

Rotationally Symmetric Deformations of a Spherical Cap *

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Abstract

We prove the existence and uniqueness of rotationally symmetric solutions to a nonlinear boundary value problem representing the elastic deformation of a spherical cap.

1 Introduction

Suppose that a rotationally symmetric membrane is subjected to a vertical pressure as well as a prescribed displacement at its boundary. In [6], Wayne Dickey derived a model to describe the deformation of this membrane. Under the assumption of small strains, i.e. Hooke's laws, Dickey derived the following problem.

$$\begin{aligned} \left(\frac{r}{m}(rT)'\right)' &= \frac{mT}{\sqrt{G^2+T^2}} - 1 + mT + \frac{\nu(r^2G^2)'}{2r\sqrt{G^2+T^2}} \text{ in } (0, 1), \\ |T(0)| &< \infty, \text{ and} \\ \left(\frac{(rT)'}{m} - \nu\sqrt{G^2+T^2}\right)_{r=1} &= \mu, \end{aligned} \tag{1}$$

where $(r, z(r))$ represents the profile of the undeformed membrane in cylindrical coordinates, $m = \sqrt{1+(z')^2}$, $G = \frac{1}{Ehr} \int_0^r \rho m P d\rho$, E is Young's modulus, h is the thickness, $P(r)$ is the pressure, ν is the Poisson ratio, and μ is the displacement at the boundary. T is an auxiliary function that can be used to derive the exact shape of the deformed membrane, as well as the internal stresses and strains. For the case of small pressure and shallow caps T can be thought of as a rescaled radial stress, and the substitution $T = \sigma_r$ is often used in the literature. (See [6].)

This problem is usually studied with additional simplifying assumptions. Specifically, assuming a shallow membrane with undeformed shape $z(r) = c(1 - r^\gamma)$, where $\gamma > 1$ and $0 \leq c \ll 1$, and assuming that the membrane is subjected

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to a small constant pressure with the property that $\lim_{G \rightarrow 0} \frac{G}{T} = -z'$, then Dickey derived an approximate theory where T must satisfy

$$\begin{aligned} r^2 T'' + 3rT' &= \frac{\lambda^2 r^{2\gamma-2}}{2} + \frac{\beta \nu r^2}{T} - \frac{r^2}{8T^2} \text{ in } (0, 1), \\ |T(0)| &< \infty, \text{ and} \\ T'(1) + (1 - \nu)T(1) &= \mu, \end{aligned} \tag{2}$$

where λ and β are positive constants depending on P, h , and E . In the case that $\gamma = 2$ the model serves as a good approximation of the spherical cap. The assumption on $\frac{G}{T}$ is motivated by a search for deformations that are small when pressure is small. This problem has been studied extensively in recent literature. A relatively complete treatment of the problem is contained in [12, 11, 2, 3], although there are still some open questions, such as the existence of multiple solutions under certain conditions, and the stability of radial solutions. Earlier fundamental work is contained in papers such as [13, 10, 5, 7, 8, 1].

In this paper we examine a special case of the more general model (1), with the assumptions that $\nu = 0$ and that $z(r) \in C^2[0, 1]$ is a positive decreasing concave function with $z'(0) = 0$ and $z''(0) < 0$. Observe that this includes spherical caps as a special case. The model reduces to

$$\begin{aligned} \left(\frac{r}{m}(rT)'\right)' &= \frac{mT}{\sqrt{G^2+T^2}} - 1 + mT \text{ in } (0, 1), \\ |T(0)| &< \infty, \text{ and} \\ T'(1) + T(1) &= m(1)\mu. \end{aligned} \tag{3}$$

It should be noted that the derivation of this model assumes small strains and stresses. As a consequence the physical relevance of the model is lost for large values of $|\mu|$. Also, the assumption that $\nu = 0$ indicates that we are modeling a material that is easily compressed.

We will show that problem (3) has a unique solution for all $\mu \in \mathbb{R}$. This conclusion agrees well with recent investigations of the approximate problem (2). Our primary tool is the technique of upper and lower solutions, and in some ways our methods simplify the shooting arguments used in the references mentioned above.

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2 Preliminaries

In this section we rewrite (3) in a convenient form, and we establish the necessary framework for an upper and lower solutions argument. Specifically, we must understand some of the geometry associated with the nonlinear term, and then we must establish some regularity estimates and a comparison theorem for the differential operator.

Problem (3) can be rewritten as

$$\left(\frac{r^3}{m}T'\right)' = r \left[\frac{mT}{\sqrt{G^2+T^2}} - 1 + kT \right] \text{ in } (0, 1),$$

$$\begin{aligned} |T(0)| &< \infty, \text{ and} \\ T'(1) + T(1) &= m(1)\mu, \end{aligned} \quad (4)$$

where $k = (m - (\frac{r}{m})')$. For notational convenience in what follows we let

$$LT := \left(\frac{r^3}{m} T' \right)', \text{ and } f(r, T) := \frac{mT}{\sqrt{G^2 + T^2}} - 1 + kT.$$

We begin our analysis of (4) by a careful examination of $f(r, T)$. It is particularly important to describe the level set $f^{-1}(0)$, and to determine how $f(r, T)$ behaves near $r = 0$.

First, observe that

$$m' = \frac{z'z''}{m}.$$

Thus m is strictly increasing with $\lim_{r \rightarrow 0} \frac{m'}{r} = (z''(0))^2$ and $\lim_{r \rightarrow 0} m(r) = 1$. Next we see that

$$k = \frac{(z')^2 + (z')^4 + rz'z''}{m^3}.$$

Therefore $k(r) > 0$ for $r > 0$ and $\lim_{r \rightarrow 0} \frac{k}{r^2} = 2(z''(0))^2$. Since P is assumed to be constant, G can be simplified as follows.

$$G = \frac{P}{Ehr} \int_0^r m\rho d\rho.$$

And so $G \in C^1[0, 1]$ with

$$\begin{aligned} \frac{Pr}{2Eh} &\leq G \leq \frac{Prm(r)}{2Eh}, \text{ and} \\ G'(0) &= \frac{P}{2Eh}. \end{aligned}$$

It is certainly possible to obtain more detailed information about the properties of m, k , and G , but the given information will suffice.

The following lemmas describe the level set $f^{-1}(0)$.

Lemma 1 $f^{-1}(0) \cap ((0, 1] \times \mathbb{R})$ is the graph of a strictly positive smooth function $\tau(r)$.

Proof: A straight forward computation gives

$$f_T = \frac{mG^2}{(G^2 + T^2)^{3/2}} + k.$$

Since $f_T \geq k(r) > 0$ for all $r > 0$, it follows that there is a function $\tau \in C^1(0, 1]$ such that $\tau(r)$ is the unique solution of $f(r, \tau(r)) = 0$ and $\tau' = -f_r/f_T$. It is clear that $f(r, T) \leq -1$ for $T \leq 0$, so $\tau(r) > 0$. \diamond

It should be clear that $f(r, T) < 0$ for $T < \tau(r)$ and $f(r, T) > 0$ for $T > \tau(r)$.

Lemma 2 $\lim_{r \rightarrow 0} \tau(r)$ exists and is strictly positive.

Proof: We begin by claiming that $\lim_{r \rightarrow 0} k(r)\tau(r) = 0$. If not, there must be a sequence $\{r_n\}$ such that $r_n \rightarrow 0$ and $\liminf_{n \rightarrow \infty} k(r_n)\tau(r_n) > 0$. Since $k(r_n) \rightarrow 0$, we know that $\tau(r_n) \rightarrow \infty$. Since $m(r_n) \rightarrow 1$ and $G(r_n) \rightarrow 0$, it follows that

$$\frac{m(r_n)\tau(r_n)}{\sqrt{G(r_n)^2 + \tau(r_n)^2}} \rightarrow 1$$

, and, since $f(r_n, \tau(r_n)) = 0$, we get $k(r_n)\tau(r_n) \rightarrow 0$, a contradiction.

Now rewrite the equation $f(\tau, r) = 0$ as

$$m^2\tau^2 = (1 - k\tau)^2(G^2 + \tau^2).$$

After a rearrangement of terms we get

$$((z')^2 + 2k\tau - k^2\tau^2)\tau^2 = G^2(1 - k\tau)^2.$$

If we divide through by $r^2 z''(0)^2$ we get a polynomial in τ whose coefficients converge as $r \rightarrow 0$ yielding the limiting polynomial

$$4\tau^3 + \tau^2 = \left(\frac{P}{2Ehz''(0)} \right)^2.$$

It is not hard to see that $\tau(r)$ must converge to the unique positive root, τ_0 , of the limiting polynomial as $r \rightarrow 0$. \diamond

The next lemmas describe $f(r, T)$ near $r = 0$.

Lemma 3 *Let $D_\epsilon := \{(r, T) : T \geq \sqrt{\epsilon^2 - r^2} \text{ for } r < \epsilon\}$. There is a continuous function $h : D_\epsilon \rightarrow \mathbb{R}$, which is continuously differentiable in T , such that $f(r, T) = r^2 h(r, T)$ in D_ϵ .*

Proof: This assertion is clear for $r \neq 0$. For points where $r = 0$ and $T \geq \epsilon$ the result is a consequence of the following limits.

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{f(r, T)}{r^2} &= \lim_{r \rightarrow 0} \frac{1}{r^2} \left(\frac{mT}{\sqrt{G^2 + T^2}} - 1 + kT \right) \\ &= \lim_{r \rightarrow 0} \frac{1}{r^2} \left(\frac{m - \sqrt{1 + \frac{G^2}{T^2}}}{\sqrt{1 + \frac{G^2}{T^2}}} \right) + 2(z''(0))^2 T \\ &= \lim_{r \rightarrow 0} \frac{1}{r^2} \left(\frac{1 + \frac{1}{2}(z')^2 + o((z')^2) - 1 - \frac{1}{2}\frac{G^2}{T^2} - o(\frac{G^2}{T^2})}{\sqrt{1 + \frac{G^2}{T^2}}} \right) \\ &\quad + 2(z''(0))^2 T \\ &= \frac{1}{2}(z''(0))^2 - \frac{1}{2} \left(\frac{P}{2Eh} \right)^2 T^{-2} + 2(z''(0))^2 T, \end{aligned}$$

and similarly

$$\lim_{r \rightarrow 0} \frac{f_T(r, T)}{r^2} = \lim_{r \rightarrow 0} \frac{1}{r^2} \left(\frac{mG^2}{(G^2 + T^2)^{\frac{3}{2}}} + k \right) = \left(\frac{P}{2Eh} \right)^2 T^{-3} + 2(z''(0))^2.$$

◇

The properties established so far are elementary but have important consequences. Knowledge of $f^{-1}(0)$ allows us to choose constant upper and lower solutions when $\mu > 0$. The monotonicity of $f(r, \cdot)$ will imply the uniqueness of solutions. The previous lemma allows us to think of the differential equation as $LT = r^3 h(r, T)$, which has useful consequences in terms of regularity, as we shall see below. Moreover, it will allow us to show that, on any compact subset of D_ϵ , there is a $\delta > 0$ such that $rf(r, T) - \delta r^3 T = r^3(h(r, T) - \delta T)$ is decreasing as a function of T . This last detail will be crucial to the success of our iteration scheme.

This is a good point for a remark on the derivation of the approximate problem (2), and its relation to problem (4). In the previous proof we wrote

$$f(r, T) = \frac{\frac{1}{2}(z')^2 - \frac{1}{2}\frac{G^2}{T^2} + o((z')^2) + o(\frac{G^2}{T^2})}{\sqrt{1 + \frac{G^2}{T^2}}} + kT.$$

Under the assumptions of small pressure and shallow caps, as in (2), it is quite reasonable to drop the $o((z')^2)$ and $o(\frac{G^2}{T^2})$ terms, and to substitute 1 for $\sqrt{1 + \frac{G^2}{T^2}}$. Also, if we are looking for solutions such that $T \rightarrow 0$ as $P \rightarrow 0$, then it is reasonable to drop the kT term. What remains is in agreement with Dickey's derivation for the case $\nu = 0$.

Next we study the differential operator L . Consider the problem

$$\begin{aligned} Lv - r^3 g(r) &= 0 \text{ in } (0, 1), \\ |v(0)| &< \infty, \text{ and} \\ T'(1) + T(1) &= m(1)\mu, \end{aligned} \tag{5}$$

where $g \in C[0, 1]$. We will show that this problem is uniquely solvable and establish estimates for the solutions.

Lemma 4 *If v is a solution of (5), then $\lim_{r \rightarrow 0} r^3 v'(r) = 0$.*

Proof: Observe that for $a, r \in (0, 1)$ v must satisfy

$$\frac{a^3}{m(a)} v'(a) - \frac{r^3}{m(r)} v'(r) = \int_r^a t^3 g(t) dt.$$

Since the limit of the integral clearly exists as $r \rightarrow 0$, and $m(r) \rightarrow 1$, then $\lim_{r \rightarrow 0} r^3 v'(r)$ must also exist. However, if this limit is not zero, then in some neighborhood of 0 there will be an $\epsilon > 0$ such that $v' \geq \frac{\epsilon}{r^3}$, or $v' \leq -\frac{\epsilon}{r^3}$. This implies that $\lim_{r \rightarrow 0} |v| = \infty$, contradicting the given boundary condition. ◇

By integrating the differential equation and using Lemma 4 we derive the equivalent problem

$$v'(r) = \frac{m}{r^3} \int_0^r t^3 g(t) dt, \text{ and } v'(1) + v(1) = m(1)\mu.$$

This implies

$$|v'(r)| \leq m \frac{r}{4} |g|_0,$$

where $|\cdot|_0$ represents the usual sup-norm on $C[0, 1]$. Notice that this estimate provides a more precise description of the behavior of v' near 0 than the given boundary condition indicates. It also helps determine the smoothness of the solution. In particular, this indicates that $|v'|_0 \leq \frac{m(1)}{4} |g|_0$. Observe that this bound is independent of the boundary data.

Differentiating the formula for $v'(r)$ leads to

$$v''(r) = \left(\frac{m}{r^3}\right)' \int_0^r t^3 g(t) dt + m(r)g(r).$$

It follows that $v \in C^2[0, 1]$ with $|v''|_0 \leq (\max_{[0,1]} |z'z''| + \frac{7}{4}m(1))|g|_0$, and $v''(0) = g(0)/4$. Now integrate $v'(r)$ to get

$$v(r) = v(0) + \int_0^r \frac{m}{t^3} \int_0^t s^3 g(s) ds dt.$$

We can solve for $v(0)$ as follows.

$$\begin{aligned} v(0) &= v(1) - \int_0^1 \frac{m}{t^3} \int_0^t s^3 g(s) ds dt \\ &= m(1)\mu - v'(1) - \int_0^1 \frac{m}{t^3} \int_0^t s^3 g(s) ds dt \\ &= m(1)\mu - m(1) \int_0^1 t^3 g(t) dt - \int_0^1 \frac{m}{t^3} \int_0^t s^3 g(s) ds dt. \end{aligned}$$

Thus we have an explicit formula for $v(r)$ and it is clear that $|v|_0 \leq c_1 + c_2|g|_0$ for some constants c_1, c_2 .

For convenience we use the notation $v = L^{-1}g$. Our estimates indicate that

$$L^{-1} : C[0, 1] \rightarrow C^2[0, 1],$$

is a continuous affine map sending bounded sets to bounded sets. Moreover, an application of the Arzela-Ascoli theorem implies that

$$L^{-1} : C[0, 1] \rightarrow C^1[0, 1]$$

is compact.

In order to apply an upper/lower solution technique we need a comparison result.

Lemma 5 *Suppose $\delta \geq 0$ and let $v_1, v_2 \in C^2[0, 1]$ such that $Lv_1 - \delta r^3 v_1 \leq Lv_2 - \delta r^3 v_2$ in $(0, 1)$, and $v_1'(1) + v_1(1) \geq v_2'(1) + v_2(1)$. Then $v_1 \geq v_2$ on all of $[0, 1]$.*

Proof: Suppose $v_1 < v_2$ at some point in $[0, 1]$, and let $[a, b] \subset [0, 1]$ be the maximal subinterval, containing this point, where the inequality $v_1 \leq v_2$ is satisfied. Observe that $L(v_1 - v_2) \leq \delta r^3(v_1 - v_2) \leq 0$. It follows that $v_1 - v_2$ cannot achieve its negative minimum in (a, b) . Suppose that minimum is achieved at $r = b$. It must be that $b = 1$. Further, $v_1(1) < v_2(1)$ and $v_1'(1) \leq v_2'(1)$. Thus $v_2'(1) + v_2(1) > v_1'(1) + v_1(1)$, a contradiction. The only remaining possibility is that the negative minimum is achieved at a with $a = 0$. If this is the case then $L(v_1 - v_2) < 0$ in a neighborhood of $r = 0$. Integrating gives $\frac{r^3}{m}(v_1' - v_2') < 0$ and so $(v_1 - v_2)$ must be decreasing in a neighborhood of $r = 0$, which contradicts the fact that a minimum is achieved at 0. \diamond

An immediate consequence of Lemma 5 is that $I - \delta L^{-1}$ is injective for any $\delta \geq 0$. Since L^{-1} is compact it follows from the Fredholm Alternative that $I - \delta L^{-1}$ is invertible. Hence the problem

$$\begin{aligned} Lv - r^3\delta v &= r^3g \text{ in } (0, 1), \\ |v(0)| &< \infty, \text{ and} \\ v'(1) + v(1) &= m(1)\mu \end{aligned} \tag{6}$$

has a solution operator $(L - r^3\delta)^{-1}$ such that

$$(L - r^3\delta)^{-1} : C[0, 1] \rightarrow C^2[0, 1]$$

is continuous, and is thus compact as a map from $C[0, 1]$ into $C^1[0, 1]$. (We are indebted to the referee for the preceding application of the Fredholm Alternative, which simplified and clarified this portion of the argument.)

We finish this section with a lemma that provides qualitative information about the desired solutions.

Lemma 6 *Suppose that T is a solution of (4). Then $T(0) > 0$ and $T'(0) = 0$.*

Proof: Suppose that $T(0) \leq 0$. Choose a maximal $\epsilon \in (0, 1]$ such that $T(r) \leq \tau(r)$ for $r \in [0, \epsilon]$, where $\tau(r)$ is the curve describing $f^{-1}(0)$. Since $f(r, T) \leq 0$ on this interval, integrating the DE yields

$$\frac{r^3}{m}T'(r) \leq 0.$$

Thus T is nonincreasing and $T \leq 0 < \tau(r)$ in $[0, \epsilon]$, and thus in $[0, \epsilon]$. It must be the case that $\epsilon = 1$. Since $T \leq 0$ we know that $f(r, T) \leq -1$ and integrating the DE yields

$$\frac{r^3}{m}T'(r) < -\frac{r^2}{2}.$$

Hence,

$$T'(r) < -\frac{m}{2r},$$

which implies $\lim_{r \rightarrow 0} T(r) = -\infty$, a contradiction.

Since $T(0) > 0$ we know that the graph of T lies in D_ϵ for some ϵ . Thus $T = L^{-1}(rf(r, T)) = L^{-1}(r^3h(r, T))$, where $h(r, T(r)) \in C[0, 1]$ so our estimates imply that $T'(0) = 0$. \diamond

Similar arguments show that if $T(0) \leq \min\{T : f(r, T) = 0\}$, then T is decreasing on $[0, 1]$, and if $T(0) \geq \max\{T : f(r, T) = 0\}$, then T is increasing on $[0, 1]$.

Finally, we remark that the previous arguments are valid for much more general boundary data. Given any smooth boundary operator $B(\alpha, \beta)$ such that $B_\alpha \geq 0$ and $B_\beta \geq c > 0$ for some constant c , then the results above can all be proved with analogous arguments using the boundary condition $B(v'(1), v(1)) = 0$. In our case we are using $B = \alpha + \beta - m(1)\mu$.

3 Existence and Uniqueness

In this section we prove the main theorems using the method of upper and lower solutions. Recall that an upper solution of (4) is defined as a function u satisfying

$$\begin{aligned} \left(\frac{r^3}{m}u'\right)' &\leq r \left[\frac{mu}{\sqrt{G^2+u^2}} - 1 + ku \right] \text{ in } (0, 1), \\ |u(0)| &< \infty, \text{ and} \\ u'(1) + u(1) &\geq m(1)\mu, \end{aligned} \tag{7}$$

and a lower solution is defined similarly with the inequalities reversed. In our proofs we will identify upper and lower solutions u and l , respectively, such that $l \leq u$, and then we will show that a sequence of approximate solutions, starting with l , increases monotonically to a solution T such that $l \leq T \leq u$. In many cases we can choose constant functions for u and l . In general it will be important that we can choose u and l lying in D_ϵ for some $\epsilon > 0$.

Theorem 1 *The boundary value problem (4) has at most one solution.*

Proof: Suppose that T_1 and T_2 are distinct solutions. Without loss of generality, $T_1 > T_2$ on some interval (a, b) , and we can assume that this interval is maximal. By the monotonicity of $f(r, \cdot)$ we have that $L(T_2) = rf(r, T_2) < rf(r, T_1) = L(T_1)$ in (a, b) . We also know that $T_1'(1) + T_1(1) = T_2'(1) + T_2(1)$. In each of the cases $0 < a < b < 1$, $0 < a < b = 1$, or $0 = a < b < 1$, the comparison lemma, or a minor modification of it, is valid. Thus $T_1 \leq T_2$, a contradiction. \diamond

Theorem 2 *(4) has a solution for all $\mu \in \mathbb{R}$.*

First, we assume that $\mu > 0$. Let $\tau(r)$ describe the level set $f^{-1}(0)$ as in the previous section. Let a and b be positive constants such that $0 < a \leq \tau(r) \leq b$ for all r , and such that $a < m(1)\mu < b$. Choose $\delta > \max_{[0,1] \times [a,b]} |h_T(r, T)|$. It

follows that $f(r, a) \leq 0 \leq f(r, b)$ for all r , and that $rf(r, T) - r^3\delta T$ is strictly decreasing as a function of T for $a \leq T \leq b$.

Consider the following iteration scheme. Let $T_0 \equiv a$, and for integers $n \geq 0$ let T_{n+1} satisfy

$$\begin{aligned} LT_{n+1} - r^3\delta T_{n+1} &= rf(r, T_n) - \delta r^3 T_n \text{ in } (0, 1), \\ |T_{n+1}(0)| &< \infty, \text{ and} \\ T'_{n+1}(1) + T_{n+1}(1) &= m(1)\mu. \end{aligned} \tag{8}$$

We will show that this sequence is well-defined, is bounded between a and b , and increases monotonically to a solution of (4).

Lemma 7 T_1 exists and satisfies $a \leq T_1 \leq b$.

Proof: Since $T_0 \equiv a > 0$ we can write $LT_1 - \delta r^3 T_1 = r^3(h(r, a) - \delta a)$ and our comments in the previous section guarantee the existence of T_1 . Moreover,

$$LT_1 - r^3\delta T_1 = rf(r, T_0) - \delta r^3 T_0 < LT_0 - \delta r^3 T_0,$$

and

$$T'_1(1) + T_1(1) = m(1)\mu > a = T'_0(1) + T_0(1).$$

Hence, $T_1 \geq T_0 \equiv a$ by the comparison lemma. A similar comparison yields the upper bound. \diamond

Now we continue by induction

Lemma 8 T_n exists for all n , and $a \leq T_{n-1} \leq T_n \leq b$.

Proof: Assume the statement is true for T_1, \dots, T_n . Since $T_n \geq a$ we know that T_{n+1} exists, just as in the previous proof. Moreover,

$$LT_{n+1} - \delta r^3 T_{n+1} = rf(r, T_n) - \delta r^3 T_n \leq rf(r, T_{n-1}) - \delta r^3 T_{n-1} = LT_n - \delta r^3 T_n,$$

where we have used the inductive hypothesis and the fact that $rf(r, \cdot) - \delta r^3 \cdot$ is decreasing. We also have $T'_{n+1}(1) + T_{n+1}(1) = T'_n(1) + T_n(1)$, and thus $T_{n+1} \geq T_n$. A similar comparison yields the upper bound $T_{n+1} \leq b$. \diamond

Thus $\{T_n\}$ is a bounded and monotonically increasing sequence in $C[0, 1]$. By regularity it follows that $\{T_n\}$ is bounded in $C^2[0, 1]$. By compactness we know that $T_n \rightarrow T$ in $C^1[0, 1]$, and we can bootstrap to get $T_n \rightarrow T$ in $C^2[0, 1]$. T is clearly a solution of (4).

We have established existence for $\mu > 0$. Now we can extend this result to $\mu \in \mathbb{R}$ by repeating the iteration scheme with new choices of upper and lower solutions. It is a nice property of this method that we can use previously established solutions as upper or lower solutions that extend the results.

Let $M = \{\mu \in \mathbb{R} : (4) \text{ has a solution}\}$. For $\mu \in M$ let T_μ represent the corresponding solution. Our work so far has shown that $(0, \infty) \subset M$. The following lemmas establish that $M = \mathbb{R}$.

Lemma 9 *If $\mu \in M$, then $[\mu, \infty) \subset M$.*

Proof: If $\mu_1 > \mu$ then T_μ can replace the constant a as the lower solution in the iteration scheme, and the constant b can be chosen as before to be an upper solution. The set $J := \{(r, T) : T_\mu(r) \leq T \leq b\}$ is a compact subset of D_ϵ for some $\epsilon > 0$, so we may choose $\delta > \max_J |h_T(r, T)|$, and conclude that $rf(r, T) - \delta r^3 T$ is decreasing as a function of T in J . The iteration scheme will work precisely as before to yield a solution for μ_1 . \diamond

Lemma 10 *M is open.*

Proof: Suppose that $\mu \in M$. Consider $l := T_\mu - \frac{1}{2}T_\mu(0)$ and $u := T_\mu$. These will serve as lower and upper solutions, respectively, and the iteration scheme implies the existence of solutions for $\mu_0 \geq \mu - \frac{T_\mu(0)}{2}$. \diamond

Lemma 11 *$M = \mathbb{R}$.*

Proof: Suppose that $M = (\mu, \infty)$ with $\mu > -\infty$, and suppose that μ_n is a decreasing sequence converging to μ . By arguments similar to those above we can show that the corresponding sequence of solutions is monotonically decreasing. Each member of this sequence is positive at $r = 0$ and cannot attain a negative interior minimum, so if $T_{\mu_n}(1)$ is bounded below, then the solution sequence is also bounded below. If $T_{\mu_n}(1) < 0$, then T_{μ_n} achieves its minimum at $r = 1$, so $T'_{\mu_n}(1) \leq 0$. Thus $\mu \leq \frac{1}{m(1)}(T'_{\mu_n}(1) + T_{\mu_n}(1)) \leq T_{\mu_n}(1)$. Hence the solution sequence is bounded below, and converges monotonically to a function T_μ . Standard arguments can now be applied to show that $T_\mu \in C[0, 1]$, T_μ satisfies the differential equation in $(0, 1]$, and T_μ satisfies the boundary data at $r = 1$. Thus $\mu \in M$, a contradiction. \diamond

Thus Theorem 2 is proved. This theorem agrees in many respects with the spherical cap results in the references. It remains to be seen how the problem will behave for more general membranes and for $\nu > 0$. We remark that an application of the Implicit Function Theorem extends existence and uniqueness to small $\nu > 0$, but a thorough investigation of these matters is left for future work.

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