z_0),
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
y_\xi &= -2a_2b_1w_1SH(\eta w_2)[c_1C(b_2, a_2, -\eta w_2) + c_2a_1SH(\eta w_2)]/(z_1z_0) \\
&- 4\delta SH(\eta w_2) \{ 2w_1h[c_1a_2CH(\xi w_1) - c_2S(a_1, b_1, \Delta_1\xi w_1)] - \\
&- h_\xi[c_1a_2SH(\xi w_1) - c_2C(b_1, a_1, \xi w_1)] \}/(z_1^2z_0),
\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}
h_\xi &= 2w_1 \left\{ c_1^2SH(\xi w_1)CH(\xi w_1)[b_2^2CH^2(\eta w_2) + a_2^2SH(\eta w_2)] \right. \\
&+ c_2^2SH^2(\eta w_2)SH(\xi w_1)CH(\xi w_1)[a_1^2 - \Delta_1b_1^2] \\
&\left. - c_1c_2SH(\eta w_2)[CH(\xi w_1)\bar{S}_1 - SH(\xi w_1)\bar{C}_1] \right\}.
\end{aligned}$$

Next we calculate x_η and y_η . Using (3.26) we find

$$\begin{aligned}
\frac{\partial}{\partial \eta} [SH(\eta w_2)/z_1] &= w_2 [CH(\eta w_2)\bar{C} + SH(\eta w_2)\bar{S}_2]/(z_1z_0) \\
&+ 2\delta [2w_2hCH(\eta w_2) - h_\eta SH(\eta w_2)]/(z_1^2z_0), \\
\frac{\partial}{\partial \eta} [CH(\eta w_2)/z_1] &= -w_2 [\Delta_2SH(\eta w_2)\bar{C} - CH(\eta w_2)\bar{S}_2]/(z_1z_0) \\
&- 2\delta [2\Delta_2w_2hSH(\eta w_2) + h_\eta CH(\eta w_2)]/(z_1^2z_0).
\end{aligned}$$

It is also clear that

$$\begin{aligned}
CH(\eta w_2)\bar{C} + SH(\eta w_2)\bar{S}_2 &= b_2C(b_1, a_1, \xi w_1), \\
\Delta_2SH(\eta w_2)\bar{C} - CH(\eta w_2)\bar{S}_2 &= -a_2S(b_1, a_1, \xi w_1),
\end{aligned}$$

and we therefore have

$$\begin{aligned}
x_\eta &= 2b_2w_2S(b_1, a_1, \xi w_1)[-c_1a_2SH(\xi w_1) + c_2C(b_1, a_1, \xi w_1)]/(z_1z_0) \\
&+ 4\delta \{ 2w_2h[\Delta_2c_1b_2SH(\xi w_1)SH(\eta w_2) + c_2S(b_1, a_1, \xi w_1)CH(\eta w_2)] \\
&+ h_\eta[c_1b_2SH(\xi w_1)CH(\eta w_2) - c_2S(b_1, a_1, \xi w_1)SH(\eta w_2)] \}/(z_1^2z_0), \\
y_\eta &= [c_2C(b_1, a_1, \xi w_1) - c_1a_2SH(\xi w_1)] \left\{ 2b_2w_2C(b_1, a_1, \xi w_1)/(z_1z_0) \right. \\
&\left. + 4\delta [2w_2hCH(\eta w_2) - h_\eta SH(\eta w_2)] \right\}/(z_1^2z_0),
\end{aligned} \tag{3.30}$$

where

$$\begin{aligned} h_\eta = & 2w_2 \left\{ c_1^2 SH^2(\xi w_1) SH(\eta w_2) CH(\eta w_2) [a_2^2 - \Delta_2 b_2^2] \right. \\ & + c_2^2 SH^2(\eta w_2) CH(\eta w_2) [b_1^2 CH^2(\xi w_1) + a_1^2 SH^2(\xi w_1)] \\ & \left. - c_1 c_2 SH(\xi w_1) [CH(\eta w_2) \bar{S} + SH(\eta w_2) \bar{C}_2] \right\}. \end{aligned}$$

Thus, introducing functions (3.16)–(3.20) and (3.21), (3.22) we have found $x_\xi, x_\eta, y_\xi, y_\eta$ and therefore coefficients of the metric tensor

$$g_{11} = x_\xi^2 + y_\xi^2, \quad g_{22} = x_\eta^2 + y_\eta^2, \quad g_{12} = x_\xi x_\eta + y_\xi y_\eta \quad (3.31)$$

for all $(\xi, \eta) \in \mathcal{R}$.

3.6 Fixed boundary points

Consider a geodesic quadrangle \mathcal{P} with sides $A_i, i = 1, \dots, 4$ and its four boundary points $(x_d, y_d) \in A_1, (x_r, y_r) \in A_2, (x_u, y_u) \in A_3$ and $(x_l, y_l) \in A_4$ such that A_4 or A_1 will not contain both points (x_d, y_d) and (x_l, y_l) .

Then equations of the geodesics passing through (x_d, y_d) and (x_u, y_u) , and through (x_l, y_l) and (x_r, y_r) respectively, are as follows:

$$a_v x + b_v y - [1 - \delta(x^2 + y^2)] = 0, \quad (3.32)$$

$$a_h x + b_h y + [1 - \delta(x^2 + y^2)] = 0. \quad (3.33)$$

Now we will find the point $x = x_{vh}, y = y_{vh}$ of intersection of (3.32) and (3.33). Denoting

$$a = \frac{a_h b_v - a_v b_h}{2}, \quad b = (a_h + a_v)^2 + (b_h + b_v)^2,$$

we find:

$$x_{vh} = \frac{-(b_h + b_v)}{a + \sqrt{a^2 + \delta b}}, \quad y_{vh} = \frac{a_h + a_v}{a + \sqrt{a^2 + \delta b}}. \quad (3.34)$$

In order to calculate the coefficients g_{ik} of the metric tensor of the mentioned quasi-isometric transformation (3.34) that maps \mathcal{R} onto \mathcal{P} we can use different explicit formulas. For this purpose we place one of the vertices of a cell of the grid in the origin by means of the motion group (2.2) and use formulas (3.6) or (3.27)–(3.28), (3.29)–(3.31).

4 Variational method for the generation of quasi-isometric grids

4.1 A certain class of Riemannian manifolds

Consider a geodesic quadrangle \mathcal{P} with given angles $\alpha_1, \dots, \alpha_4$ and sides of Euclidean lengths r_1, \dots, r_4 . Let $\sigma \in \Sigma_4, \alpha = (\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(4)})$, and $r =$

$r_{\sigma(1)}$. Consider one-parameter family of geodesic quadrangles \mathcal{P}_r with angles $\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(4)}$ and parameter $r \in (r_{\sigma(1)}^{\min}, r_{\sigma(1)}^{\max})$. Every mapping of the form (3.4), (3.23)–(3.24) or (3.34) of the unit square \mathcal{R} onto a quadrangle \mathcal{P}_r has the following form:

$$x = x(\xi, \eta, r), \quad y = y(\xi, \eta, r), \quad (4.1)$$

and the mapping generates the class of conformally equivalent metrics. Let the following metric be a representative of the class:

$$ds^2 = g_{11}(\xi, \eta, r)d\xi^2 + 2g_{12}(\xi, \eta, r)d\xi d\eta + g_{22}(\xi, \eta, r)d\eta^2. \quad (4.2)$$

So we can define the Riemannian manifold $\mathcal{N}(g_{ij}(\xi, \eta, r), \mathcal{R})$ with the coordinate domain \mathcal{R} , the metric tensor g_{ij} and the parameter r . Then the Riemann mapping theorem implies the unique existence of the parameter r^* such that the Riemannian manifold $\mathcal{N}(g_{ij}(\xi, \eta, r^*), \mathcal{R})$ can be mapped onto given curvilinear quadrangle \mathcal{D} conformally with respect to the metric (4.2). The mapping is quasi-isometric if all sides of \mathcal{D} are smooth enough (belong to C^2) and in addition \mathcal{P}_{r^*} and \mathcal{D} have the same angles [19, 14].

Thus the main problem consists of finding the parameter r^* and a mapping $X^*(\xi, \eta), Y^*(\xi, \eta)$ such that this mapping is conformal with respect to the metric (4.2) with metric tensor $g_{ij}(\xi, \eta, r^*)$.

4.2 Functional Φ

Using the the class of functions $g_{ij} = g_{ij}(\xi, \eta, r)$ we can define the class of admitted functions $A = A(\xi, \eta, r)$, $B = B(\xi, \eta, r)$, $C = C(\xi, \eta, r)$, where

$$\begin{aligned} A(\xi, \eta, r) &= \frac{g_{22}(\xi, \eta, r)}{g}, & B(\xi, \eta, r) &= \frac{g_{12}(\xi, \eta, r)}{g}, \\ C(\xi, \eta, r) &= \frac{g_{11}(\xi, \eta, r)}{g}, & g^2 &= g_{11}g_{22} - g_{12}^2. \end{aligned}$$

Let us also introduce the class of admitted mappings $X = X(\xi, \eta)$, $Y = Y(\xi, \eta)$ of the computational region \mathcal{R} onto \mathcal{D} that has the following properties:

1. $X(\xi, \eta)$, $Y(\xi, \eta)$ define a quasi-isometric correspondence between $\partial\mathcal{R}$ and $\partial\mathcal{D}$;
2. $X(\xi, \eta)$, $Y(\xi, \eta)$ can be continued inside \mathcal{R} in such a way that the functional

$$\begin{aligned} \Phi(X, Y, r) &= \int_0^1 \int_0^1 A(\xi, \eta, r)[X_\xi^2 + Y_\xi^2] - 2B(\xi, \eta, r)[X_\xi X_\eta + Y_\xi Y_\eta] \\ &\quad + C(\xi, \eta, r)[X_\eta^2 + Y_\eta^2] d\xi d\eta, \end{aligned} \quad (4.3)$$

is bounded.

The minimum value of the functional is equal to the area $S_{\mathcal{D}}$ of the domain \mathcal{D} , as it was shown in [10]. Functions $X^*(\xi, \eta)$, $Y^*(\xi, \eta)$ from the described class and the number r^* that provide the minimum of the functional Φ , give us the desired mapping of the Riemannian manifold $\mathcal{N}(g_{ij}(\xi, \eta, r), \mathcal{R})$ onto \mathcal{D} .

4.3 The variational principle

In order to find $X^*(\xi, \eta)$, $Y^*(\xi, \eta)$ and r^* we will construct a minimizing sequence $\{X^n, Y^n, r^n\}$ that has the following property:

$$\Phi(X^{n+1}, Y^{n+1}, r^{n+1}) < \Phi(X^n, Y^n, r^n), \quad \lim_{n \rightarrow \infty} \mathcal{M}(\mathcal{P}_{r^n}) = \mathcal{M}(\mathcal{P}_{r^*}) = \mathcal{M}(\mathcal{D}).$$

To begin the minimization process we assume that the functions $X^n(\xi, \eta)$, $Y^n(\xi, \eta)$ and r^n are known to us.

On the first step, using the Montel variational principle from the conformal mapping theory, we obtain r^{n+1} such that

$$\Phi(X^n, Y^n, r^{n+1}) < \Phi(X^n, Y^n, r^n).$$

In particular,

$$r^{n+1} = r^n + \frac{\Lambda(X^n, Y^n, r^n) - V(X^n, Y^n, r^n)}{\nu\Lambda(X^n, Y^n, r^n) + \mu V(X^n, Y^n, r^n)}, \quad (4.4)$$

where

$$\begin{aligned} \Lambda^2(X, Y, r) &= \int_0^1 \int_0^1 A(\xi, \eta, r) [X_\xi^2 + Y_\xi^2] d\xi d\eta, \\ V^2(X, Y, r) &= \int_0^1 \int_0^1 C(\xi, \eta, r) [X_\eta^2 + Y_\eta^2] d\xi d\eta. \end{aligned}$$

Then we calculate the new coefficients of the functional

$$A = A(\xi, \eta, r^{n+1}), \quad B = B(\xi, \eta, r^{n+1}), \quad C = C(\xi, \eta, r^{n+1}).$$

In (4.4) we can use $\mu = 1/\sqrt{g_{11}(1, 0, r)}$ and $\nu = -\frac{dr_{\sigma(4)}}{dr}/\sqrt{g_{22}(0, 1, r)}$ [7].

On the second step for computation of new approximation X^{n+1} , Y^{n+1} such that

$$\Phi(X^{n+1}, Y^{n+1}, r^{n+1}) < \Phi(X^n, Y^n, r^{n+1}),$$

we use obtained A , B , C as coefficients of the elliptic equations that represent the variational Euler – Lagrange equations for the functional (4.3) being minimized on X and Y :

$$-\frac{\partial}{\partial \xi} A \frac{\partial X}{\partial \xi} - \frac{\partial}{\partial \eta} C \frac{\partial X}{\partial \eta} + \left(\frac{\partial}{\partial \xi} B \frac{\partial X}{\partial \eta} + \frac{\partial}{\partial \eta} B \frac{\partial X}{\partial \xi} \right) = 0, \quad (4.5)$$

$$-\frac{\partial}{\partial \xi} A \frac{\partial Y}{\partial \xi} - \frac{\partial}{\partial \eta} C \frac{\partial Y}{\partial \eta} + \left(\frac{\partial}{\partial \xi} B \frac{\partial Y}{\partial \eta} + \frac{\partial}{\partial \eta} B \frac{\partial Y}{\partial \xi} \right) = 0. \quad (4.6)$$

The solution of the system (4.5)–(4.6) with appropriate boundary conditions can be used as a new approximation X^{n+1} , Y^{n+1}

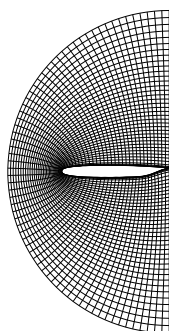


Figure 6.

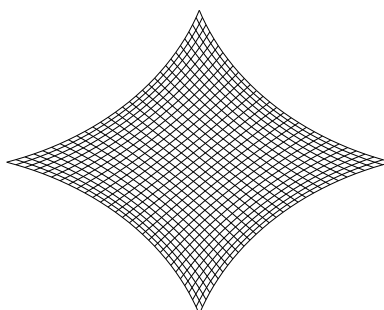


Figure 7.

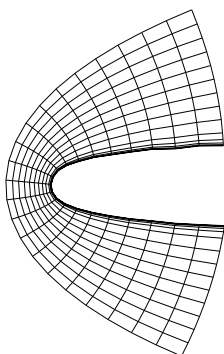


Figure 8.

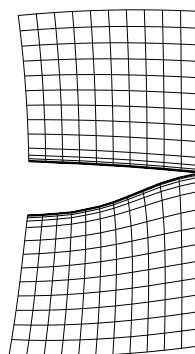


Figure 9.

Steps 1 and 2 are to be repeated till the desired accuracy of determining the solution of the variational problem is achieved.

Note that the formula (4.4) is invariant with respect to the metric chosen, i.e., the iteration process does not depend on the way the metric depend on the varying parameter. This feature is one of major developments in our paper comparing with works [10, 5].

5 Grid examples

In Figure 6, the boundary points are free on the straight boundaries, fixed on the outer half-circle and "same as opposite" on the wing surface. In figure 7, all boundary points are fixed. In figure 8, the grid is nearly orthogonal. And figure 9 demonstrates that the grid cells remain parallelograms near corners as the grid is refined.

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