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Connectedness of number theoretic tilings

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Let $T = T(A, D)$ be a self-affine tile in $\mathbb{R}^n$ defined by an integral expanding matrix A and a digit set D. In connection with canonical number systems, we study connectedness of T when D corresponds to the set of consecutive integers $\{0, 1, \ldots, |\det(A)| - 1\}$. It is shown that in $\mathbb{R}^3$ and $\mathbb{R}^4$, for any integral expanding matrix A, $T(A, D)$ is connected.

We also study the connectedness of Pisot dual tilings which play an important role in the study of β-expansion, substitution and symbolic dynamical system. It is shown that each tile generated by a Pisot unit of degree 3 is arcwise connected. This is naturally expected since the digit set consists of consecutive integers as above. However surprisingly, we found families of disconnected Pisot dual tiles of degree 4. Also we give a simple necessary and sufficient condition for the connectedness of the Pisot dual tiles of degree 4. As a byproduct, a complete classification of the β-expansion of 1 for quartic Pisot units is given.

Keywords: Tiling, Connectedness, Pisot Number, Fractal

1 Introduction

A non empty set in $\mathbb{R}^n$ is called a *tile* [(i)] if it coincides with the closure of its interior. If a finite set of tiles and their translations covers the space $\mathbb{R}^n$ without overlapping, then we say it forms a *tiling*. By 'without overlapping' we mean that the translated tiles are mutually disjoint up to an n-dimensional set of Lebesgue measure zero.

In this paper, we will discuss the connectedness of tiles which arise from two different kinds of number systems. Although the systems are pretty different in nature and could be separately discussed, we decided to put them together in a single paper since the underlying ideas are close and the reader can find the sharp contrast between them.

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(i) In some cases, it is called a *protile* when the following tiling properties are not yet proved.

1.1 Tiles associated to expanding integral matrices.

Let $M_n(\mathbb{Z})$ denotes the set of $n \times n$ matrices with entries in $\mathbb{Z}$. Let A be an expanding integral matrix in $M_n(\mathbb{Z})$. The word 'expanding' means that all its eigenvalues have modulus greater than 1. We also say that a monic polynomial in $\mathbb{Z}[x]$ is expanding if all roots have modulus greater than one. By definition, the characteristic polynomial of the expanding matrix is expanding and vice versa. Let $|detA| = q$ and let $D = \{d_1, \dots d_q\} \subset \mathbb{R}^n$ be a set of q distinct vectors, called a *$q-$digit set*. If we let $S_j(x) = A^{-1}(x+d_j)$, $1 \le j \le q$, then they are contractive maps under a suitable norm in $\mathbb{R}^n$ [28] and it is well known that there is a unique compact set T satisfying $T = \bigcup_{j=1}^q S_j(T)$ [15, 22], which is explicitly given by

$$T := T(A, D) = \left\{ \sum_{i=1}^{\infty} A^{-i} d_{j_i} : d_{j_i} \in D \right\}.$$

T is called an *attractor* of the system $\{S_j\}_{j=1}^q$, and it is called a *self-affine tile* if its Lebesgue measure $\mu(T)$ is positive. Indeed this positiveness is equivalent to the fact that T and its translations form a tiling. Basic questions and detailed studies on the tiling generated by T are found for example in J. C. Lagarias-Y. Wang [28], R. Kenyon [25], C. Bandt [10], Y. Wang [43], A. Vince [42] and their references.

One of the important aspects of self-affine tiles is connectedness. Hata [21] has shown that if $\{f_j\}_{1 \le j \le m}$ is a finite set of contractive maps [(ii)] of X, then the attractor $K = K(f_1, \cdots, f_m)$ is a locally connected continuum if and only if, for any $1 \le i < j \le m$, there exists a sequence $\{r_0, r_1, \cdots, r_n, r_{n+1}\} \subset \{1, 2, \cdots, m\}$ with $r_0 = i$ and $r_{n+1} = j$ such that $f_{r_k}(K) \cap f_{r_{k+1}}(K) \neq \emptyset$ for $k \in \{0, 1, \cdots, n\}$. Note that if a tile is connected then it must be arcwise connected. This is seen in the same proof by Hata [21]. Thus after all

Arcwise connectedness and connectedness are equivalent

in our framework. We will confirm this point also in the Pisot case in the proof of Theorem 4.1 on page 287 in a slightly different context, the graph directed sets case (c.f., Luo-Akiyama-Thuswaldner [29]). Hacon-Saldanha-Veerman [20] have shown that, if $|\det A| = 2$ and $D = \{0, v\} \subset \mathbb{Z}^n$ is a complete set of coset representatives of the quotient group $\mathbb{Z}^n / A\mathbb{Z}^n$, then $T(A, D)$ is a connected tile. Gröchenig-Haas [19] have proved the existence of connected self-similar lattice tilings for parabolic and elliptic dilations in dimension two. Kirat-Lau [26], using a graph argument on D, have rediscovered Hata's above criterion of connectedness. Also they have shown the following sufficient criterion, which we will use in the proof of Theorem 3.1 on page 279 and Theorem 3.2 on page 281. Afterwards we will call it a Kirat-Lau Criterion.

Let $A \in M_n(\mathbb{Z})$ be an expanding matrix with $|\det A| = q$ and $p(x)$ be its characteristic polynomial. Let $D = \{0, v, \cdots, (q-1)v\}$ with $v \in \mathbb{R}^n \setminus \{0\}$. Suppose that there exists a polynomial $g(x) \in \mathbb{Z}[x]$ (which will be called multiplying factor) such that

$$h(x) = g(x)p(x) = x^k + a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_1 x \pm q$$

with $|a_i| \le q - 1$, for $1 \le i \le k-1$. Then $T(A, D)$ is connected.

The idea of this criterion is to find a common point on consecutive two tiles $T + kv$ and $T + (k+1)v$ and to apply Hata's type criterion mentioned above. As it is easy to describe in this way all expanding

(ii) Hata [21] studied 'weak' contractions, a slightly general concept.

polynomials of degree 2, Kirat and Lau succeeded in proving the connectedness of a tile for a suitable digit set in dimension 2.

In the first part of this paper, we are interested in generalizing their results to higher dimensional cases using the digit sets corresponding to consecutive integers $\{0, 1, \ldots, |\det(A)| - 1\}$. We will show the following theorem, using the Schur-Cohn criterion reviewed in Section 2 on page 275.

Theorem 1.1 *Let $d = 3, 4$ and $A \in M_d(\mathbb{Z})$ be an expanding matrix with $|detA| = q$ and $D = \{0, v, \cdots, (q-1)v\}$ with $v \in \mathbb{R}^d \setminus \{0\}$. Then $T(A, D)$ is connected.*

The proofs are settled separately in Theorem 3.1 on page 279 and Theorem 3.2 on page 281. They are almost done by brute force and are quite complicated having lots of subcases. However this result gives an evidence of a widely believed speculation that all such 'consecutive integer digit tiles' may be connected. This is in good contrast to the later part of this paper.

We do not intend to consider general digit sets but only use digits which correspond to consecutive integers. One reason of this restriction is that this case is essential and widely studied in relation to canonical number systems. For canonical number systems and attached tilings, see Kátai-Kőrnyei [24], Kovács-Pethő [27], Gilbert [18]. Recent progress on topological studies on this tiling are seen in Akiyama-Thuswaldner [6, 7].

Another reason is as follows. As it is easy to find a disconnected tile when we choose 'scattered' digit sets, an interesting direction is to find a connected tile for a given expanding matrix A. Thus it may be just awkward to consider general digit sets in higher dimensional cases, since we are already able to show the connectedness only by using consecutive integers.

1.2 Tiles associated to Pisot units.

Now we will explain the later part of this paper. Let $\beta > 1$ be a real number which is not an integer. A *greedy expansion* of a positive real x in base β is an expansion of the form:

$$x = \sum_{i=-N_0}^{\infty} a_{-i}\beta^{-i} = a_{N_0}, a_{N_0-1}, \cdots a_0.a_{-1}a_{-2}\ldots$$

with $a_{-i} \in \mathcal{A}_\beta = [0, \beta) \cap \mathbb{Z}$ and a *greedy condition*

$$0 \le x - \sum_{-N_0}^{N} a_{-i}\beta^{-i} < \beta^{-N} \qquad \forall N \ge -N_0.$$

The *integer part* of x is $a_{N_0}, a_{N_0-1}, \cdots a_0. = \sum_{i=-N_0}^{0} a_{-i}\beta^{-i}$ and the *fractional part* is defined similarly. This expansion for $x \in [0, 1)$ is produced by iterating the *beta transform* (c.f. [37]):

$$U_\beta : x \to \beta x - \lfloor \beta x \rfloor$$

keeping track its carries $\lfloor \beta x \rfloor \in \mathcal{A}_\beta$. Basic properties of this expansion are summarized in [30]. To fix our notations we briefly review them. Denote by $\mathcal{A}_\beta^*$ (resp. $\mathcal{A}_\beta^\omega$) the set of finite words on $\mathcal{A}_\beta$ (resp. the set of right infinite words on $\mathcal{A}_\beta$). Let $1 = d_{-1}\beta^{-1} + d_{-2}\beta^{-2} + \cdots$ be an expansion of 1 defined by the algorithm

$$c_{-i} = \beta c_{-i+1} - \lfloor \beta c_{-i+1} \rfloor, \quad d_{-i} = \lfloor \beta c_{-i+1} \rfloor$$

with $c_0 = 1$, where $\lfloor x \rfloor$ denotes the maximal integer not exceeding x. In other words, this expansion is achieved as the trajectory of $U_\beta^n(1)$ $(n = 1, 2, \ldots)$. $d_\beta(1) = .d_{-1}, d_{-2}, \cdots$ is called *$\beta-$expansion of* 1. Let $u^\omega \in \mathcal{A}_\beta^\omega$ denote the right infinite word generated by repetition of u, that is, $u, u, \cdots$. Parry [33] has shown that the β-expansion of 1 can be characterized by the conditions of lexicographic order, as follows:

Let $d = (d_{-i})_{i\geq 1}$ be a sequence of nonnegative integers different from $1, 0^\omega$, such that $\sum_{i\geq 1} d_{-i}\beta^{-i} = 1$, with $d_{-1} \geq 1$ and for $i \geq 2$, $d_{-i} \leq d_{-1}$, then d is the β-expansion of 1 if and only if:

$$\forall p \geq 1,\ \sigma^p(d) <_{\text{lex}} d, \tag{1.1}$$

where σ is the shift defined by $\sigma((x_i)_{i\leq M}) = (x_{i-1})_{i\leq M}$. He also has shown that a sequence $x = x_1, x_2, \cdots$ of nonnegative integers is realized as a β-expansion of some positive real number if and only if it satisfies the following lexicographical condition:

$$\forall p \geq 0, \quad \sigma^p(x) <_{\text{lex}} d^*(1) \tag{1.2}$$

with $d^*(1) = \begin{cases} d_\beta(1), & \text{if } d_\beta(1) \text{ is infinite;} \\ (d_{-1}, d_{-2}, \cdots, d_{-n+1}, (d_{-n}-1))^\omega, & \text{if } d_\beta(1) = d_{-1}, \cdots, d_{-n}. \end{cases}$

In this case this sequence $x = x_1, x_2, \cdots$ is called *admissible*.

Hereafter let β be a *Pisot number* which is an algebraic integer greater than 1 whose Galois conjugates other than itself have modulus smaller than 1. Let $\mathbb{Q}(\beta)_{\geq 0}$ be nonnegative elements of the minimum field containing the rational numbers $\mathbb{Q}$ and β. Bertrand [12] and Schmidt [36] showed that any greedy expansion of $x \in \mathbb{Q}(\beta)_{\geq 0}$ is eventually periodic, which means that there exists a positive integer L such that $a_{-N} = a_{-N-L}$ for sufficiently large N. We call a *Pisot unit* a Pisot number which is also a unit of the integer ring of $\mathbb{Q}(\beta)$. The symbolic dynamical system X_β attached to β-expansion is the subshift of the full shift $\mathcal{A}_\beta^{\mathbb{N}}$ whose language consists of all admissible words in $\mathcal{A}_\beta^*$. X_β is sofic if and only if the expansion of 1 is eventually periodic (see [13]). Especially when β is a Pisot number it gives a sofic system. Thurston [41] introduced an idea to construct a self-affine tiling generated by a Pisot unit β which is a geometric realization of this sofic system X_β. Akiyama [2] and Praggastis [34] studied in detail such self-affine tilings. G. Rauzy [35] already constructed this kind of tiling in a different approach closely related to substitutions. This tiling has a strong connection to the explicit construction of Markov partitions of dynamical systems, hopefully toral automorphisms. See also P. Arnoux-Sh. Ito [9].

Let us recall this tiling by Pisot units, which is called *dual tiling*, following the notation of [2]. Let

$$\beta = \beta^{(1)}, \beta^{(2)}, \cdots, \beta^{(r_1)} \text{ and } \beta^{(r_1+1)}, \overline{\beta^{(r_1+1)}}, \cdots, \beta^{(r_1+r_2)}, \overline{\beta^{(r_1+r_2)}}$$

be the real and the complex conjugates of β, respectively. We also denote by $x^{(j)}$ $(j = 1, 2, \cdots, r_1+2r_2)$ the corresponding conjugates of $x \in \mathbb{Q}(\beta)$. Define a map $\Phi\colon \mathbb{Q}(\beta) \to \mathbb{R}^{r_1+2r_2-1}$ by:

$$\Phi(x) = \left(x^{(2)}, \cdots, x^{(r_1)}, \Re(x^{(r_1+1)}), \Im(x^{(r_1+1)}), \cdots, \Re(x^{(r_1+r_2)}), \Im(x^{(r_1+r_2)})\right).$$

Let $A = a_{-1}, a_{-2}, \cdots$ be a greedy expansion in base β of an element $\mathbb{Z}[\beta] \cap [0, 1)$. Define S_A to be the set of elements of $\mathbb{Z}[\beta]_{\geq 0}$ whose greedy expansion has the suffix A. In other words we classify all elements of $\mathbb{Z}[\beta]_{\geq 0}$ by their fractional part and map them via Φ to have a tile $T_A = \overline{\Phi(S_A)}$. An empty word is designated by λ and the tile T_λ is called the central tile. As already noticed in Thurston [41], the Pisot condition guarantees that T_A is compact and the restriction to units is necessary to have a tiling by this construction. Therefore we restrict ourselves to Pisot units. Under this restriction, it is not so easy to show that these T_A give a tiling of the space $\mathbb{R}^{r_1+2r_2-1}$ though we expect it is always valid. Let $\mathbf{Fin}(\beta)$ be the set of all finite beta expansions. This is obviously a subset of $\mathbb{Z}[1/\beta]_{\geq 0}$. If β satisfies

$$\mathbf{Fin}(\beta) = \mathbb{Z}[1/\beta]_{\geq 0},$$

then we say that β has finitely expansible property (F). This property (F) implies that β is a Pisot number (see [16]). It is comparatively easy to construct a tiling defined by Pisot units with (F), in the above sense ([2]). In [5], we introduced a wider class of Pisot units with this tiling property called *weakly finiteness*. It is conjectured that this property holds even for all Pisot numbers (c.f. [8], [38], [39]). In this paper, we do not discuss further this tiling property.

The second aim of this paper is to explore the problem of connectedness of Pisot dual tiles of low degree using again the Schur-Cohn criterion discussed in Section 2 on page 275. A general arcwise connectedness criterion for Pisot dual tiles is established in Theorem 4.1 on page 287.

Furthermore we can prove the following theorem.

Theorem 1.2 *Each tile corresponding to a Pisot unit β is arcwise connected if $d_\beta(1)$ terminates with* 1.

The proof is found after the one of Theorem 4.1 on page 287. Our conjecture is that for all Pisot units with finite β-expansion of 1 , the last non zero digit of $d_\beta(1)$ must be one. The conjecture is true especially for cubic Pisot units β with finite β-expansion of 1 , (see [4], [11]) and as we prove in Theorem 4.9 on page 307 it is also true for quartic Pisot units β with finite β-expansion of 1 .

To treat all Pisot units, Theorem 1.2 is not enough since the β-expansion of 1 is not finite in general. Let p be the characteristic polynomial of β. If $p(0) = 1$ then the β-expansion of 1 cannot be finite (see Proposition 1 of [1]). Even when $p(0) = -1$ there are many such cases. Including these cases, we can generalize the above conjecture:

Conjecture 1 *Let β be a Pisot unit and consider its eventually periodic β-expansion of 1 : $d_\beta(1) = .d_{-1}, \cdots, d_{-n}, (d_{-n-1}, \cdots, d_{-n-k})^\omega$. Then*

$$d_{-n-k} - d_{-n} = \pm 1.$$

This conjecture is shown to be valid for degree less than 5 in this paper. More challenging would be the following conjecture:

Conjecture 2 *Let $\beta > 1$ be a real number and assume that its β-expansion of 1 is eventually periodic with $d_\beta(1) = .d_{-1}, \cdots, d_{-n}, (d_{-n-1}, \cdots, d_{-n-k})^\omega$. Then $|d_{-n-k} - d_{-n}|$ coincides with the absolute value of the norm of β.*

This conjecture was first formulated in [3]. Strong numerical evidence exists for this conjecture. However, unfortunately the Pisot dual tile can be disconnected even if this conjecture is true. We summarize our main results in the following theorem.

Theorem 1.3 *Let β be a Pisot unit of degree 3 or 4 defined by the monic polynomial $p(x) \in \mathbb{Z}[x]$. If $\deg \beta = 3$ or $p(0) = 1$ then each tile is connected. If $\deg \beta = 4$ and $p(0) = -1$ then each tile is connected if and only if*

$$a + c - 2\lfloor \beta \rfloor \neq 1$$

for $p(x) = x^4 - ax^3 - bx^2 - cx - 1$.

These statements are a combination of Theorem 4.4 on page 291, Theorem 4.5 on page 292, Theorem 4.7 on page 301 and Theorem 4.8 on page 307. In spite of the quite simple nature of the statement, the proof is pretty involved having lots of subcases. However we may say that this result gives us a breakthrough.

In fact, if $\deg \beta = 4$, $p(0) = -1$ and $a + c - 2[\beta] = 1$, there exists a disconnected tile. As far as we know, no example of disconnected Pisot dual tiles was known before. As these tiles are generated by consecutive integers, it was even expected that Pisot dual tiles are always connected. Thus this result gives an unfortunate surprise that there exists a concrete family of Pisot units one of whose dual tiles is disconnected. (See a remark after Theorem 4.8 on page 307.)

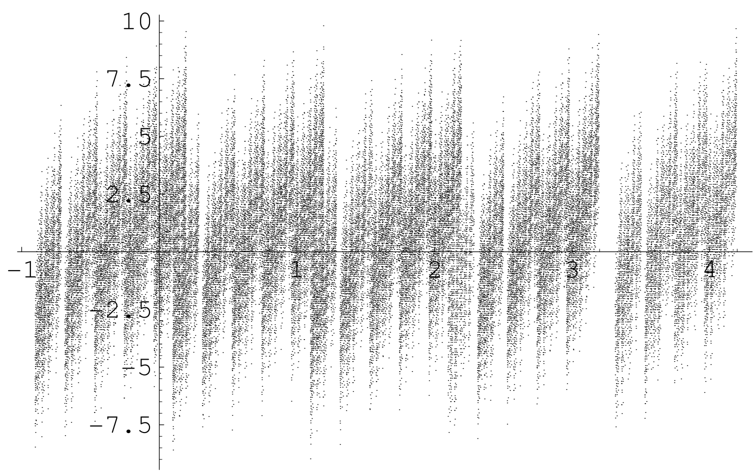

Fig. 1: The projection of the central tile (disconnected) generated by the Pisot unit β with minimal equation $x^4 - 3x^3 - 7x^2 - 6x - 1 = 0$

When β-expansion of 1 is eventually periodic, write it as

$$d_\beta(1) = c_{-1}, \dots c_{-M}(c_{-M-1} \dots c_{-M-L})^\omega$$

with $c_{-M} \neq c_{-M-L}$. We say that the period (resp. preperiod) of β-expansion of 1 is L (resp. M).

As a byproduct, we will give a complete classification of the β-expansion of 1 for cubic and quartic Pisot units in Theorem 4.3 on page 290, Theorem 4.9 on page 307 and Theorem 4.6 on page 298 which are naturally proven during our proofs. Theorem 4.3 on page 290 was proved by Bassino [11]. She computed the β-expansion of 1 for any cubic Pisot number, including non units. In view of the prominent role of the expansion of 1 in symbolic dynamics of beta expansion, it is worthy to state independently Theorem 4.9 on page 307 and Theorem 4.6 on page 298. It is also an unfortunate surprise that there is no uniform bound on the length of the expansion of 1 for quartic Pisot units with finite β-expansion of 1 . Also, there is no uniform bound on period and preperiod of the expansion of 1 for quartic Pisot units with infinite β-expansion of 1 . The next table makes the situation clearer.

Further study of connectedness may be explored in a different setting. Pisot dual tilings under a certain condition are formulated as a geometric realization of substitutive dynamical system. Canterini [14] studied connectedness of such substitutive tilings and gave general criteria which works for these tiles. It

Degree	Length of finite $d_\beta(1)$	Preperiod of infinite $d_\beta(1)$	Period of infinite $d_\beta(1)$
2	2	1	1
3	5	2	2
4	∞	∞	∞

Tab. 1: Length bounds related to the expansion of 1.

may be fruitful to extend the above conjectures to his situation and to study the connectedness of a family of substitutive tiles.

This paper is organized as follows: In Section 2, we prepare some results related to the Schur-Cohn criteria to count the number of roots inside/outside the unit circle. Section 3 on page 279 is devoted to the connectedness of tiles associated to expanding integral matrices of low degree by the Kirat-Lau criterion. Tiles associated to Pisot numbers are treated in Section 4 on page 287. The beginning of Section 4 on page 287 is of importance. We give a proof of Theorem 1.2 on page 273 and describe a method to prove connectedness of Pisot dual tiles. This is more complicated than the one in Section 3 on page 279 but the underlying spirit is similar. Then we show in the subsections 4.1 and 4.2 the connectedness for quadratic and cubic Pisot units. Later subsections are for the quartic Pisot units. The idea of the proof of disconnectedness is found in Lemma 3 on page 300 in this last section. In few words, we show the disconnectedness of a projection of the tile along the direction of the negative real root and use the forbidden words for beta expansions in $\mathcal{A}_\beta^*$ to 'cut' the tile. Convenient lists are found in Figure 2 on page 309 and Figure 3 on page 310. In the shaded box, the expansion of one is not written in a fixed length. Readers find the explicit form in Theorem 4.9 on page 307 and Theorem 4.6 on page 298. The four disconnected cases are also indicated in Figure 3 on page 310.

2 Expanding polynomials and Pisot polynomials

Let $f(x) = \sum_{i=0}^{n} a_i x^{n-i}$ be a polynomial with *complex* coefficients a_i within this section. Admitting an abuse of terminology, we say that $f(x)$ is an *expanding polynomial* if each root has modulus greater than one. A monic *real* polynomial f is a *Pisot polynomial* if it has a real root greater than one and other roots are inside the unit circle and additionally $|a_n| \geq 1$. These definitions agree with the original situation when $f(x)$ is the irreducible polynomial over $\mathbb{Z}$ of an algebraic integer.

We briefly review the Schur-Cohn criterion to count the number of zeros inside/outside the unit circle. In the literature, the Schur-Cohn criterion is often explained in the simplest case that all the determinants are non zero [iii]. In general, this restriction leads us to a difficulty to characterize polynomials with prescribed location of zeros, in terms of a single family of polynomial inequalities. However for expanding polynomials, such a characterization is well known. Further a characterization of Pisot polynomials will be given (Theorem 2.2 and Corollary 2.2 on page 278), which will be used later on.

The reciprocal polynomial of f is defined by $f^*(x) = x^{\deg f} f(1/x)$. Let $D_n = D_n(f)$ be the determinant of following $2n \times 2n$ matrix with coefficients:

$$b_{i,j} = \begin{cases} a_{j-i}, & \text{for } 1 \leq i \leq n \text{ and } i \leq j \leq i+n \\ \overline{a}_{i-j}, & \text{for } n+1 \leq i \leq 2n \text{ and } i-n \leq j \leq i \\ 0, & \text{otherwise} \end{cases}$$

[iii] A clear and original description including such degenerate cases is found in [40]. An earlier version of this section was based on this Japanese book, without noticing the standard name after Schur-Cohn.

We will write it in the following form where the empty entries represent 0.

$$
\begin{array}{c}
D_n = \left|\begin{array}{cccc|c|ccccc|c|}
a_0 & a_1 & \cdot & \cdot & \cdot & a_n & & & & & \\
 & a_0 & \cdot & \cdot & \cdot & a_{n-1} & a_n & & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
\hline
 & & & & a_0 & a_1 & \cdot & \cdot & \cdot & \cdot & a_n \\
\hline
\overline{a}_n & \overline{a}_{n-1} & \cdot & \cdot & \cdot & \overline{a}_0 & & & & & \\
 & \overline{a}_n & \cdot & \cdot & \cdot & \overline{a}_1 & \overline{a}_0 & & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
 & & & & \overline{a}_n & \overline{a}_{n-1} & \cdot & \cdot & \cdot & \cdot & \overline{a}_0
\end{array}\right|
\begin{array}{l}
\\ \\ \\ \to n \\ \\ \\ \\ \to 2n
\end{array}
\\
\begin{array}{cc}
\downarrow \qquad\qquad\qquad & \downarrow \\
n \qquad\qquad\qquad & 2n
\end{array}
\end{array}
$$

which is the resultant of f and $\overline{f^*}$. Hence $D_n = 0$ if and only of there exists an *inversible root* β, that is, $f(\beta) = f(1/\overline{\beta}) = 0$. Especially if a real polynomial f has a root on the unit circle then $D_n = 0$. By definition, $D_n \neq 0$ for expanding polynomials and Pisot polynomials with $n \geq 3$, since $|a_n| \geq 1$ does not allow an inversible root. Delete the n-th, $2n$-th rows and columns from D_n to get a $2(n-1) \times 2(n-1)$ matrix with determinant D_{n-1}. From D_{n-1} we create D_{n-2} in the same way. Continue like this till we get

$$
D_2 = \left|\begin{array}{cc|cc}
a_0 & a_1 & a_n & \\
 & a_0 & a_{n-1} & a_n \\
\hline
\overline{a}_n & \overline{a}_{n-1} & \overline{a}_0 & \\
 & \overline{a}_n & \overline{a}_1 & \overline{a}_0
\end{array}\right|, \qquad D_1 = \left|\begin{array}{cc}
a_0 & a_n \\
\overline{a}_n & \overline{a}_0
\end{array}\right|.
$$

Then the famous Schur-Cohn's criterion (c.f. [31]) is

Theorem 2.1 *Assume that $D_i \neq 0$ $(i = 1, \ldots, n)$ and let p be the number of sign changes of the sequence $1, -D_1, D_2, \ldots, (-1)^n D_n$. Then $f(x) \in \mathbb{C}[x]$ has p zeros inside the unit circle and no zeros on the unit circle.*

A technical problem arises from the non vanishing assumption on D_i.

Example 1 *We have $(D_0, D_1, \ldots, D_5) = (1, 0, 0, 0, 1, 5)$ for $x^5 - 2x^4 - 2x^3 - x^2 - 2x + 1$ and $(1, 0, 0, 0, 1, -5)$ for $x^5 - 2x^4 - x^3 - 2x^2 - 2x + 1$. However the situation of zeros is the same: there are exactly two roots in the unit circle and three outside for both polynomials. When consecutive zeros appear in $D_1, D_2, \ldots, D_n$, the number of sign changes of $1, -D_1, D_2, \ldots, (-1)^n D_n$ does not tell how many roots lie in the unit circle.*

The classical theory of Schur-Cohn assures that there is a way to escape from such a situation by taking different principal minors of the corresponding quadratic form (c.f. [40]), or by replacing f with other polynomials which have as many zeros as f (c.f. Theorem 45.1 and Theorem 45.2 of [31]).

However this is not convenient in practice. As we wish to derive results on families of polynomials, exceptional treatments should be reduced to a minimum. For this purpose, we prepare some necessary and sufficient conditions of expanding polynomials and Pisot polynomials.

Corollary 2.1 *The polynomial $f(x) \in \mathbb{C}[x]$ is expanding if and only if $\mathrm{sgn}(D_i) = (-1)^i$ for $i = 1, \ldots, n$,* which is also called the Schur-Cohn criterion. Here we define

$$\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \\ 0 & x = 0. \end{cases}$$

The origin of this Corollary dates back to Hermite and Hurwitz who connected the root distribution problem with the invariants of Hermitian forms. The determinants D_i do not vanish because they are principal minors of a positive definite Hermitian forms. We derive this Corollary 2.1 by slightly extending Marden's argument in page 194–200 of [31] (c.f. [17]). Define $f_0(x) = f(x)$ and $f_{j+1}(x) = \overline{a}^{(j)}_{n-j} f_j(x) - a^{(j)}_0 f^*_j(x)$ for $j = 0, 1, \ldots, n-1$ with $f_j(x) = \sum_{k=0}^{n-j} a^{(j)}_k x^{n-j-k}$. Direct determinant computation yields

$$f_{k+1}(0) D_k = -f_1(0) \ldots f_k(0) D_{k+1}$$

and hence

$$\operatorname{sgn}(D_k D_{k+1}) = -\operatorname{sgn}(f_1(0) \ldots f_{k+1}(0)) \tag{2.1}$$

provided $f_1(0) \ldots f_{k+1}(0) \neq 0$, which is (43.4) in [31] [(iv)]. A crucial fact is

If f_j has p_j zeros inside the unit circle and $f_{j+1}(0) \neq 0$, then f_{j+1} has

$$p_{j+1} = \begin{cases} p_j & \text{if } f_{j+1}(0) > 0 \\ n - j - p_j & \text{if } f_{j+1}(0) < 0 \end{cases} \tag{2.2}$$

zeros inside the unit circle. The set of zeros on the unit circle of f_j coincides with that of f_{j+1}.

which is a consequence of Rouché's theorem for circles of radius $1 + \varepsilon$ with small ε's, using the equality $|f(z)| = |f^*(\overline{z})|$ valid on the unit circle.

Proof of Corollary 2.1. The sufficiency of the condition $\operatorname{sgn}(D_i) = (-1)^i$ is a direct consequence of Theorem 2.1. Let us prove the necessity. We claim that that if f_{j+1} has a root in the closed unit disk then f_j also does. To show this, we divide the situation into three cases. If $|a^{(j)}_{n-j}| > |a^{(j)}_0|$ then (2.2) gives $p_j = p_{j+1} > 0$. If $|a^{(j)}_{n-j}| < |a^{(j)}_0|$ then $p_j = n - j - p_{j+1} > 0$ since $p_{j+1} \leq n - j - 1$. Finally if $|a^{(j)}_{n-j}| = |a^{(j)}_0|$ then the leading coefficient and the constant term of f_j have the same absolute value, proving that at least one root of f_j is in the closed unit disk. This shows the claim. As f is expanding, this claim shows that f_j is also expanding for $j = 1, \ldots, n$. Therefore $f_j(0)$ can not vanish for $j = 1, \ldots n$. Observing (2.2) again, since $p_j = 0$ for $j = 0, \ldots, n$, we have $f_j(0) > 0$ for $j = 1, \ldots, n$. The relation (2.1) implies that $\operatorname{sgn}(D_k D_{k+1}) = -1$, which shows the assertion. □

We give a characterization of Pisot polynomials, which does not seem to have been written down elsewhere although it follows from the above reviewed results.

(iv) $D_k = (-1)^k \Delta_k$ in [31].

Theorem 2.2 *Each Pisot polynomial satisfies $f(1) < 0$ and $D_i \le 0$ $(i = 2, \dots, n)$. Conversely a polynomial $f(x) = x^n + a_1x^{n-1} + \dots + a_n \in \mathbb{R}[x]$ is a Pisot polynomial if $f(1) < 0$ and $D_i < 0$ $(i = 2, \dots, n)$. If $a_n \neq \pm 1$ then every Pisot polynomial satisfies $f(1) < 0$ and $D_i < 0$ $(i = 2, \dots, n)$.*

In other words, provided $a_n \neq \pm 1$, a Pisot polynomial is characterized by a system of inequalities $f(1) < 0$ and $D_i < 0$ $(i = 2, \dots, n)$. It is likely that this characterization is also valid for $a_n = \pm 1$. We prove some cases of low degree in Corollary 2.2.

Proof: Assume that a monic $f \in \mathbb{R}[x]$ is a Pisot polynomial with $|a_n| > 1$. As there is only one real root greater than 1, we have $f(1) < 0$. Using $f_1(0) = |a_n|^2 - 1 > 0$ and (2.2), f_1 and f have the same number of roots inside the unit circle. As f_1 is of degree $n-1$, f_1^* must be an expanding polynomial. Thus Corollary 2.1 reads $\operatorname{sgn}(D_j(f_1^*)) = (-1)^j$ and thus $\operatorname{sgn}(D_j(f_1)) = (-1)^j \operatorname{sgn}(D_j(f_1^*)) = 1$ for $j = 1, \dots, n-1$. Employing the formula (43.3) in [31]:

$$f_1(0)^{j+2} D_j(f) = -D_{j-1}(f_1)$$

with $f_1(0) = |a_n|^2 - 1 > 0$, we get $D_j = D_j(f) < 0$ for $j = 2, \dots, n$, proving the last statement. Now we consider the case $a_n = \pm 1$. We replace a_i by $a_i + \varepsilon_i$ with small ε_i's, and we write the corresponding Schur-Cohn determinants as $D_i^{(\varepsilon_1, \dots, \varepsilon_n)}$. If $|a_n + \varepsilon_n| > 1$ then $D_i^{(\varepsilon_1, \dots, \varepsilon_n)} < 0$ by the above discussion. As $D_i^{(\varepsilon_1, \dots, \varepsilon_n)} \to D_i$ when $(\varepsilon_1, \dots, \varepsilon_n)$ tends to 0, we have $D_i \le 0$ for $i = 2, \dots, n$. This proves the first statement of the Theorem.

It remains to show that $f(1) < 0$ and $D_i < 0$ $(i = 2, \dots, n)$ is a sufficient condition to have a Pisot polynomial. Let us start with the case $|a_n| > 1$. Since $f(x)$ is a monic polynomial in $\mathbb{R}[x]$ and $D_i < 0$ $(i = 1, \dots, n)$, Theorem 2.1 implies that there are exactly $n-1$ zeros inside the unit circle. $f(1) < 0$ shows the existence of at least one positive root greater than 1, proving that f is a Pisot polynomial. Finally let us assume that $f(x) \in \mathbb{R}[x]$, $|a_n| = 1$, $D_i < 0$ $(i = 2, \dots, n)$ and $f(1) < 0$. Choose a small real ε such that $|a_n + \varepsilon|^2 - 1 > 0$. Substitute a_n by $a_n + \varepsilon$ and denote by $D_i^{(\varepsilon)}$ the corresponding Schur-Cohn determinants. Then following the same discussion, $D_i^{(\varepsilon)} < 0$ for $i = 1, 2, \dots n$ implies that $f(x) + \varepsilon$ is a Pisot polynomial. On the other hand, $D_n \neq 0$ implies there are no zeros of f on the unit circle, because, by definition, D_n is the resultant of f and f^*. As the roots are continuous functions with respect to coefficients, this shows that f is a Pisot polynomial. □

Corollary 2.2 *If $n = 3$ or $n = 4$ then a monic polynomial $f(x) = x^n + a_1x^{n-1} + \dots + a_n \in \mathbb{R}[x]$ is a Pisot polynomial if and only if $f(1) < 0$ and $D_i < 0$ $(i = 2, \dots, n)$.*

Proof: According to Theorem 2.2, it remains to show that if f is a Pisot polynomial with $a_n = \pm 1$, then $D_i \neq 0$ $(i = 2, \dots, n)$. Recall that $D_n \neq 0$ for Pisot polynomials with $n \ge 3$. Note that

$$D_2 = \begin{vmatrix} 1 & a_1 & a_n & \\ & 1 & a_{n-1} & a_n \\ a_n & a_{n-1} & 1 & \\ & a_n & a_1 & 1 \end{vmatrix} = (-1 + a_n^2 + a_{n-1} - a_n a_1)(-1 + a_n^2 - a_{n-1} + a_n a_1).$$

$D_2 = 0$ implies $a_{n-1} = a_n a_1$. From the two equalities $a_n = \pm 1$ and $a_{n-1} = a_n a_1$ we deduce $D_3 = 0$, which shows the case for $n = 3$. For the quartic case, we have

$$\begin{aligned} D_4 &= -(a_1-a_2+a_3)(a_1+a_2+a_3)(a_1^2-4a_2-a_3^2)^2 \\ D_3 &= -(a_1+a_3)^2(a_1^2-4a_2-a_3^2) \qquad \text{for } a_4=-1 \\ D_2 &= -(a_1+a_3)^2 \end{aligned}$$

and

$$\begin{aligned} D_4 &= -(a_1-a_3)^4(-2+a_1-a_2+a_3)(2+a_1+a_2+a_3) \\ D_3 &= -(a_1-a_3)^3(a_1+a_3) \qquad \text{for } a_4=1. \\ D_2 &= -(a_1-a_3)^2 \end{aligned}$$

If $a_4=-1$, then $D_2=0$ or $D_3=0$ happens only when $a_3=-a_1$, since $D_4\neq 0$. But this implies $D_4=16a_2^4\geq 0$. Together with the fact that Theorem 2.2 gives $D_4\leq 0$, we have $D_4=0$, a contradiction. If $a_4=1$, then $D_2=0$ or $D_3=0$ happens only when $a_3=-a_1$. This gives $D_4=16a_1^4(2+a_2)^2\geq 0$ which leads us to the same contradiction. □

3 Connectedness of self-affine tilings generated by an expanding matrix

In this section we shall prove connectedness of tiles generated by an expanding matrix, up to degree 4.

3.1 Connectedness of self-affine tilings generated by an expanding cubic matrix

The next lemma is an explicit form of Corollary 2.1 on page 276.

Lemma 1 *The polynomial $p(x)=x^3+ax^2+bx+c$ with integer coefficients is expanding if and only if*

$$\begin{cases} |\,b-ac\,|<c^2-1, \\ |\,b+1\,|<|\,a+c\,|. \end{cases} \tag{3.1}$$

Theorem 3.1 *Let $A\in M_3(\mathbb{Z})$ be an expanding matrix with $|detA|=q$ and $D=\{0,v,\cdots,(q-1)v\}$ with $v\in\mathbb{R}^3\backslash\{0\}$. Then $T(A,D)$ is connected.*

Proof: Let $p(x)=x^3+ax^2+bx+c$ with $a,b,c\in\mathbb{Z}$ be the characteristic polynomial of A, which is expanding. We study the following two systems of inequalities, equivalent to (3.1):

$$\begin{cases} b-ac-c^2\leq -2, \\ b-ac+c^2\geq 2, \\ a-b+c\geq 2, \\ a+b+c\geq 0, \end{cases} \quad \text{and} \quad \begin{cases} b-ac-c^2\leq -2, \\ b-ac+c^2\geq 2, \\ a-b+c\leq 0, \\ a+b+c\leq -2\ . \end{cases} \tag{3.2}$$

From the first one, we get the following bounds for the coefficients :

$$c\geq 2 \qquad -2c+2,\leq b\leq 2c-1, \qquad -c+1\leq a\leq c+1,$$

while from the second we have:

$$c\leq -2, \qquad 2c+2\leq b\leq -2c-1, \qquad c-1\leq a\leq -c-1.$$

To show the connectedness of $T(A,D)$, we use the Kirat-Lau Criterion. Since the way of finding the multiplying factor is the same for both systems, here we solve only the first system. We can divide the classification into the following cases:
Case 1. Suppose that $-2c+2 \le b \le -c$. From the system (3.2) in this case we get $-b-c \le a \le c-1$ and $0 \le 1+a < c$.

♦ If $a > -b-c$ then $-c+1 \le a+b \le 0$. We also have that $-c < b+c \le 0$. So the required polynomial is $h(x) = (1+x)p(x) = x^4+(1+a)x^3+(a+b)x^2+(b+c)x+c$.

♦ If $a = -b-c$ then the required polynomial $h(x)$ is:

$$x^5+(1+a)x^4+(1-c)x^3+(b+c)x+c = (x^2+x+1)p(x).$$

Case 2. Suppose that $-c+1 \le b \le -1$. From the system (3.2) in this case we get $-b-c \le a \le c-1$ which implies that $-c+1 \le a \le c-1$. So in this case the multiplying factor is $g(x) \equiv 1$.
Case 3. Suppose that $0 \le b \le c-1$. From the system (3.2) in this case we get $2+b-c \le a \le c$ which implies that $-c+2 \le a \le c$.

♦ If $a \le c-1$ the multiplying factor is $g(x) \equiv 1$.

♦ If $a = c$ then $b > 1$ and $1-c < b-c < 0$, so the polynomial $h(x)$ is

$$x^4+(c-1)x^3+(b-c)x^2+(c-b)x-c = (x-1)p(x).$$

Case 4. Suppose that $c \le b \le 2c-1$ which implies that $-c < c-b \le 0$. From the system (3.2) in this case we get $-1 \le b-a \le c-2$ and $1 \le 1-a \le c$.

♦ If $a < 1+c$ the required polynomial $h(x)$ is

$$x^4+(a-1)x^3+(b-a)x^2+(c-b)x-c = (x-1)p(x).$$

♦ If $a = 1+c$ then $b \ge c+2$, $-c+1 < b-2c < 0$, $-2c+2 \le 2c-2b+1 \le 0$.

♢ If $-c+1 \le 2c-2b+1 \le 0$ then the required polynomial $h(x)$ is

$$x^5+(c-1)x^4+(b-2c-1)x^3+(2c-2b+1)x^2+(b-2c)x+c = (x-1)^2p(x).$$

♢ If $-2c+2 \le 2c-2b+1 \le -c$ then $-c+1 < 3c-2b+1 \le 0$ and $-c \le 2b-4c-1 < -1$.

⋄ If $2b-4c-1 > -c$ then the required polynomial $h(x)$ is

$$x^7+(c-1)x^6+(b-2c)x^5+(3c-2b)x^4+(2b-4c-1)x^3+(3c-2b+1)x^2+(b-2c)x+c = (x^2+1)(x-1)^2p(x).$$

⋄ If $2b-4c-1 = -c$ then the required polynomial $h(x)$ is

$$x^6+(c-1)x^5+(b-2c)x^4+(2c-b)x-c = (x^3-2x^2+2x-1)p(x).$$

□

3.2 Connectedness of self-affine tilings generated by an expanding quartic matrix

From Corollary 2.1 on page 276, we deduce

Lemma 2 *The polynomial $p(x) = x^4 + ax^3 + bx^2 + cx + d$ with integer coefficients is expanding if and only if*

$$\begin{cases} d \geq 2, \\ |c - ad| \leq d^2 - 2, \\ |a + c| < 1 + b + d, \\ -1+b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3 < 0 \end{cases}$$
$$or \tag{3.3}$$
$$\begin{cases} d \leq -2, \\ |c - ad| \leq d^2 - 2, \\ |a + c| < -1 - b - d, \\ -1+b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3 > 0. \end{cases}$$

Proof: From Corollary 2.1 on page 276 we observe that:

$$\begin{cases} D_1 < 0 \\ D_2 > 0 \end{cases} \Longleftrightarrow |c - ad| \leq d^2 - 2,$$

$$D_3 < 0 \Longleftrightarrow \begin{array}{c} \begin{cases} (d^2-1)(1+b+d) + (a+c)(c-ad) < 0 \\ (-1+b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3) > 0 \end{cases} \\ \text{or} \\ \begin{cases} (d^2-1)(1+b+d) + (a+c)(c-ad) > 0 \\ (-1+b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3) < 0, \end{cases} \end{array}$$

$$D_4 > 0 \Longleftrightarrow |a + c| < 1 + b + d \qquad \text{or} \qquad |a + c| < -1 - b - d.$$

and that:

$$\text{if } \begin{cases} |a+c| < 1+b+d \\ |c-ad| \leq d^2-2 \end{cases} \text{ then } (d^2-1)(1+b+d) + (a+c)(c-ad) > 0,$$
$$\text{if } \begin{cases} |a+c| < -1-b-d \\ |c-ad| \leq d^2-2 \end{cases} \text{ then } (d^2-1)(1+b+d) + (a+c)(c-ad) < 0.$$

Second, since for the expanding polynomial $p(0),\ p(1)$ and $p(-1)$ have the same sign,

$$\begin{array}{ll} d \geq 2 & \Longrightarrow |a+c| < 1+b+d \\ d \leq -2 & \Longrightarrow |a+c| < -(1+b+d). \end{array}$$

We get the desired result (3.3). □

Theorem 3.2 *Let $A \in M_4(\mathbb{Z})$ be an expanding matrix with $|detA| = d$ and $D = \{0, v, \cdots, (d-1)v\}$ with $v \in \mathbb{R}^4 \backslash \{0\}$. Then $T(A, D)$ is connected.*

Proof: Let $p(x) = x^4+ax^3+bx^2+cx+d$ with $a, b, c, d \in \mathbb{Z}$ be the characteristic polynomial of A, which is expanding. From the systems of inequalities (3.3) we get the following bounds for the coefficients :

$$|d| \geq 2, \quad -|d| \leq a \leq |d|, \quad -3|d| + 8 \leq b \leq 3|d| - 8, \quad -3|d| + 6 \leq c \leq 3|d| - 6.$$

We can divide the classification into the following cases:

<u>*Conditions 1*</u> $\begin{cases} d \le -2, \\ |a+c| < -1-b-d, \\ |c-ad| \le d^2-2, \\ -1+b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3 > 0. \end{cases}$

<u>*Conditions 2*</u> $\begin{cases} d \ge 2, \\ |a+c| < 1+b+d, \\ |c-ad| \le d^2-2, \\ -1+b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3 < 0. \end{cases}$

Since the matrix A is expanding if and only if $-A$ is expanding and the characteristic polynomials of both matrices are monic polynomials, we may choose $p(x)$ or $p(-x)$ appropriately in the proof, which enables us to assume that $a \ge 0$. Now we use the Kirat-Lau Criterion again.

<u>First</u> suppose that the coefficients of the polynomial $p(x)$ satisfy Conditions 1 with $a \ge 0$. Here we have 2 possibilities:

Case 1 $\begin{cases} d \le -2, \\ b+d+1 < a+c \le 0, \\ 1-d^2 < c-ad < d^2-1, \\ -1+b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3 > 0, \end{cases}$

or

Case 2 $\begin{cases} d \le -2, \\ 0 < a+c \le -b-d-1, \\ 1-d^2 < c-ad < d^2-1, \\ b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3 > 0. \end{cases}$

Let us see the range of the coefficients in Case 1. We get that

$$\begin{cases} d \le -2, \\ 0 \le a \le -d, \\ (a-d)(1+d) < b \le -2-d, \\ 1-a+b+d < c \le -a. \end{cases}$$

- For $a = -d$ we get that $b \ge 0$. So the required polynomial $h(x)$ is

  $$x^6-(1+d)x^5+(b+d+1)x^4+(c-b-d)x^3+(b+d-c)x^2+(c-d)x+d,$$

  where the multiplying factor is x^2-x+1.

- For $0 \le a \le -d-1$ we have that $2d \le c \le -d-1$ and $d-2 \le b \le a-2$.
  - If $c = 2d$ then $d \le -7,\ b = d-2,\ a = 0$. The required polynomial $h(x)$ is
    $$x^9-2x^8+(d+1)x^7+x^6-4x^5+(d+5)x^4-(d+4)x^3+2x^2-d,$$
    where the multiplying factor is $(x^2+1)(x-1)(x^2-x+1)$.
  - If $2d+1 \le c \le d$ we get $d-2 \le b \le -d-2$.
    - $*$ If $b \ge d+1$ the required polynomial $h(x)$ is
      $$x^6+(a-1)x^5+(1+b-a)x^4+(a-b+c)x^3+(b-c+d)x^2+(c-d)x+d,$$

where the multiplying factor is $x^2 - x + 1$.

- $*$ If $d - 2 \le b \le d$ then $a = 0$ or $a = 1$.
  - ⋄ For $b - a = d$ we have that the polynomial $h(x)$ is
  $$x^7+(a-1)x^6+(d+1)x^5+(c-1-d)x^4+(2d-c)x^3+(c-b-d)x^2+(d-c)x-d,$$
  where the multiplying factor is $(x^2+1)(x-1)$.
  - ⋄ For $b-a\le d-1$, $a=0$ and $b=d-2$ the multiplying factor is $(x^2-x+1)(x^2+1)(x-1)$.
  For $b-a\le d-1$, $a=0$ and $b=d-1$ the multiplying factor is $(x^2-x+1)(x-1)$.
  For $b-a\le d-1$, $a=1$, and $b=d$ the multiplying factor is $(x^2-x+1)(x^2+1)(x-1)$.

– If $c\ge d+1$ then $|a|, |b|, |c|$ are less than $|d|$ so the multiplying factor is $g(x) \equiv 1$.

Now let us see the Case 2 of the Conditions 1 which leads to:

$$\begin{cases} d=-2, \\ a=0, \\ b=-1, \\ c=1, \end{cases} \quad \text{or} \quad \begin{cases} d\le -3, \\ 0\le a\le -d, \\ -(a+d)(1+d) < b \le -3-d, \\ 1-a\le c\le -2-a-b-d. \end{cases}$$

In this case we have two subcases:

⧫ For $a=-d$ we have that $b\ge 1$ and $d+1\le c\le -3$. So the polynomial $h(x)$ is
$$x^6-(1+d)x^5+(1+b+d)x^4+(c-b-d)x^3+(b-c+d)x^2+(c-d)x+d,$$
where the multiplying factor is $x^2 - x + 1$.

⧫ For $0\le a\le -d-1$, we have that $3d+8\le b\le -d-3$ and $2+d\le c\le -3d-6$.

- If $-2d \le c \le -3d-6$ we have that $d \le 2+2a+b \le -d-1$
  - If $2+2a+b \ge d+1$ the polynomial $h(x)$ is
  $$x^7+(2+a)x^6+(2+2a+b)x^5+(1+2a+2b+c)x^4+(a+2b+2c+d)x^3+(b+2c+2d)x^2+(c+2d)x+d,$$
  where the multiplying factor is $(x^2+x+1)(x+1)$.
  - If $2+2a+b = d$ then the polynomial $h(x)$ is
  $$x^9+(2+a)x^8+(1+d)x^7+(a+b+c+d+1)x^6+(a+2b+2c+2d)x^5+(2b+3c+3d-1)x^4+(a+2b+3c+3d)x^3+(b+2c+3d)x^2+(c+2d)x+d,$$
  where the multiplying factor is $(x^2+x+1)(x^2+1)(x+1)$.
- If $-d \le c \le -2d-1$ we have that $2d+3 \le b \le -2$ and $a \le -d-2$.
  - If $d+1 \le b \le -2$ then the polynomial $h(x)$ is
  $$x^5+(1+a)x^4+(a+b)x^3+(b+c)x^2+(c+d)x+d,$$
  where the multiplying factor is $x+1$.
  - If $2d+3 \le b \le d$ then $2d+3 \le a+b \le -2$.
    - $*$ If $a+b \ge d+1$ then the polynomial $h(x)$ is
    $$x^5+(1+a)x^4+(a+b)x^3+(b+c)x^2+(c+d)x+d,$$

where the multiplying factor is $x+1$.

- $*$ If $a+b \le d$ then $a \le -d-3$, $c \ge -d+1$ and $d \le 2a+b+2 \le -1$.
  - $\diamond$ If $d+1 \le 2a+b+2$ then the polynomial $h(x)$ is
    $x^7+(2+a)x^6+(2a+b+2)x^5+(2a+2b+c+1)x^4+(a+2b+2c+d)x^3+(b+2c+2d)x^2+(c+2d)x+d,$
    where the multiplying factor is $(x^2+x+1)(x+1)$.
  - $\diamond$ If $2a+b+2=d$ then the polynomial $h(x)$ is
    $$x^9+(a+2)x^8+(d+1)x^7+(1+a+b+c+d)x^6+(a+2b+2c+2d)x^5+(2b+3c+3d-1)x^4+ (a+2b+3c+3d)x^3+(b+2c+3d)x^2+(c+2d)x+d,$$
    where the multiplying factor is $(x^2+x+1)(x^2+1)(x+1)$.

- If $d+2 \le c \le -d-1$ then $2d+6 \le b \le -3-d$.
  - If $b \le d$ then $a \le -d-2$, and the polynomial $h(x)$ is
    $$x^5+(1+a)x^4+(a+b)x^3+(b+c)x^2+(d+c)x+d,$$
    where the multiplying factor is $x+1$.
  - If $b \ge d+1$ then the multiplying factor is $g(x) \equiv 1$.

<u>Second</u> suppose that the coefficients of the polynomial $p(x)$ satisfy Conditions 2 with $a \ge 0$. Here we have 2 possibilities:

$$\text{Case 1} \begin{cases} d \ge 2, \\ -b-d \le a+c \le 0, \\ 2-d^2 \le c-ad \le d^2-2, \\ b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3 \le 0, \end{cases}$$

or

$$\text{Case 2} \begin{cases} d \ge 2, \\ 1 \le a+c \le b+d, \\ 1-d^2 < c-ad < d^2-1, \\ b-ac+c^2+d+a^2d-2bd-acd+d^2+bd^2-d^3 \le 0. \end{cases}$$

In Case 1 we get 3 subcases:

- $d \ge 2$, $0 \le a \le d-2$, $b=-d$, $c=-a$. Here the polynomial $h(x)$ is
  $$x^6+ax^5+(1-d)x^4-ax+d,$$
  where the multiplying factor is x^2+1.

- $$\begin{cases} d \ge 2, \\ 0 \le a \le d-2, \\ -d+1 \le b \le -3-a-d-ad+d^2, \\ -a-b-d \le c \le -a. \end{cases}$$
  In this case we get that the bounds for the coefficients are
  $$-d+1 \le b \le 2d-3 \quad \text{and} \quad -2d+3 \le c \le 0.$$

- If $-2d+3 \le c \le -d$ then $d \ge 3$, $b \ge -a$, and $-2d+2 \le b+c \le 0$.
  - If $b+c \le -d$ then the polynomial $h(x)$ is
    $$x^7+(1+a)x^6+(1+a+b)x^5+(1+a+b+c)x^4+(d+a+b+c)x^3+(d+b+c)x^2+(c+d)x+d,$$
    where the multiplying factor is $(x^2+1)(x+1)$.
  - If $b+c \ge -d+1$ then the polynomial $h(x)$ is
    $$x^5+(1+a)x^4+(a+b)x^3+(b+c)x^2+(c+d)x+d,$$
    where the multiplying factor is $x+1$.
- If $-d+1 \le c \le 0$ we have that $-d+1 \le b \le d$.
  - If $b < d$ the multiplying factor is $g(x) \equiv 1$.
  - If $b = d$ then the polynomial $h(x)$ is
    $$x^6+ax^5+(d-1)x^4+(c-a)x^3-cx-d,$$
    where the multiplying factor is x^2-1.

- $\begin{cases} d \ge 2, \\ 0 \le a \le d-2, \\ b \ge -2-a-d-ad+d^2, \\ 2+ad-d^2 \le c \le -a. \end{cases}$

  This case is possible only if $2 \le d \le 4$ and the multiplying factor is $g(x) \equiv 1$ except when $d=2$, $a=0$, $b=2$, $c=0$. In this case the multiplying factor is x^2+1.

Now let us consider the Case 2 of the Conditions 2.
Here we get that $0 \le a \le d$, $-d+1 \le b \le 3d-3$, and $-d+1 \le c \le 3d-3$.

- If $c \ge 2d$ then $d \ge 3$, $b \ge a+d$ and $c \le -a+b+d$ and for $d=3,4$ we have that $b=a+d$ and $c=2d$. In this case the polynomial $h(x)$ is
  $$x^7+(a-2)x^6+(b+2-2a)x^5+(2a+c-2b-1)x^4+(2b+d-a-2c)x^3+(2c-b-2d)x^2+(2d-c)x-d,$$
  where the multiplying factor is $(x^2-x+1)(x-1)$.
- If $d \le c \le 2d-1$ then $b \ge a$ and there are three cases to be studied:
  - If $b=a$ then $c=d$ and the polynomial $h(x)$ is
    $$x^5+(a-1)x^4+(d-a)x^2-d,$$
    where the multiplying factor is $x-1$.
  - If $a+1 \le b \le a+d-1$ then $d \le c \le -a+b+d$. Here we see that $b-c \ge -d$.
    - If $b-c=-d$ then $a=0$, $c=b+d$ and $b \le d-3$. The polynomial $h(x)$ is
      $$x^7-x^6+(b+1)x^5+(d-1)x^4-bx-d,$$
      where the multiplying factor is $(x^2+1)(x-1)$.
    - If $b-c \ge -d+1$ then the polynomial $h(x)$ is
      $$x^5+(a-1)x^4+(b-a)x^3+(c-b)x^2+(d-c)x-d,$$

where the multiplying factor is $x-1$.

- If $b \geq a+d$ then $c \geq d+1$, $b \leq 2d$, $-2d+2 \leq b+d-2c \leq d-1$.
  - $*$ If $b+d-2c \geq -d+1$ then $a \geq 2$ and $-2d+1 \leq a+c-2b+1 \leq 0$.
    - $\diamond$ If $a+c-2b \geq -d$ the polynomial $h(x)$ is
      $$x^9+(a-2)x^8+(1-2a+b)x^7+(a+c+1-2b)x^6+(b+d-2c+a-2)x^5+(1-2a+c+b-2d)x^4+(a-2b+c+d)x^3+(b+d-2c)x^2+(c-2d)x+d,$$
      where the multiplying factor is $(x^3+1)(x-1)^2$.
    - $\diamond$ If $a+c-2b+1 \leq -d$ then the polynomial $h(x)$ is
      $$x^{11}+(a-2)x^{10}+(b+2-2a)x^9+(2a+c-2b-1)x^8+(2b-a-2c+d-1)x^7+(2-a+2c-b-2d)x^6+(2a-b-c+2d-2)x^5+(1-2a+2b-c-d)x^4+(a+2c-2b-d)x^3+(b+2d-2c)x^2+(c-2d)x+d,$$
      where the multiplying factor is $(x^3+1)(x^2+1)(x-1)^2$.
  - $*$ If $b+d-2c \leq -d$ then $d \geq 5$, $a \leq d-1$ and the polynomial $h(x)$ is
    $$x^7+(a-2)x^6+(b-2a+2)x^5+(2a-2b+c-1)x^4+(2b-a-2c+d)x^3+(2c-b-2d)x^2+(2d-c)x-, d$$
    where the multiplying factor is $(x^2-x+1)(x-1)$.

• If $-d+1 \leq c \leq d-1$ then $-d+1 \leq b \leq 2d-1$.

- If $b \geq d$ then $a \leq d-1$ and $c \geq 2-d$.
  - $*$ If $c \leq 1$ then $a \geq 1-c$ and $b=d$. The polynomial $h(x)$ is
    $$x^6+ax^5+(d-1)x^4+(c-a)x^3-cx-d,$$
    where the multiplying factor is x^2-1.
  - $*$ If $c \geq 2$ then $d \geq 3$ and $-d \leq a-b \leq 0$.
    - $\diamond$ If $a-b \geq -d+1$ the polynomial $h(x)$ is
      $$x^5+(a-1)x^4+(b-a)x^3+(c-b)x^2+(d-c)x-d,$$
      where the multiplying factor is $x-1$.
    - $\diamond$ If $a-b=-d$ then the polynomial $h(x)$ is
      $$x^7+(a-1)x^6+(d-1)x^5+(1-2a-d+c)x^4-cx^3+(a-c)x^2+(c-d)x+d,$$
      where the multiplying factor is $(x-1)^2(x+1)$.
- If $b \leq d-1$ and $a \leq d$ then the multiplying factor is $g(x) \equiv 1$.
- If $b \leq d-1$ and $a=d$ then the polynomial $h(x)$ is
  $$x^5+(d-1)x^4+(b-d)x^3+(c-b)x^2+(d-c)x-d,$$
  where the multiplying factor is $x-1$.

□

Remark 1 *Here we do not restrict ourselves only in the case when the characteristic polynomial of the matrix A is irreducible. This fact is in contrast with the following section.*

4 Connectedness of self-affine tilings generated by a Pisot unit.

We give a sufficient condition for the tiling generated by a Pisot unit to be arcwise connected. Let β be a Pisot unit whose minimal polynomial is $p(x) = x^n - a_1 x^{n-1} - \cdots - a_{n-1}x - a_n \in \mathbb{Z}[x]$ with $a_n = \pm 1$. It follows immediately from Thurston's construction that there are only finitely many tiles up to translation, that the number of tiles coincides with that of different suffix of the β-expansion of 1 , i.e., the cardinality of $\{U_\beta^k(1)\}_{k=1,2,\dots} \cup \{0\}$. Recall the convention used in the introduction that the symbol A stands for the greedy expansion of elements of $\mathbb{Z}[\beta] \cap [0,1)$, which is identified with a right infinite (or finite) admissible word in $\mathcal{A}_\beta^* \cup \mathcal{A}_\beta^\omega$. The tile T_A was defined as $\overline{\Phi(S_A)}$. Symbolically the set T_A is the collection of left infinite admissible sequences

$$\dots a_3 a_2 a_1 a_0 \oplus A = \dots a_3 a_2 a_1 a_0 . A$$

realized by the map Φ into $\mathbb{R}^{n-1}$. Here we denote by $a \oplus b$ the concatenation of words $a \in \mathcal{A}_\beta^*$ and $b \in \mathcal{A}_\beta^*$ and say that a left infinite word is *admissible* when all finite suffixes are admissible. The interval $[0,1)$ is subdivided by $\{U_\beta^k(1)\}_{k=1,2,\dots} \cup \{0\}$ into $0 = t_0 < t_1 < t_2 < \cdots < 1$ and the shape of T_A depends on the interval $[t_i, t_{i+1})$ where A belongs (c.f. [5]). In the sense of $n-1$ dimensional Lebesgue measure, the smallest tile T_A corresponds to a suffix A which satisfies $\max_{k \ge 1} U_\beta^k(1) \le A < 1$ by the lexicographical ordering. The larger the suffix the stricter the restriction on the integer parts $\dots a_3 a_2 a_1 a_0$ by the admissibility condition (1.2). Conversely T_A becomes biggest when $0 \le A < \min_{U_\beta^k(1) \ne 0} U_\beta^k(1)$, identifying 0 with λ. Especially the central tile T_λ is the biggest tile.

Theorem 4.1 *Let $\beta > 1$ be a Pisot unit. Set $\eta = \max_{k \ge 1} U_\beta^k(1)$ which gives the smallest tile T_η. If*

$$T_\eta \cap (T_\eta - \Phi(\beta^{-1})) \ne \emptyset, \tag{4.1}$$

then each Pisot dual tile is arcwise connected.

To begin our proof, we recall graph directed attractors and graph directed iterated function system (GIFS for short). Let $G = G(V, E)$ be a strongly connected graph where $V = \{1, \dots, q\}$ is the set of vertices and E is the set of directed edges. Let $E_{i,j}$ be the set of edges from i to j. Now for each $e \in E$ define a uniformly contractive map $F_e : \mathbb{R}^d \to \mathbb{R}^d$. Then by [32, Theorem 1] there exists a unique family $K_1, \dots, K_q$ of compact non-empty sets satisfying

$$K_i = \bigcup_{j=1}^{q} \bigcup_{e \in E_{i,j}} F_e(K_j). \tag{4.2}$$

The set of contractions $\{F_e \mid e \in E\}$ is called a *graph directed iterated function system* and the sets K_i are called *graph directed attractors*. Connectedness and arcwise connectedness criteria for these graph directed attractors are found in [29] as well. We claim that Pisot dual tiles form graph directed self affine attractors. Let G_t be the natural map defined by the following commutative diagram:

$$\begin{array}{ccc}
\mathbb{Q}(\beta) & \xrightarrow{\times \beta^t} & \mathbb{Q}(\beta) \\
\Phi \big\downarrow & & \big\downarrow \Phi \\
\mathbb{R}^{d-1} & \xrightarrow[G_t]{} & \mathbb{R}^{d-1}.
\end{array}$$

Then G_t is contractive for $t > 0$ since β is a Pisot number. The set equations are given in the following form:

$$T_A = \bigcup_{i \oplus A} G_1(T_{i \oplus A}), \tag{4.3}$$

where the summation is taken over all possible $i \in [0, \beta) \cap \mathbb{Z}$ such that $i \oplus A$ is admissible (see [5]). Note that we identify $i \oplus A$ with the corresponding $\beta-$ expansion to realize it as a nonnegative real number. Since there are finitely many tiles up to translation, it is easy to show that they form graph directed self affine attractors by using (1.2). This proves the claim.

Proof of Theorem 4.1. To prove that all tiles are connected, it suffices to show that two neighboring tiles $T_{(i-1)\oplus A}$ and $T_{i\oplus A}$ have nonempty intersection. Indeed, if this is true, then for any two points on a tile it is easy to find an ε-chain connecting these by repeated applications of (4.3) (see [21]).

Since the admissibility condition (1.2) is described by the lexicographic order, for a word $u \in \mathcal{A}_\beta^*$, if $u \oplus \eta$ is admissible then $u \oplus \kappa$ is admissible for any admissible word κ. Hence putting $w = i \oplus A - \eta$, we have

$$S_{i\oplus A} \supset S_\eta + w \text{ and } S_{(i-1)\oplus A} \supset S_\eta + w - \beta^{-1}.$$

This shows that

$$T_{(i-1)\oplus A} \cap T_{i\oplus A} \supset (T_\eta \cap (T_\eta - \Phi(\beta^{-1}))) + \Phi(w).$$

Thus, by the assumption, each tile is connected.

Finally we discuss shortly the local connectedness and arcwise connectedness. Recalling the theorem of Hahn and Mazurkiewicz, it suffices to show that each tile is a locally connected set having at least two points. Local connectedness is shown easily by (4.3), since each tile is reconstructed as a finite union of sufficiently small connected compact sets. □

From Theorem 4.1 on the previous page we immediately get a

Corollary 4.1 *If for the Pisot unit β, $\exists a_i \in \mathbb{Z}$ $(i = 1, 2, \cdots)$ such that $|a_i| < \lfloor \beta \rfloor$ and $\Phi(1) + \sum_{i=1}^{\infty} a_i \Phi(\beta^i) = 0$ then each Pisot dual tile is arcwise connected.*

which is akin to the Kirat-Lau criterion. In practice, this Corollary is quite useful but not enough in some cases.

Proof: Let $x_i = \max(a_i, 0)$ and $y_i = \max(-a_i, 0)$. Then we have

$$\sum_{i=1}^{\infty} x_i \Phi(\beta^{i-1}) + \Phi(\eta) = \sum_{i=1}^{\infty} y_i \Phi(\beta^{i-1}) + \Phi(\eta) - \Phi(1/\beta)$$

Since the maximal digit $\lfloor \beta \rfloor \in \mathcal{A}_\beta$ does not appear in x_i and y_i, both $\dots x_2 x_1 x_0 \oplus \eta$ and $\dots y_2 y_1 y_0 \oplus \eta$ are admissible by (1.2). Therefore the left hand side belongs to T_η and the right to $T_\eta - \Phi(1/\beta)$. □

For a string of symbols $\varpi = a_1, a_2, \cdots, a_n$ let us write ϖ^ω for the right periodic expansion

$$a_1, a_2, \cdots, a_n, a_1, a_2, \cdots, a_n, \cdots, a_1, a_2, \cdots, a_n, \cdots$$

and ${}^\omega\varpi$ for the left periodic expansion

$$\cdots, a_1, a_2, \cdots, a_n, a_1, a_2, \cdots, a_n, \cdots, a_1, a_2, \cdots, a_n$$

Here we shall prove Theorem 1.2 on page 273 that each tile is connected if $d_\beta(1)$ terminates with 1.

Proof of Theorem 1.2: By the assumption, $1 = \sum_{i=1}^{d} c_{-i}\beta^{-i}$ with $c_{-d} = 1$, which gives rise to a relation $P(\beta) = 0$ with

$$P(x) = x^d - c_{-1}x^{d-1} - c_{-2}x^{d-2} - \cdots - c_{-d+1}x - 1.$$

Let k be the greatest integer less than d such that $c_{-k} = \lfloor\beta\rfloor$. Since β is also a root of $P(x)(1 - x^{d-k})\sum_{i=0}^{\infty} x^{d\,i} = 0$ we get that

$$^{\omega}\left(\lfloor\beta\rfloor, c_{-k-1}, \cdots, c_{-d+1}, 0, \lfloor\beta\rfloor, c_{-2}, \cdots, c_{-k+1}\right), \lfloor\beta\rfloor - 1, c_{-k-1}, \cdots, c_{-d+1}.\eta =$$
$$^{\omega}\left(c_{-1}, c_{-2}, \cdots, c_{-d+1}, 0\right), \underbrace{0, \cdots, 0}_{d-k-1}.\eta \;\; - 0.1$$

is a common point of T_η and $T_\eta - \Phi(\beta^{-1})$, where η is the biggest suffix in the β-expansion of 1 . □

We remark here that this Theorem 1.2 is a generalization of the same result proved under the finiteness condition (F) (see [2]).

4.1 Connectedness of self-affine tilings generated by a quadratic Pisot unit.

It is well known that Pisot dual tiles for quadratic Pisot units are nothing but intervals. For the sake of the completeness, we describe them in this subsection. Let β be a quadratic Pisot unit. Its minimal polynomial is either $x^2 - ax - 1$ $(a \geq 1)$ or $x^2 - ax + 1$ $(a \geq 3)$.

Case $x^2 - ax - 1$ $(a \geq 1)$. In this case $d_\beta(1) = a, 1$ which satisfies the condition of Theorem 1.2 on page 273. Therefore T_A is a non empty compact connected set in $\mathbb{R}^1$, that is, a closed interval. One can obtain their concrete shapes by computing extremal values. Take the conjugate $\beta' = (a - \sqrt{a^2+4})/2 \in (-1, 0)$. Then

$$\begin{aligned} T_\lambda &= \left\{ \sum_{i=0}^{\infty} a_i(\beta')^i \;\middle|\; a_{i+1}, a_i <_{\text{lex}} a, 1 \right\} \\ &= \left[\sum_{k=1}^{\infty} a(\beta')^{2k-1}, \sum_{k=0}^{\infty} a(\beta')^{2k} \right] \\ &= \left[\frac{a\beta'}{1-(\beta')^2}, \frac{a}{1-(\beta')^2} \right] = [-1, \beta] \end{aligned}$$

The other tile is

$$\begin{aligned} T_1 - \frac{1}{\beta'} &= \left\{ \sum_{i=0}^{\infty} a_i(\beta')^i \in T_\lambda \;\middle|\; a_0 \neq a \right\} \\ &= \left[\frac{a\beta'}{1-(\beta')^2}, \frac{a}{1-(\beta')^2} - 1 \right] = [-1, \beta - 1]. \end{aligned}$$

The translation $-1/\beta'$ was performed to make clearer the situation.

Case $x^2 - ax + 1$ $(a \geq 3)$. We have $d_\beta(1) = (a-1), (a-2)^\omega$ and $\eta = \max_{k\geq 1} U_\beta^k(1) = (a-2)^\omega$. Take the conjugate $\beta' = (a - \sqrt{a^2-4})/2 \in (0, 1)$. By (4.3) we have

$$G_{-1}(T_\lambda) = \beta'^{-1}T_\lambda = T_\lambda \cup T_1 \cup \cdots \cup T_{a-2} \cup T_{a-1}$$

and

$$G_{-1}(T_{a-1}) = T_{0,a-1} \cup T_{1,a-1} \dots T_{a-3,a-1} \cup T_{a-2,a-1}.$$

Up to translation, there are only two tiles T_λ and T_η. If $A <_{\text{lex}} \eta$ then T_A is congruent to T_λ and if $A \geq_{\text{lex}} \eta$ then T_A is congruent to T_η. Observing the above set of equations, the smaller tile T_η appears only at the last terms T_{a-1} and $T_{a-2,a-1}$. Therefore in view of the proof of Theorem 4.1 on page 287, to prove the connectedness of tiles, we only need to show a weaker condition:

$$T_{a-1} \cap T_{a-2} \neq \emptyset,$$

which is shown by

$$T_{a-1} \ni \frac{a-1}{\beta'} = \frac{a-2}{\beta'} + a - 1 + \sum_{i=2}^{\infty} (a-2)(\beta')^i \in T_{a-2}.$$

As a result, the condition of Theorem 4.1 on page 287 is sufficient but not necessary to have connectedness. A similar computation yields:

$$T_\lambda = \left[0, 1 + \frac{a-2}{1-\beta'}\right] = [0, \beta] \quad \text{and} \quad T_{a-1} - \frac{a-1}{\beta'} = \left[0, \frac{a-2}{1-\beta'}\right] = [0, \beta - 1].$$

4.2 Connectedness of self-affine tilings generated by a cubic Pisot unit.

Let β be a Pisot unit of degree 3 defined by the monic polynomial $p(x) \in \mathbb{Z}[x]$. In this subsection we prove that the dual tiling generated by β is connected, i.e. each tile is connected. To make explicit the cubic case of Corollary 2.2 on page 278, we have

Theorem 4.2 *A monic polynomial*

$$p(x) = x^3 - ax^2 - bx - c \in \mathbb{Z}[x]$$

is a Pisot polynomial if and only if three inequalities

$$1 < a + b + c, \ |b - 1| < a + c \quad \textit{and} \quad (c^2 - b) < \operatorname{sgn}(c)(1 + ac)$$

hold.

The following Theorem due to Akiyama [4] and Bassino [11] gives the β-expansion of 1 for the cubic Pisot units. Note that [11] also dealt with non unit Pisot cases.

Theorem 4.3 *Let β be a cubic Pisot unit and let*

$$p(x) = x^3 - ax^2 - bx - c$$

with $c = \pm 1$ be its minimal polynomial. Then the β-expansion of 1 is given by the following table.

$c=1$	
b	$d_\beta(1)$
$-a+1\le b\le -2$	$a-1, a+b-1, (a+b)^\omega$
$b=-1$	$a-1, a-1, 0, 1$
$0\le b\le a$	$a, b, 1$
$b=a+1$	$a+1, 0, 0, a, 1$

$c=-1$	
b	$d_\beta(1)$
$-a+3\le b\le 0$	$a-1, a+b-1, (a+b-2)^\omega$
$1\le b\le a-1$	$a, (b-1, a-1)^\omega$

From now on, for simplicity we denote $\beta_i = \Phi(\beta^i)$.

Theorem 4.4 *Let β be a Pisot unit of degree 3. Then each tile is arcwise connected.*

Proof:
We only need to prove this theorem for the cases when the β-expansion of 1 is infinite because the other cases are shown by Theorem 1.2 on page 273 (c.f. [4]). We use Corollary 4.1 on page 288 to prove the connectedness of each tile.

<u>*Case 1.*</u> $c=1$ and $-a+1\le b\le -2$.
Here $d_\beta(1) = .a-1, a+b-1, (a+b)^\omega$, $\lfloor\beta\rfloor = a-1$ and the smallest tile in this case is T_η for $\eta=(a+b)^\omega$. Since every conjugate of β is also a root of $p(x)(x^3-1)(1+x)(1+x^6+\cdots+x^{6n}+\cdots)$, we have

$$1+(b+1)\beta_1+\sum_{i=0}^{\infty}\left((a+b)\beta_2+(a-2)\beta_3-(b+2)\beta_4-(a+b)\beta_5-(a-2)\beta_6+(b+2)\beta_7\right)\beta_{6i}=0$$

and all the coefficients have absolute value less than $\lfloor\beta\rfloor = a-1$.
<u>*Case 2.*</u> $c=-1$ and $-a+3\le b\le 0$.
Here $d_\beta(1) = .a-1, a+b-1, (a+b-2)^\omega$, $\lfloor\beta\rfloor = a-1$ and the smallest tile in this case is T_η for $\eta = a+b-1, (a+b-2)^\omega$.
 ♦ Suppose that $b\le -1$.
Since every conjugate of β is a root of $p(x)\sum_{i=0}^{\infty}x^i$, we have

$$1+(1-b)\beta_1+(1-a-b)\beta_2+(2-a-b)\sum_{i=3}^{\infty}\beta_i = 0$$

and all the coefficients have absolute value less than $a-1$.
 ♦ Suppose that $b=0$.
Since every conjugate of β is a root of $p(x)\sum_{i=0}^{\infty}x^{2i}$, we have ${}^\omega(1,1-a),0,1.=0$ and ${}^\omega(1,0),0.1 = {}^\omega(a-1,0).1 \; - 0.1$. Adding $.(a-2)^\omega$ we get that a common point of T_η and $T_\eta-\Phi(\beta^{-1})$ is

$${}^\omega(1,0),0.\eta = {}^\omega(a-1,0).\eta \;\; - \;\; 0.1$$

According to (1.2), both expansions are admissible.
 <u>*Case 3.*</u> $c=-1$ and $1\le b\le a-1$.
Here $d_\beta(1) = .a, (b-1, a-1)^\omega$ and the smallest tile in this case is T_η for $\eta=(a-1,b-1)^\omega$. Since every conjugate of β is also a root of $p(x)\sum_{i=0}^{\infty}x^{2i}$, we have

$$1-b\beta_1+\sum_{i=0}^{\infty}\left((1-a)\beta_2+(1-b)\beta_3\right)\beta_{2i}=0.$$

and all the coefficients have absolute value less than $\lfloor\beta\rfloor = a$. □

4.3 Connectedness of self-affine tilings generated by a quartic Pisot unit.

Let β be a Pisot unit of degree 4 defined by the monic polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1 \in \mathbb{Z}[x]$. We prove that the dual tiling generated by β is connected, i.e. each tile is connected, if $p(0) = 1$. We also prove that if $p(0) = -1$ then $a+c-2\lfloor\beta\rfloor \le 1$ and that each tile is connected if and only if $a+c-2\lfloor\beta\rfloor \ne 1$. If $p(0) = -1$ and $a + c - 2[\beta] = 1$, we prove the existence of a disconnected tile. As a byproduct, we give a complete classification of the β-expansion of 1 for quartic Pisot units. Let us start with a

Proposition 4.1 *A monic polynomial*

$$p(x) = x^4 - ax^3 - bx^2 - cx - d$$

with $d = \pm 1$ *is a Pisot polynomial if and only if*

$$\begin{cases} |b-2| < a+c, \\ a-c > 0, \end{cases} \quad \textit{for}\ \ d=-1; \qquad \begin{cases} |b| < a+c, \\ a^2+4b-c^2 > 0, \end{cases} \quad \textit{for}\ \ d=1.$$

which is just an explicit form of the quartic case of Corollary 2.2 on page 278. In the Theorem 4.5 and Theorem 4.7 on page 301, we frequently use Parry's conditions (1.1) and (1.2) on admissible words.

Theorem 4.5 *Let β be a Pisot unit of degree 4 with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx + 1$. Then each tile is arcwise connected.*

Proof: To prove this and the following Theorem we use Theorem 4.1, Corollary 4.1 on page 288. If β is a Pisot unit of degree 4 then, according to the Proposition 4.1, we have that the coefficients satisfy the system of inequalities:

$$\begin{cases} |b-2| \le a+c-1, \\ a-c \ge 1, \end{cases} \implies \begin{cases} a \ge 1, \\ 1-a \le c \le a-1, \\ 3-a-c \le b \le a+c+1. \end{cases}$$

<u>*Case 1.*</u> If $1-a \le c \le -1$ then $4-a \le b \le a$.

- If $2 \le b \le a$, we have $a \ge 2$. Here, $\lfloor\beta\rfloor = a$,

$$d_\beta(1) = .a, b-1, (a+c, b-2)^\omega,$$

and the smallest tile is T_η for

$$\eta = \begin{cases} (a+c, b-2)^\omega, & \text{if } b-1 < a+c; \\ b-1, (b-1, b-2)^\omega, & \text{if } b-1 = a+c. \end{cases}$$

Since every conjugate of β is also a root of $p(x)\sum_{i=0}^{\infty} x^{2i}$, we have

$$1 - c\beta_1 + (1-b)\beta_2 + \sum_{i=0}^{\infty} \left(-(a+c)\beta_3 + (2-b)\beta_4\right) \beta_{2i} = 0.$$

Here, all the coefficients have absolute value less than $\lfloor\beta\rfloor$, so according to Corollary 4.1 on page 288, each tile is arcwise connected.

- If $b = 1$ then $a \geq 3$, $c \geq 2-a$. In this case $\lfloor\beta\rfloor = a$,

$$d_\beta(1) = .a, 0, a+c-1, a-1, a+c, (a+c-1)^\omega,$$

and the smallest tile is T_η for $\eta = a-1, a+c, (a+c-1)^\omega$. Since every conjugate of β is also a root of $p(x)\sum_{i=0}^{\infty} x^{3i}$, we have

$$1-c\beta_1+\sum_{i=0}^{\infty} \left(-\beta_2+(1-a)\beta_3+(1-c)\beta_4\right) \beta_{3i} = 0,$$

and all the coefficients have absolute value less than $\lfloor\beta\rfloor$.

- If $4 - a \leq b \leq 0$ then $a \geq 4$ and $c \geq 3 - a$. Here $\lfloor\beta\rfloor = a-1$,

$$d_\beta(1) = .a-1, a+b-1, a+b+c-1, (a+b+c-2)^\omega,$$

and the smallest tile is T_η for $\eta = a+b-1, a+b+c-1, (a+b+c-2)^\omega$. Since every conjugate of β is also a root of $p(x)\sum_{i=0}^{\infty} x^{3i}$, we have

$$1-c\beta_1+\sum_{i=0}^{\infty} \left(-b\beta_2+(1-a)\beta_3+(1-c)\beta_4\right) \beta_{3i} = 0,$$

and all the coefficients except $1-a$ have absolute value less than $\lfloor\beta\rfloor$. A common point of T_η and $T_\eta - \Phi(\beta^{-1})$ is

$$^\omega(1-c, 0, -b), -c.\eta = {}^\omega(a-1, 0, 0).\eta \;\; - \;\; 0.1.$$

<u>*Case 2.*</u> If $0 \leq c \leq a-1$ then $4-2a \leq b \leq 2a$.

- If $4 - 2a \leq b \leq -a$, we have $a \geq 4$, $c \geq 3$, $2 \leq 2a + b - 2 \leq a - 2$ and $\lfloor\beta\rfloor = a - 2$.

  $*$ If $b \leq -a - 1$, then $c \geq 4$ and $a \geq 5$.
  First, let us find the β-expansion of 1 . Since $1 \leq a+b+c-2 < a-2$, there exists an integer $2 \leq k \leq a-2$ with $\frac{a-2}{k} \leq a+b+c-2 < \frac{a-2}{k-1}$, which implies that $(k-1)(a+b+c-2) < a-2 \leq k(a+b+c-2)$.

  $\diamond$ If $(k-1)(a+b+c-2) \geq c-2$ we get $k \geq 3$. Let m be the integer defined by $m = \inf\{i : (i+1)(a+b+c-2) \geq c-2\}$. Since $b < -a$, we have $m \geq 1$. By the definition $m \leq k-2$ and $(m+1)(a+b+c-2) \leq a-3$.

  ⧫ If $(m+1)(a+b+c-2) < a-3$ let us show that the β-expansion of 1 is eventually periodic with period 1 and preperiod $m+3$, so let us write it as $d_\beta(1) = .d_1, d_2, \cdots, d_{m+3}, d_{m+4}^\omega$.
  When $m = 1$, since

$$p(x)(1+x) = x^5-(a-1)x^4-(a+b)x^3-(b+c)x^2-(c-1)x+1,$$

we get that

$$1 = .a-2, 2a+b-2, 2a+2b+c-2, 2a+2b+2c-3, (2a+2b+2c-4)^{\omega}.$$

Here $d_5 = d_{m+4},\ d_4 = d_{m+3}, d_3 = d_{m+2}$.
When $m = 2$, since

$$p(x)(1+x+x^2) = x^6-(a-1)x^5-(a+b-1)x^4-(a+b+c)x^3-(b+c-1)x^2-(c-1)x+1,$$

we get that

$$1 = .a-2, 2a+b-3, 3a+2b+c-3, 3a+3b+2c-4, 3(a+b+c)-5, 3(a+b+c-2)^{\omega}.$$

Here $d_6 = d_{m+4},\ d_5 = d_{m+3},\ d_4 = d_{m+2},\ d_3 = d_{m+1}$, where the formulas of d_i will be given later.
When $m \geq 3$, since

$$p(x)\sum_{i=0}^{m} x^i = x^{m+4}-(a-1)x^{m+3}-(a+b-1)x^{m+2}-(a+b+c-1)x^{m+1}-\sum_{i=4}^{m}(a+b+c-2)x^i-(a+b+c-1)x^3-(b+c-1)x^2-(c-1)x+1$$

(where the terms $\sum_{i=4}^{m}(a+b+c-2)x^i$ do not appear for $m = 3$), we have that

$$\begin{aligned} &d_1 = a-2, \qquad d_2 = 2a+b-3, \qquad d_3 = 3a+2b+c-4,\\ &d_i = d_{i-1} + (a+b+c-2) \quad \text{for } i \in \{4, 5, \cdots m\},\\ &d_{m+4} = (m+1)(a+b+c-2), \qquad d_{m+3} = d_{m+4} + 1, \qquad\qquad (**)\\ &d_{m+2} = d_{m+3} - (c-1), \qquad d_{m+1} = d_{m+2} - (b+c-1). \end{aligned}$$

We now verify that the conditions of lexicographic order on $d_\beta(1)$ are satisfied. Since $a+b+c-1 \geq 2$, we have that $d_2 < d_3 < \cdots < d_m < d_{m+1}$. Here we get that $d_2 \geq b+2a-3 \geq 1$ and $d_{m+1} =\leq a-2$. Since $d_{m+1} > d_{m+2}$ and $d_{m+2} \geq 0$ we need to check only the case when $d_{m+1} = a-2$, which implies that $d_2 - d_{m+2} = a-c > 0$. So the conditions of lexicographic order are satisfied.

♦ If $(m+1)(a+b+c-2) = a-3$ then $m = k+2$. As a result we have

$$m(a+b+c-2) < c-2 \leq (m+1)(a+b+c-2) = a-3 < a-2 \leq (m+2)(a+b+c-2),$$

which implies that $b+2c-2 \geq 0$.
For $m = 1$ we get $a+2b+2c-1 = 0$.
Since $p(x)(x+1)(x^2+x+1) = x^7 - \sum_{i=1}^{5} d_i x^{7-i} - (c+2)x + 1$ is equal to

$$x^7-(a-2)x^6-(b+a-2)x^5-(2a+2b+c-1)x^4-(b+2c-2)x^2-(c+2)x+1,$$

and $d_1 = a-2 > d_2 > d_3 > d_4 = 0$, $0 \leq d_5 \leq c-3 \leq a-4$, we get that

$$d_\beta(1) = .a-2, (b+2a-2, 2a+2b+c-1,\ 0\ , 2c+b-2, c-3, a-3)^{\omega}.$$

For $m \geq 2$ we will show that
▼ If $b+2c-2 > 0$, the β-expansion of 1 is eventually periodic with period $2m+4$ and preperiod 1. So

$$d_\beta(1) = .a-2, (d_2, \cdots, d_{2m+3}, c-3, a-3)^{\omega}.$$

▼ If $b+2c-2 = 0$, the β-expansion of 1 is eventually periodic with period 1 and preperiod $2m+4$. So

$$d_\beta(1) = .a-2, d_2, \cdots, d_{2m+1}, d_{2m+2}-1, a-2, a+b+c-3, (a+b+c-2)^{\omega}.$$

In both cases, d_i's satisfy
$p(x)\sum_{i=0}^{m} x^i \sum_{i=0}^{m+1} x^i = x^{2m+5} - \sum_{i=1}^{2m+3} d_i\, x^{2m+5-i} - (c-2)x + 1.$
Since $ma+(m+1)b+(m+1)c-2m+1 = 0$, we have
$d_1 = a-2, \qquad d_2 = 2a+b-3,$
$d_i = ia + (i-1)b + (i-2)c - 2(i-1) \quad$ for $3 \leq i \leq m$, (these terms do not appear for $m = 2$)

$d_{m+1}=a-b-2c,\qquad d_{m+2}=a-c,\qquad d_{m+3}=0,\qquad d_{m+4}=-a-b,$
$d_{2m+3-i}=ia+(i+1)b+(i+2)c-2(i+1)$ for $1\le i\le m-2$, (these terms do not appear for $m=2$)
$d_{2m+3}=b+2c-3$

Since $a+b+c-2>0$, we have $d_i<d_{i+1}$ and $d_{m+2+i}>d_{m+3+i}$ for $2\le i\le m$. We also have that $d_2>0$, $0\le d_{m+2}<d_{m+1}$, $d_{m+4}\le a-4$ and $d_{2m+2}<a-2$. Since $d_{m+1}\le a-3$ for $b+2c-3\ge 0$, for $2\le i\le 2m+3$ we have that $0\le d_i\le a-3$.
For $b+2c-2=0$ we have that $m(a+b+c-2)=c-3$, $d_{m+1}=a-2$ and $d_3=3a-3c$ for $m\ge 3$. Since $a+b+c-3=a-c-1$, $d_{m+2}=a-c$ and $d_2-(a-c)=a-c-1\ge 0$, for $c=a-1$ we need to compare d_3 with $d_{m+3}=0$. Since $d_3>0$, the conditions of lexicographic order on $d_\beta(1)$ are satisfied in this case also.

- If $(k-1)(a+b+c-2)<c-2$, let us show that the β-expansion of 1 is eventually periodic with preperiod 1 and period $2k+2$. So
$$d_\beta(1)=.a-2,(d_2,\cdots,d_{2k+1},c-3,a-3)^\omega,$$
where d_i's are as follows:
$p(x)\sum_{i=0}^{k-1}x^i\sum_{i=0}^{k}x^i=x^{2k+3}-\sum_{i=1}^{2k+1}d_i\,x^{2k+3-i}-(c-2)x+1$. So we have
$d_1=a-2,\qquad d_2=2a+b-3,$
$d_i=ia+(i-1)b+(i-2)c-2(i-1)$ for $3\le i\le k-1$, (these terms do not appear for $k=3$)
$d_k=ka+(k-1)b+(k-2)c-2k+3,\quad d_{k+1}=ka+kb+(k-1)c-2k+3,$
$d_{k+2}=(k-1)a+kb+kc-2k+3,\quad d_{k+3}=(k-2)a+(k-1)b+kc-2k+3,$
$d_{2k+1-i}=ia+(i+1)b+(i+2)c-2(i+1)$ for $1\le i\le k-3$, (these terms do not appear for $k=3$)
$d_{2k+1}=b+2c-3.$
So we have that $d_1>d_2$, $d_i<d_{i+1}$ for $2\le i\le k-1$, $d_k>d_{k+1}>d_{k+2}$, $d_{k+2}<d_{k+3}$, $d_{k+i}>d_{k+1+i}$ for $3\le i\le k$. First we notice that $d_2\ge 1$, $d_k\le a-2$, $d_{k+2}\ge 1$, $d_{k+3}<d_k$ and $d_{2k+1}\ge d_{k+1}-1$. So all the d_i's are nonnegative and smaller than d_1, only d_k can be equal to d_1. But, if $(k-1)(a+b+c-2)=c-3$ we have that $d_3>d_2\ge d_{k+1}>d_{k+2}$. So conditions of lexicographic order are satisfied.

Second we find the common point of T_η and $T_\eta-\Phi(\beta^{-1})$. Since every conjugate of β is also a root of $p(x)(x^2+x+1)(x+1)\sum_{i=0}^{\infty}x^{6i}$, we have
$$1+(2-c)\beta_1+\sum_{i=0}^{\infty}((2-2c-b)\beta_2+(1-a-2b-2c)\beta_3+(1-2a-2b-c)\beta_4$$
$$+(2-2a-b)\beta_5+(3-a)\beta_6+(3-c)\beta_7)\,\beta_{6i}=0$$
and all the coefficients have absolute value less than $\lfloor\beta\rfloor$.

$*$ If $b=-a$, we have $5-a\le 2c-a-1\le a-3$. We get that
$$d_\beta(1)=.a-2,(a-2,c-1,2c-a-1,2c-a-2,c-3,a-3)^\omega$$
for $1\le 2c-a-1\le a-3$, while
$$d_\beta(1)=.a-2,a-2,c-2,2c-3,(2c-4)^\omega$$

for $5-a \le 2c-a-1 \le 0$.

- ◇ If $3 \le c \le a-2$, since every conjugate of β is also root of $p(x)(x^2+x+1)\sum_{i=0}^{\infty} x^{3i}$, we have

$$1+(1-c)\beta_1+(a+1-c)\beta_2+(1-c)\beta_3+(2-c)\beta_4\sum_{i=0}^{\infty}\beta_i=0.$$

For $4 \le c \le a-2$ all the coefficients have absolute value less than $\lfloor\beta\rfloor$, so according to Corollary 4.1 on page 288, each tile is arcwise connected.
For $c=3$ we get that

$$a-2\,,\,0\,.\,1 = {}^{\omega}1\,,\,2\,,\,0\,,\,2\,.\,0$$

which shows that $a-2,0.\eta = {}^{\omega}1,2,0,2.\eta-0.1$ is a common point of T_η and $T_\eta-\Phi(\beta^{-1})$ for $a \ge 5$.

- ◇ If $c=a-1$, since every conjugate of β is also a root of $p(x)(x^3-1)(x+1)^2\sum_{i=0}^{\infty}x^{6i}=0$, then

$$1+(3-a)\beta_1+(3-a)\beta_2+(-2\beta_4-\beta_5+\beta_6+2\beta_7+\beta_8-\beta_9)\sum_{i=0}^{\infty}\beta_{6i}=0$$

and for $c \ge 4$ all the coefficients have absolute value less than $\lfloor\beta\rfloor$, so, according to Corollary 4.1 on page 288, each tile is arcwise connected.
If $b=-a$, $c=3$ and $a=4$ we have that

$$d_\beta(1) = .2,(2,2,1,0,0,1)^\omega,$$

and for $\eta=(2,2,1,0,0,1)^\omega$ we get that

$${}^{\omega}(1,2,1,0,0,0),0,0.\eta = {}^{\omega}(1,0,0,0,1,2),0,1,1.\eta \quad -0.1$$

is a common point of T_η and $T_\eta-\Phi(\beta^{-1})$.

- • If $-a+1 \le b \le -1$, we have $3-a \le b+c \le a-2$.

◇ If $b+c \ge 0$ and $c \ge 2$, we have $a \ge 3$ and

$$d_\beta(1) = .a-1,(b+a,c+b,c-2,a-2)^\omega.$$

Since every conjugate of β is also a root of $p(x)(x+1)\sum_{i=0}^{\infty}x^{4i}=0$, we have

$$1+(1-c)\beta_1+(-b-c)\beta_2-(a+b)\beta_3+(2-a)\beta_4+(2-c)\beta_5)\sum_{i=0}^{\infty}\beta_{4i}=0$$

and for $b \le -2$ all the coefficients have absolute value less than $\lfloor\beta\rfloor$. So, according to Corollary 4.1 on page 288, each tile is arcwise connected.
For $b=-1$, since $d_\beta(1) = .a-1,(a-1,c-1,c-2,a-2)^\omega$, the smallest tile is T_η for $\eta=(a-1,c-1,c-2,a-2)^\omega$. Thus we get that

$$.\eta = {}^{\omega}(c-2,a-2,a-1,c-1),c-1.\eta \quad -0.1$$

is a common point of T_η and $T_\eta-\Phi(\beta^{-1})$.

◇ If $b+c \ge 0$ and $c=1$, we have $b=-1$, $a \ge 3$ and

$$d_\beta(1) = .a-1,a-2,a-1,(a-2)^\omega.$$

Hence the smallest tile is T_η for $\eta=a-1,(a-2)^\omega$. Since every conjugate of β is also a root of $p(x)(1+x^3)\sum_{i=0}^{\infty}x^{6i}=0$, we have

$$1-\beta_1+(\beta_2+(1-a)\beta_3)\sum_{i=0}^{\infty}\beta_{3i}=0.$$

So a common point of T_η and $T_\eta-\Phi(\beta^{-1})$ is

$${}^{\omega}(0,1,0).\eta = {}^{\omega}(0,a-1,0),1.\eta - \quad 0.1.$$

◇ If $b+c\le -1$, we have $a\ge 4$ and

$$d_\beta(1) = .a-1,a+b-1,a+b+c-1,(a+b+c-2)^{\omega}.$$

So the smallest tile in this case is T_η for

$$\eta = \begin{cases} .a+b+c-1,(a+b+c-2)^{\omega}, & \text{for } c\ge 1;\\ .a+b-1,a+b-1,(a+b-2)^{\omega}, & \text{for } c=0. \end{cases}$$

Since every conjugate of β is also a root of $p(x)(x+1)\sum_{i=0}^{\infty}x^{4i}=0$, we have

$$1+(1-c)\beta_1+(-b-c)\beta_2-(a+b)\beta_3+(2-a)\beta_4+(2-c)\beta_5)\sum_{i=0}^{\infty}\beta_{4i}=0$$

and for $b\le -2$ all the coefficients have absolute value less than $\lfloor\beta\rfloor$. So, according to Corollary 4.1 on page 288, each tile is arcwise connected.
For $b=-1$ we have that $c=0$ and $\eta=a-2,a-2,(a-3)^{\omega}$. So

$${}^{\omega}(2,0,0,1),\ 1.\eta = {}^{\omega}(a-2,a-1,0,0).\eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta-\Phi(\beta^{-1})$.

• If $0\le b\le a$, we have $a\ge 2$ and

$$d_\beta(1) = \begin{cases} .a,(b,c-1,a-1)^{\omega}, & \text{if } a\ge 2 \text{ and } 1\le c\le a-1;\\ .a,b-1,(a,b-2)^{\omega}, & \text{if } a\ge 2,\ c=0, \text{ and } 2\le b\le a;\\ .a,0,a-1,(a,0,a-2)^{\omega}, & \text{if } a\ge 2,\ c=0 \text{ and } b=1;\\ .a-1,a-1,a-1,(a-2)^{\omega}, & \text{if } a\ge 3,\ c=0 \text{ and } b=0. \end{cases}$$

So $\lfloor\beta\rfloor = \begin{cases} a-1, & \text{if } b=c=0;\\ a, & \text{otherwise;} \end{cases}$ and the smallest tile is T_η for

$$\eta = \begin{cases} (a,c-1,a-1)^{\omega}, & \text{for } a\ge 2,\ b=a, \text{ and } 1\le c\le a-1;\\ (a-1,b,c-1)^{\omega}, & \text{for } a\ge 2,\ 0\le b\le a-1, \text{ and } 1\le c\le a-1;\\ (a,b-2)^{\omega}, & \text{for } a\ge 2,\ c=0, \text{ and } 2\le b\le a;\\ (a,0,a-2)^{\omega}, & \text{for } a\ge 2,\ b=1, \text{ and } c=0;\\ a-1,a-1,(a-2)^{\omega}, & \text{for } a\ge 3,\ b=0, \text{ and } c=0. \end{cases}$$

Since every conjugate of β is also a root of $p(x)\sum_{i=0}^{\infty}x^{3i}=0$, we have

$$1-c\beta_1-(b\beta_2+(a-1)\beta_3+(c-1)\beta_4)\sum_{i=0}^{\infty}\beta_{3i}=0. \tag{4.4}$$

∗ For $b\le a-1$, with the exception of the case where $b=c=0$, all the coefficients have absolute value less than $\lfloor\beta\rfloor=a$. So, according to Corollary 4.1 on page 288, each tile is arcwise connected.
For $b=c=0$, from (4.4), we get that

$$^{\omega}(1,0,0),0.\eta = {}^{\omega}(a-1,0,0).\eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$, where $\eta = a{-}1, a{-}1, (a{-}2)^\omega$.

∗ For $b = a$ and $c \geq 1$, from (4.4), we get that

$$.\eta = {}^{\omega}(c-1,a-1,a),c.\eta - \quad 0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$ where $\eta = (a, c{-}1, a{-}1)^\omega$.

∗ For $b = a$ and $c = 0$, from (4.4), we get that

$$^{\omega}(1,0,0),0.\eta = (a-1,a,0).\eta - \quad 0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$ where $\eta = (a, a{-}2)^\omega$.

- If $a{+}1 \leq b \leq 2a$, we have $c{-}b{+}a \geq -1$.

  ∗ If $c{-}b{+}a \geq 0$, we have $\lfloor\beta\rfloor = a+1$ and

  $$d_\beta(1) = .a{+}1,(b{-}a{-}1, c{+}a{-}b, b{-}c{-}1, c, a)^\omega.$$

  Since every conjugate of β is also a root of $p(x)(x{-}1)(x^4{-}1)\sum_{i=0}^{\infty} x^{8i} {=} 0$, we have

  $$1-(c{+}1)\beta_1+((c{-}b)\beta_2{+}(b{-}a)\beta_3{+}a\beta_4{+}c\beta_5)\,(1{-}x^4)\sum_{i=0}^{\infty}\beta_{8i} = 0$$

  and all the coefficients have absolute value less than $\lfloor\beta\rfloor$.

  ∗ If $c{-}b{+}a = -1$ and $b \geq a{+}2$, we have $\lfloor\beta\rfloor = a+1$ and

  $$d_\beta(1) = .a{+}1,(b{-}a{-}2, a{+}1, b{-}a{-}2, 0, a{-}1, b{-}a, a{-}1, b{-}a{-}1, a)^\omega.$$

  Since every conjugate of β is also a root of $p(x)(x^2{-}x{+}1)\sum_{i=0}^{\infty} x^{5i} {=} 0$, we have

  $$1-(c{+}1)\beta_1-(a\beta_2-\beta_3+c\beta_4+a\beta_5+c\beta_6)\sum_{i=0}^{\infty}\beta_{5i} = 0$$

  and all the coefficients have absolute value less than $\lfloor\beta\rfloor$.

  ∗ If $c{-}b{+}a = -1$ and $b = a{+}1$, we have $c = 0$, $\lfloor\beta\rfloor = a$ and

  $$d_\beta(1) = .a, a, (a, a{-}1)^\omega.$$

  Since every conjugate of β is also a root of the $p(x)(x^2{+}1)\sum_{i=0}^{\infty}\beta_{4i} {=} 0$, we have

  $$.\eta = {}^{\omega}(a-1,a),a,0.\eta \quad -0.1$$

  is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$.

□

From the proof of the Theorem 4.5 on page 292 we get also the following theorem which gives the β-expansion of 1 for any Pisot unit of degree four with minimal polynomial $x^4 - ax^3 - bx^2 - cx + 1 = 0$.

Theorem 4.6 *Let β be a Pisot unit of degree four with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx + 1 = 0$. The β-expansion of 1 is:*

- *When $1 - a \leq c \leq -1$,*

- $\star$ *and* $4-a \le b \le 0$, $d_\beta(1) = .a-1, a+b-1, a+b+c-1, (a+b+c-2)^\omega$
- $\star$ *and* $b = 1$, $d_\beta(1) = .a, 0, a+c-1, a-1, a+c, (a+c-1)^\omega$
- $\star$ *and* $2 \le b \le a$, $d_\beta(1) = .a, b-1, (a+c, b-2)^\omega$

- *When* $0 \le c \le a-1$
  - $\star$ *and* $4-2a \le b \le -a-1$, *let k be the integer of* $\{2, 3, \cdots, a-2\}$ *with* $(k-1)(a+b+c-2) < a-2 \le k(a+b+c-2)$.
    - $*$ *If* $(k-1)(a+b+c-2) \ge c-2$ *let* $m = \inf\{i \in \mathbb{N}$ *such that* $(i+1)(a+b+c-2) \ge c-2\}$. *We have* $1 \le m \le k-2$.
      - $\diamond$ *If* $(m+1)(a+b+c-2) < a-3$, *the* β*-expansion of* 1 *is eventually periodic with period 1 and preperiod* $m+3$.
        $m=1 \Rightarrow d_\beta(1) = .a-2, 2a+b-2, 2a+2b+c-2, 2a+2b+2c-3, (2a+2b+2c-4)^\omega$,
        $m=2 \Rightarrow d_\beta(1) = .a-2, 2a+b-3, 3a+2b+c-3, 3a+2b+2c-4, 3(a+b+c)-5, (3a+3b+3c-6)^\omega$,
        $m \ge 3 \Rightarrow d_\beta(1) = .a-2, 2a+b-3, 3a+2b+c-4, d_4, \cdots, d_{m+3}, (d_{m+4})^\omega$,
        with $d_i = d_{i-1}+a+b+c-2$ *for* $4 \le i \le m$ *(these terms do not appear for* $m = 3$*)*
        and $\begin{cases} d_{m+1} = d_{m+2}-b-c+1, \\ d_{m+2} = d_{m+3}-c+1, \\ d_{m+3} = d_{m+4}+1, \\ d_{m+4} = (m+1)(a+b+c-2). \end{cases}$
      - $\diamond$ *If* $(m+1)(a+b+c-2) = a-3$ *then*
        $m=1 \Rightarrow d_\beta(1) = .a-2, (2a+b-2, 2a+2b+c-1, 0, 2c+b-2, c-3, a-3)^\omega$
        If $m \ge 2$ *and* $b+2c \ge 3$, *the* β*-expansion of* 1 *is eventually periodic with preperiod 1 and period* $2m+4$. *So*
        $$d_\beta(1) = .a-2, (2a+b-3, d_3, \cdots, d_{2m+3}, c-3, a-3)^\omega.$$
        If $m \ge 2$ *and* $b+2c = 2$, *the* β*-expansion of* 1 *is eventually periodic with period 1 and preperiod* $2m+4$. *So*
        $d_\beta(1) = .a-2, 2a+b-3, d_3, \cdots, d_{2m+1}, d_{2m+2}-1, a-2, a+b+c-3, (a+b+c-2)^\omega$ *where*
        $d_i = ia+(i-1)b+(i-2)c-2(i-1)$ *for* $3 \le i \le m$, *(these terms do not appear for* $m = 2$*)*
        $d_{m+1} = a-b-2c, \quad d_{m+2} = a-c, \quad d_{m+3} = 0, \quad d_{m+4} = -a-b,$
        $d_{2m+3-i} = ia+(i+1)b+(i+2)c-2(i+1)$ *for* $1 \le i \le m-2$, *(these terms do not appear for* $m = 2$*)*
        $d_{2m+3} = b+2c-3.$
    - $*$ *If* $(k-1)(a+b+c-2) \le c-3$, *the* β*-expansion of* 1 *is eventually periodic with preperiod 1 and period of length* $2k+2$. *So*
      $$d_\beta(1) = .a-2, (2a+b-3, d_3, \cdots, d_{2k+1}, c-3, a-3)^\omega$$
      with $d_i = ia+(i-1)b+(i-2)c-2(i-1)$ *for* $3 \le i \le k-1$, *(these terms do not appear for* $k = 3$*)*
      $d_k = ka+(k-1)b+(k-2)c-2k+3, \quad d_{k+1} = ka+kb+(k-1)c-2k+3,$
      $d_{k+2} = (k-1)a+kb+kc-2k+3, \quad d_{k+3} = (k-2)a+(k-1)b+kc-2k+3,$
      $d_{2k+1-i} = ia+(i+1)b+(i+2)c-2(i+1)$ *for* $1 \le i \le k-3$, *(these terms do not appear for* $k = 3$*)*
      $d_{2k+1} = b+2c-3.$

$\star$ *and* $b = -a$, $d_\beta(1) = \begin{cases} .a{-}2, (a{-}2, c{-}1, 2c{-}a{-}1, 2c{-}a{-}2, c{-}3, a{-}3)^\omega, & \text{if } 2c{-}a \geq 2; \\ .a{-}2, a{-}2, c{-}2, 2c{-}3, (2c{-}4)^\omega, & \text{if } 2c{-}a \leq 1; \end{cases}$

$\star$ *and* $-a+1 \leq b \leq -1$,

$*$ *for* $b+c \geq 0$ *we have* $d_\beta(1) = \begin{cases} .a{-}1, (b{+}a, c{+}b, c{-}2, a{-}2)^\omega, & \text{if } (b,c) \neq (-1,1); \\ .a{-}1, a{-}2, a{-}1, (a{-}2)^\omega, & \text{if } (b,c) = (-1,1); \end{cases}$

$*$ *if* $b+c \leq -1$, $d_\beta(1) = .a-1, a+b-1, a+b+c-1, (a+b+c-2)^\omega$,

$\star$ *and* $0 \leq b \leq a$,

$*$ *for* $c \geq 1$ *we have* $d_\beta(1) = .a, (b, c{-}1, a{-}1)^\omega$,

$*$ *for* $c = 0$ *we have* $d_\beta(1) = \begin{cases} .a{-}1, a{-}1, a{-}1, (a{-}2)^\omega & \text{if } b = 0; \\ .a\ ,\ 0\ , a{-}1, (a,\ 0\ , a{-}2)^\omega & \text{if } b = 1; \\ .a\ , b{-}1, (\ a\ , b{-}2)^\omega & \text{if } b \geq 2 \end{cases}$

$\star$ *and* $a+1 \leq b \leq 2a$,

$*$ *for* $a-b+c \geq 0$ *we have* $d_\beta(1) = .a{+}1, (b{-}a{-}1, a{-}b{+}c, b{-}c{-}1, c, a)^\omega$

$*$ *for* $a-b+c = -1$ *we have*

$d_\beta(1) = \begin{cases} .a, a, (a, a-1)^\omega, & \text{if } b{=}a{+}1; \\ .a{+}1, (b{-}a{-}2, a{+}1, b{-}a{-}2, 0, a{-}1, b{-}a, a{-}1, b{-}a{-}1, a)^\omega, & \text{if } b \geq a{+}2. \end{cases}$

Example 2 *Here we want to show that from a class of Pisot units of degree 4 which are roots of the polynomial* $x^4 - ax^3 - bx^2 - cx + 1 = 0$, *we can obtain a* β*-expansion of* 1 *with an arbitrarily long preperiod. For* $n \geq 5$, $a = n+2$, $b = 4-2a = -2n$ *and* $c = a-1 = n+1$ *we have that* $\lfloor \beta \rfloor = a-2$ *and the* β*-expansion of* 1 *is*

$$d_\beta(1) = .n, 1, \underbrace{3, 4, 5, \cdots, n{-}3, n{-}2}_{n-4 \text{ elements}}, n, 1, 0, n{-}2, \underbrace{n{-}4, n-5, \cdots, 3, 2}_{n-5 \text{ elements}}, 0, n, 0, 1^\omega.$$

Therefore the length of the preperiod is $2n$.

Lemma 3 *If* β *is a Pisot unit of degree four with minimal polynomial* $p(x) = x^4 - ax^3 - bx^2 - cx - 1$, *and if the negative root* γ *of the polynomial* $x^2 - \lfloor \beta \rfloor x - 1$ *has the property*

$$p(\gamma) > 0,$$

then at least one of the tiles is not connected.

Proof: Let $d_\beta(1) = .d_{-1}, d_{-2}, \cdots$ and $\xi = \xi_{-1}\beta^{-1} + \xi_{-2}\beta^{-2} + \cdots$ be a $\beta-$ expansion with $d_\beta(1) > .\xi_{-1}, \xi_{-2}, \cdots \geq .d_{-2}, d_{-3}, \cdots$. Since $p(-1) > 0$ and $p(0) = -1$, the polynomial $p(x)$ has at least one root in the interval $(-1, 0)$. Let $\theta \in (-1, 0)$ be the biggest among such roots. First we want to show that

$$T_\xi \cap \left(T_\lambda + \Phi(\xi - m\beta^{-1})\right) = \emptyset \tag{4.5}$$

for $m \in \{1, 2, \cdots\}$ such that $\xi_{-1} \geq m$. If we suppose the contrary, then there exists an expansion $\cdots, c_1, c_0.m$ with $c_i \in [-\lfloor \beta \rfloor, \lfloor \beta \rfloor] \cap \mathbb{Z}$ for $i = 0, 1, \cdots$ and $c_0 \leq \lfloor \beta \rfloor - 1$, which implies that $m\theta^{-1} + c_0 + \sum_{i=1}^{\infty} c_i\theta^i = 0$. The assumptions of the lemma show that $\gamma < \theta$. So θ is between two roots of the polynomial $x^2 - \lfloor \beta \rfloor x - 1$ and we have that

$$\frac{1}{\theta} + \lfloor \beta \rfloor - 1 + \frac{\lfloor \beta \rfloor \theta^2}{1-\theta^2} < \frac{1}{\theta} + \lfloor \beta \rfloor - 1 - \frac{\lfloor \beta \rfloor \theta}{1+\theta} = \frac{\theta^2 - \lfloor \beta \rfloor \theta - 1}{-\theta(1+\theta)} < 0,$$

which implies that $m\theta^{-1} + c_0 + \sum_{i=1}^{\infty} c_i\theta^i < 0$. Second, we prove the existence of a disconnected tile. Since $.d_{-2}, d_{-3}, \cdots < d_\beta(1)$, let

$$l = \min\{k \in \mathbb{N} \mid .d_{-2}, d_{-3}, \cdots, (d_{-k}+1) \text{ is admissible}\}.$$

For $l = 2$ we have by (4.3),

$$G_{-1}(T_\lambda) = T_0 \cup T_1 \cup \cdots \cup T_{\lfloor\beta\rfloor-1} \cup T_{\lfloor\beta\rfloor} \quad \text{and} \quad \lfloor\beta\rfloor \geq (d_{-2}+1) \geq d_{-2}, d_{-3}\cdots$$

By using (4.5) with $\xi = \lfloor\beta\rfloor/\beta$ and

$$T_{\lfloor\beta\rfloor-m} \subset T_\lambda + \Phi(\frac{\lfloor\beta\rfloor - m}{\beta}),$$

we deduce

$$T_{\lfloor\beta\rfloor} \cap T_{\lfloor\beta\rfloor-m} = \emptyset$$

for $m = 1, 2, \cdots, \lfloor\beta\rfloor$. Therefore the central tile T_λ is disconnected. For $l \geq 3$ let $\epsilon = d_{-3}, \cdots, (d_{-l}+1)$. Then we have

$$G_{-1}(T_\epsilon) = T_{0,\epsilon} \cup T_{1.\epsilon} \cup \cdots \cup T_{d_{-2},\epsilon}$$

with

$$d_{-2}, \epsilon >_{\text{lex}} d_{-2}, d_{-3}, \cdots = U_\beta(1).$$

Therefore the tile T_ϵ is disconnected in the same way by using (4.5). □

Theorem 4.7 *Let β be a Pisot unit of degree 4 with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. Each tile is arcwise connected except for the following cases:*

$$\begin{cases} a \geq 5, \\ c = a-3, \\ \frac{5-3a}{2} \leq b \leq -a, \end{cases} \quad \begin{cases} a \geq 3, \\ c = a-1, \\ \frac{1-a}{2} \leq b \leq -1, \end{cases} \quad \begin{cases} a \geq 3, \\ c = a+1, \\ \frac{1+a}{2} \leq b \leq a-1, \end{cases} \quad \begin{cases} a \geq 1, \\ c = a+3, \\ \frac{5+3a}{2} \leq b \leq 2a+2. \end{cases}$$

Proof: We only need to prove this theorem for the cases when the β-expansion of 1 is infinite because the other cases are shown in Theorem 1.2 on page 273. According to Proposition 4.1 on page 292, the coefficients satisfy the following system of inequalities:

$$\begin{cases} |b| \leq a + c - 1, \\ a^2 + 4b - c^2 \geq 1. \end{cases}$$

Here we have the following bounds for the coefficients:

$$\begin{cases} a \geq 1, \\ 1 - a \leq c \leq a + 3, \\ 2 - 2a \leq b \leq 2a + 2. \end{cases}$$

Case 1. For $-a+1 \leq c \leq -1$ we have $a \geq 2$, $1-a-c \leq b \leq -1+a+c$, hence $2-a \leq b \leq a-2$.

- For $b \leq 0$ we have $\lfloor\beta\rfloor = a - 1$ and

$$d_\beta(1) = \begin{cases} .a-1, a+b-1, a+b+c-1, (a+b+c)^\omega, & \text{if } (c,b) \neq (-1,0); \\ .a-1, a-1, a-1, 0, 0, 1, & \text{if } (c,b) = (-1,0). \end{cases}$$

Therefore the smallest tile in this case is T_η for

$$\eta = \begin{cases} a+b-1, a+b+c-1, (a+b+c)^\omega, & \text{for } c \leq -2; \\ (a+b-1)^\omega, & \text{for } c = -1 \text{ and } b \neq 0; \\ a-1, a-1, 0, 0, 1, & \text{for } (c,b) = (-1,0). \end{cases}$$

- For $c \geq 2-a$, since every conjugate of β is also a root of $p(x)(x^3-1)\sum_{i=0}^\infty x^{6i} = 0$, we have

  $$1+c\beta_1+ (b\beta_2+(a-1)\beta_3-(c+1)\beta_4-b\beta_5-(a-1)\beta_6+(c+1)\beta_7) \sum_{i=0}^\infty \beta_{6i} = 0$$

  and

  $${}^\omega(0, -b, -c-1, a-1, 0, 0).\,\eta = {}^\omega(-c-1, a-1, 0, 0, 0, -b), -c.\,\eta \quad -0.1$$

  is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$.

- For $c = 1-a$ we have $b = 0$. Since every conjugate of β is also a root of $p(x)\left(x^2+x+1+(x^3-1)x^5\sum_{i=0}^\infty x^{6i}\right) = 0$, then

  $$1-(a-2)\beta_1-(a-2)\beta_2+\beta_3+(a-1)\beta_4+(a-2)\beta_5+((a-2)\beta_6-(a-1)\beta_8-(a-2)\beta_9+(a-1)\beta_{11}) \sum_{i=0}^\infty \beta_{6i} = 0$$

  and

  $${}^\omega(a-1, 0, 0, 0, 0, a-2), a-2, a-1, 1, 0, 0.\,\eta = {}^\omega(a-2, a-1, 0, 0, 0, 0), 0, a-2, a-2.\,\eta \quad -0.1$$

  is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$.

- For $b \geq 1$ we have $a \geq 3$, $2-a \leq c \leq -1$ and $1 \leq b \leq a+c-1$. Here $\lfloor \beta \rfloor = a$ and

  $$d_\beta(1) = .a, b-1, (c+a, b)^\omega,$$

  therefore the smallest tile in this case is T_η for $\eta = (a+c, b)^\omega$. Since every conjugate of β is also a root of $p(x)(x^3-1)\sum_{i=0}^\infty x^{6i} = 0$, then

  $$1+c\beta_1+ (b\beta_2+(a-1)\beta_3-(c+1)\beta_4-b\beta_5-(a-1)\beta_6+(c+1)\beta_7) \sum_{i=0}^\infty \beta_{6i} = 0$$

  and all the coefficients have absolute value less than $\lfloor \beta \rfloor$.

<u>*Case 2.*</u> For $0 \leq c \leq a-1$ we have $-2a+2 \leq b \leq 2a-2$.

- If $-2a+2 \leq b \leq -a$ then $a \geq 5$ and $1 \leq c \leq a-3$.
  - If $1 \leq c \leq a-4$ then $1-a-c \leq b \leq -a$ and $\lfloor \beta \rfloor = a-2$.
    First, let us find the β-expansion of 1. Since $1 \leq a+b+c < a-2$, there exists an integer $k \in \{2, 3, \cdots, a-2\}$ with $\frac{a-2}{k} \leq a+b+c < \frac{a-2}{k-1}$, which implies that $(k-1)(a+b+c) < a-2 \leq k(a+b+c)$.

◇ If $(k-1)(a+b+c) \geq c+2$ we get $k \geq 3$ and $c < a-4$. Let m be the integer defined by $m = \inf\{i : (i+1)(a+b+c) \geq c+2\}$. By definition, $m \leq k-2$ and, since $b \leq -a$, we get $m \geq 1$. Let us show that the β-expansion of 1 is eventually periodic with period 1 and that the length of the preperiod is $m+3$. So let us write it as $d_\beta(1) = .a-2, d_2, \cdots, d_{m+3}, d_{m+4}^\omega$.
When $m=1$, since

$$p(x)(1+x) = x^5 - (a-1)x^4 - (a+b)x^3 - (b+c)x^2 - (c+1)x - 1,$$

we get that

$$1 = .a-2, 2a+b-2, 2a+2b+c-2, 2a+2b+2c-1, (2a+2b+2c)^\omega.$$

Here $d_5 = d_{m+4},\ d_4 = d_{m+3}, d_3 = d_{m+2}$.
When $m=2$, since

$$p(x)(1+x+x^2) = x^6 - (a-1)x^5 - (a+b-1)x^4 - (a+b+c)x^3 - (b+c+1)x^2 - (c+1)x - 1,$$

we get that

$$1 = .a-2, 2a+b-3, 3a+2b+c-3, 3a+3b+2c-2, 3(a+b+c)-1, (3a+3b+3c)^\omega.$$

Here $d_6 = d_{m+4},\ d_5 = d_{m+3},\ d_4 = d_{m+2},\ d_3 = d_{m+1}$, where the formulas of d_i will be given later.
When $m \geq 3$, since

$$p(x)\sum_{i=0}^m x^i = x^{m+4} - (a-1)x^{m+3} - (a+b-1)x^{m+2} - (a+b+c-1)x^{m+1} - \sum_{i=4}^m (a+b+c)x^i - (a+b+c+1)x^3 - (b+c+1)x^2 - (c+1)x - 1$$

(where the terms $\sum_{i=4}^m (a+b+c)x^i$ do not appear for $m=3$), we have that
$d_2 = 2a+b-3, \qquad d_3 = 3a+2b+c-4,$
$d_i = d_{i-1} + (a+b+c)$ for $i \in \{4, 5, \cdots m\}$
(these terms do not appear for $m=3$), $\qquad (*)$
$d_{m+4} = (m+1)(a+b+c), \qquad d_{m+3} = d_{m+4} - 1,$
$d_{m+2} = d_{m+3} - (c+1), \qquad d_{m+1} = d_{m+2} - (b+c+1).$
We now verify that the conditions of lexicographic order on $d_\beta(1)$ are satisfied. Since $a+b+c+1 \geq 0$, we have that $d_2 \leq d_3 < \cdots < d_m < d_{m+1}$. Here we get that $d_2 \geq b+2a-3 \geq 2$ and $d_{m+1} = m(a+b+c)+a-c-3 \leq c+1+a-c-3 \leq a-2$. From definition of m, we have that $d_{m+2} = (m+1)(a+b+c)-c-2 \geq 0$ and, since $m \leq k-2$, we have that $(m+1)(a+b+c) < a-2$. Since $d_{m+2} < d_{m+3} < d_{m+4}$, till now we showed that all d_i's are nonnegative and $d_i \leq a-2$.
We now study the cases where d_i is not strictly smaller than d_1. For $m=1$ only $d_2 = b+2a-2$ may be equal to $a-2$. For $m \geq 2$ only d_{m+1} may be equal to $a-2$, which means that $m(a+b+c) = c+1$ and that $d_2 - d_{m+2} = a-c-2$ is a positive integer. So we showed that the above expansions of 1 defined by $(*)$ are β-expansions of 1.

◇ If $(k-1)(a+b+c) \leq c+1$ and $k(a+b+c) = a-2$, let us show that the β-expansion of 1 is finite with length $2k+4$. Let us write it as $d_\beta(1) = a-2, d_2, \cdots, d_{2k+3}, 1$, where $p(x)\sum_{i=0}^{k-1} x^i \sum_{i=0}^{k+1} x^i = x^{2k+4} - \sum_{i=1}^{2k+3} d_i x^{2k+4-i} - 1$.
When $k=2$, since $2(a+b+c) = a-2$, we get

$$d_\beta(1) = .a-2, 2a+b-2, 2a+2b+c-2, a-2, 0, 2c+b+2, c+2, 1.$$

We now verify that the conditions of lexicographic order on $d_\beta(1)$ are satisfied. Since $1-a-c \leq b \leq -a$, we have $3 \leq a-c-1 \leq b+2a-2 \leq a-2$. Since $d_3 = d_2 + (b+c) = d_4 - (c+2)$, we have $0 \leq d_3 < a-2$. Since $d_6 = -(a+b)$, we have $0 \leq d_6 \leq a-5$.

Since $d_2 \geq 3$ and $d_8 = 1$, the conditions of lexicographic order are satisfied.
When $k \geq 3$, since $k(a+b+c) = a-2$, we get
$d_2 = 2a+b-3$,
$d_i = ia+(i-1)b+(i-2)c-4$ for $3 \leq i \leq k-1$, (these terms do not appear for $k=3$)
$d_k = ka+(k-1)b+(k-2)c-3$, $\quad d_{k+1} = ka+kb+(k-1)c-2$,
$d_{k+2} = a-2$, $\quad d_{k+3} = 0$, $\quad d_{k+4} = 1-b-a$,
$d_{2k+2-i} = ia+(i+1)b+(i+2)c+4$ for $1 \leq i \leq k-3$, (these terms do not appear for $k=3$)
$d_{2k+2} = 2c+b+3$, $\quad d_{2k+3} = c+2$, $\quad d_{2k+4} = 1$.
We now verify that the conditions of lexicographic order on $d_\beta(1)$ are satisfied. Here we have that $d_2 \leq d_3 < \cdots < d_k$, $d_2 \geq 2$ and $d_k = (k-1)(a+b+c)+a-c-3 \leq a-2$. Since $d_{k+1}+c+2 = a-2$, we have $0 \leq d_{k+1} \leq a-5$. We also have that $d_{k+4} \geq d_{k+5} > \cdots > d_{2k+2}$, $d_{k+4} = 1-b-a \leq c \leq a-4$ and $d_{2k+2} \geq b+c+2+(k-1)(a+b+c) = 0$. So we showed that all d_i are nonnegative and not greater than d_1. Since $d_2 \geq 3$ we have that $a-2$ may be followed by 1 or 2. If $d_k = a-2$ we have that $d_2 - d_{k+1} = b+a+c+1 \geq 2$. So the conditions of lexicographic order are satisfied.

◇ If $(k-1)(a+b+c) \leq c+1$ and $k(a+b+c) > a-2$, let us show that the β-expansion of 1 is finite with length $2k+3$. Let us write it as $d_\beta(1) = .d_1, d_2, \cdots, d_{2k+2}, 1$, where $p(x)\sum_{i=0}^{k-1} x^i \sum_{i=0}^{k} x^i = x^{2k+3} - \sum_{i=1}^{2k+2} d_i x^{2k+3-i} - 1$.
When $k=2$, we get

$$d_\beta(1) = .a-2, 2a+b-2, 2a+2b+c-1, a+2b+2c+1, 2c+b+2, c+2, 1$$

We now verify that the conditions of lexicographic order on $d_\beta(1)$ are satisfied. Since $1-a-c \leq b \leq -a$, then $3 \leq b+2a-2 \leq a-2$. Since $2(a+b+c) > a-2$ then $2 \leq 2a+2b+c-1 \leq c-1 \leq a-5$, $0 \leq a+2b+2c+1 \leq a-5$. Also $b+2c+2 \leq a-6$ and $b+2c+3 = 2(b+c+a)-b-2a+3 > 1-b-a \geq 1$. Only d_2 or d_6 can be equal to d_1. Since $1 < d_2$ the conditions of lexicographic order are satisfied.
When $k \geq 3$ we get
$d_1 = a-2$, $\quad d_2 = 2a+b-3$,
$d_i = ia+(i-1)b+(i-2)c-4$ for $3 \leq i \leq k-1$, (these terms do not appear for $k=3$)
$d_k = ka+(k-1)b+(k-2)c-3$, $\quad d_{k+1} = ka+kb+(k-1)c-1$,
$d_{k+2} = (k-1)a+kb+kc+1$, $\quad d_{k+3} = (k-2)a+(k-1)b+kc+3$,
$d_{2k+1-i} = ia+(i+1)b+(i+2)c+4$ for $1 \leq i \leq k-3$, (these terms do not appear for $k=3$)
$d_{2k+1} = 2c+b+3$, $\quad d_{2k+2} = c+2$.
We now verify that the conditions of lexicographic order on $d_\beta(1)$ are satisfied.
Here we have that $2 \leq d_2 \leq d_3 < \cdots < d_k$, $d_k > d_{k+1} > d_{k+2}$, $d_{k+2} < d_{k+3}$ and $d_{k+3} \geq d_{k+4} > \cdots > d_{2k+1}$. The condition $(k-1)(a+b+c) \leq c+1 < a-2$ implies that $d_k = (k-1)(a+b+c)+a-c-3 \leq a-2$ and $d_{k+3} \leq a-4$. Also, since $k(a+b+c) > a-2$, then $d_{k+2} = k(a+b+c)+1-a > -1$. Since $c+1 \geq (k-1)(a+b+c)$ then $d_{2k+1} > 0$. So we showed that all d_i's satisfy $0 \leq d_i \leq d_1$. Since $d_2 \geq 3$ we have that $a-2$ may be followed by 1 or 2. If $d_k = a-2$, which means that $(k-1)a+(k-1)b+(k-2)c-1 = 0$, then $d_2 - d_{k+1} = a-c-1 \geq 1$. So the conditions of lexicographic order are satisfied.

Second, let us find the common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$. Since every conjugate of β is also a root of $p(x)(x^5-1)(x^2-x+1)\sum_{i=0}^{\infty} x^{10i} = 0$, then

$$1+(c+1)\beta_1+((b+c+1)\beta_2+(a+b+c)\beta_3+(a+b-1)\beta_4+(a-2)\beta_5-(c+2)\beta_6$$

$$-(b+c+1)\beta_7-(a+b+c)\beta_8-(a+b-1)\beta_9-(a-2)\beta_{10}+(c+2)\beta_{11})\sum_{i=0}^{\infty}\beta_{10i}=0$$

and

$$^{\omega}(c+2,\ 0\ ,1-a-b,\ 0\ ,-b-c-1,\ 0\ ,a-2,\ 0\ ,a+b+c,\ 0\ ),c+1.\ \eta=$$
$$^{\omega}(a-2,\ 0\ ,a+b+c,\ 0\ ,c+2,\ 0\ ,1-a-b,\ 0\ ,-1-b-c,\ 0\ ).\ \eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$.

$*$ If $c = a-3$, we have $\dfrac{5-3a}{2} \leq b \leq -a$, $\lfloor\beta\rfloor = a-2$ and

$$d_\beta(1) = .a-2, b+2a-2, (3a+2b-4, 3a+2b-5, 2a+b-3, 0, 1-a-b, 2-a-b, 0, b+2a-3)^{\omega}.$$

To show that one of the tiles is not connected, according to Lemma 3 on page 300, it is enough to prove that $p(\gamma) > 0$. Since $\gamma^2-(a-2)\gamma-1=0$, we have

$$p(\gamma) \geq \gamma^4 - a\gamma^3 + a\gamma^2 - (a-3)\gamma - 1 = -\gamma^2(\gamma-2) > 0.$$

- If $-a+1 \leq b \leq -1$, we have $a \geq 3$, $0 \leq c \leq a-1$.

$*$ If $0 \leq c \leq a-3$, we have $a \geq 4$ and

$$d_\beta(1) = \begin{cases} .a-1, a+b, b+c, c+1,\ 1, & \text{if } b+c\geq 0;\\ .a-1, a+b-1, a-1,\ 0\ ,c+1,\ 1, & \text{if } b+c=-1;\\ .a-1, a+b-1, a+b+c-1, (a+b+c)^{\omega}, & \text{if } b+c\leq -2. \end{cases}$$

$\diamond$ If $c+b \leq -2$ and $b+a \geq 2$, since every conjugate of β is also a root of $p(x)(x^3-1)\sum_{i=0}^{\infty} x^{6i} = 0$, we have

$$1+c\beta_1+(b\beta_2+(a-1)\beta_3-(c+1)\beta_4-b\beta_5-(a-1)\beta_6+(c+1)\beta_7)\sum_{i=0}^{\infty}\beta_{6i}=0$$

and

$$^{\omega}(c+1,\ 0\ ,-b,\ 0\ ,a-1,\ 0\ ),c\ .\ \eta = {}^{\omega}(a-1,\ 0, c+1,\ 0\ ,-b,\ 0\ ).\ \eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$.

$\diamond$ If $b=-a+1$ and $c \geq 1$, we have $c+b \leq -2$ and the smallest tile is T_η for $\eta = (c+1)^{\omega}$. Since every conjugate of β is also a root of $p(x)(x^4-1)(x+1)\sum_{i=0}^{\infty} x^{8i}=0$, then

$$1+(c+1)\beta_1+((c+1-a)\beta_2+\beta_3+(a-2)\beta_4$$
$$-(c+2)\beta_5+(a-c-1)\beta_6-\beta_7-(a-2)\beta_8+(c+2)\beta_9)\sum_{i=0}^{\infty}\beta_{8i}=0$$

and

$$^{\omega}(c+2,\ 0\ ,\ 0\ ,a-c-1,\ 0\ ,a-2,\ 1\ ,\ 0\ ),c+1.\ \eta = {}^{\omega}(a-2,1,0,c+2,0,0,a-c-1,0).\ \eta \quad -0.1$$

is a common point of the smallest tile T_η and $T_\eta - \Phi(\beta^{-1})$.

$\diamond$ If $b=-a+1$ and $c=0$, we have $c+b \leq -2$ and the smallest tile is T_η for $\eta = 1^{\omega}$. Since every conjugate of β is also a root of $p(x)(x^2+x+1)\sum_{i=0}^{\infty} x^{3i}=0$, then

$$1+\beta_1-(a-2)\beta_2+2\beta_3+\sum_{i=4}^{\infty}\beta_i = 0$$

and all the coefficients have absolute value less than $\lfloor\beta\rfloor$.

$*$ If $c = a-2$, we have $-a+2 \leq b \leq -1$, $\lfloor\beta\rfloor = a-1$ and

$$d_\beta(1) = .a-1, a+b, a+b-2, a-1, 1.$$

To show that one of the tiles is not connected, according to Lemma 3 on page 300, it is enough to prove that $p(\gamma)>0$. Since $\gamma^2-(a-1)\gamma-1=0$, we have

$$p(\gamma) \geq \gamma^4 - a\gamma^3 + \gamma^2 - (a-1)\gamma - 1 = \gamma^2(1-\gamma) > 0.$$

- If $0 \leq b \leq a$, we have $\lfloor \beta \rfloor = a$ and

$$d_\beta(1) = .\,a\,,\; b\,,\; c\,,\; 1.$$

- If $a+1 \leq b \leq 2a-2$, we have $a \geq 3$, $2 \leq c \leq a-1$ and $1+a \leq b \leq a+c-1$. $\lfloor \beta \rfloor = a+1$ and

$$d_\beta(1) = .a{+}1, b{-}a{-}1, c{+}a{-}b, b{+}1{-}c, c{-}1,\; 1.$$

<u>*Case 3.*</u> If $a \leq c \leq a+3$, we have $a \geq 1$ and $\dfrac{1-a^2+c^2}{4} \leq b \leq a+c-1$.

- If $c = a$, we get $1 \leq b \leq 2a-1$ and

$$d_\beta(1) = \begin{cases} .a, b, 1, & \text{if } b \leq a; \\ .a{+}1, b{-}a{-}1, 2a{-}b, b{-}a{+}1, a{-}1,\; 1, & \text{if } b > a. \end{cases}$$

- If $c = a+1$, we get $\frac{a+1}{2} \leq b \leq 2a$ and

$$d_\beta(1) = \begin{cases} .a\,, b{+}1, (0\,, a{-}b,\; b\,,\; b\,, a{-}b{+}1,\; 0\,,\; b)^\omega, & \text{if } b \leq a{-}1; \\ .a{+}1,\; 0\,,\; 0\,, (0\,,\; a\,,\; 0\,,\; 0\,,\; a\,,\; a\,,\; 1)^\omega, & \text{if } b = a; \\ .a{+}1, b{-}a{-}1, 2a{-}b{+}1, b{-}a,\; a\,,\; 1, & \text{if } b \geq a{+}1. \end{cases}$$

  * For $b \leq a-1$, to show that one of the tiles is not connected, according to Lemma 3 on page 300, it is enough to prove that $p(\gamma) > 0$. Since $\gamma^2 - a\gamma - 1 = 0$ we have

$$(\gamma) \geq \gamma^4 - a\gamma^3 - (a-1)\gamma^2 - (a+1)\gamma - 1 = \gamma^2(1-\gamma) > 0.$$

  * For $b = a$, since every conjugate of β is also a root of $p(x)(x{-}1)\sum_{i=0}^{\infty} x^{3i} = 0$, we have

$$1{+}a\beta_1 - \beta_2 {+} (\beta_3 {-} \beta_4) \textstyle\sum_{i=0}^{\infty} \beta_{3i} = 0$$

  and all the coefficients have absolute value less than $\lfloor \beta \rfloor = a+1$.

- If $c = a+2$, we get $a{+}2 \leq b \leq 2a{+}1$, $\lfloor \beta \rfloor = a{+}1$ and

$$d_\beta(1) = .a{+}1, b{-}a{-}1, 2a{-}b{+}2, b{-}a{-}1, a{+}1, 1.$$

- If $c = a+3$, we get $a{+}2{+}\dfrac{a+1}{2} \leq b \leq 2a{+}2$, $\lfloor \beta \rfloor = a+1$ and

$$d_\beta(1) = .a{+}1, b{-}a{-}1,(2a{-}b{+}3, b{-}a{-}1,\; 0\,, 2a{-}b{+}3, 2b{-}3a{-}5, 4a{-}2b{+}6, 2b{-}3a{-}4, 2a{-}b{+}3, 0, b{-}a{-}2)^\omega.$$

  To show that one of the tiles is not connected, according to Lemma 3 on page 300, it is enough to prove that $p(\gamma)>0$. Since $\gamma^2-(a+1)\gamma-1=0$ we have that

$$p(\gamma) \geq \gamma^4 - a\gamma^3 - (2a+2)\gamma^2 - (a+3)\gamma - 1 = -\gamma^3 > 0.$$

□

From the proof of this theorem we can easily see that $a + c - 2\lfloor\beta\rfloor = 1$ for the cases when at least one of the tiles is disconnected and $a + c - 2\lfloor\beta\rfloor \leq 0$ for the cases when each tile is connected. So, the above theorem can be written in the following equivalent way:

Theorem 4.8 *Let β be a Pisot unit of degree 4 with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1$. Then $a + c - 2\lfloor\beta\rfloor \leq 1$, and each tile is arcwise connected if and only if $a + c - 2\lfloor\beta\rfloor \leq 0$.*

In [14], Canterini gave an interesting example of GIFS substitutive tiles that the union $\bigcup_i K_i$ in (4.2) is connected although each K_i is disconnected. In our setting, $\bigcup_i K_i$ corresponds to the central tile T_λ. As the proof of disconnectedness relies on Lemma 3 on page 300, the readers see that T_λ is disconnected provided there exists a disconnected tile and $d_{-1} > d_{-2}$. After submission of this paper, we could further show that *all* the tiles are disconnected, provided there *exists* a disconnected tile. As this paper is already of this length, this fact will be published elsewhere. Therefore we can not find examples like Canterini's among quartic Pisot dual tiles.

Finally from the proof of Theorem 4.7 on page 301 we extract the following theorem which gives the β-expansion of 1 for any Pisot unit of degree four with the minimal polynomial $x^4 - ax^3 - bx^2 - cx - 1 = 0$.

Theorem 4.9 *Let β be a Pisot unit of degree four with minimal polynomial $p(x) = x^4 - ax^3 - bx^2 - cx - 1 = 0$. Then the β-expansion of 1 is:*

- *When $-a + 1 \leq c \leq -1$,*
  - ⋄ *for $b \leq 0$ we have*
  $$d_\beta(1) = \begin{cases} .a-1, a+b-1, a+b+c-1, (a+b+c)^\omega, & \text{for } (c,b) \neq (-1,0); \\ .a-1, a-1, a-1, 0, 0, 1, & \text{for } (c,b) = (-1,0); \end{cases}$$
  - ⋄ *for $b \geq 1$ we have $d_\beta(1) = .a, b-1, (a+c, b)^\omega$.*
- *When $0 \leq c \leq a$,*
  - ⋄ *for $b \leq -a$ and $c \leq a-4$, let k be the integer of $\{2, 3, \cdots, a-2\}$ with $(k-1)(a+b+c) < a-2 \leq k(a+b+c)$.*
    - ∗ *If $(k-1)(a+b+c) \geq c+2$, let $m = \inf\{i \in \mathbb{N}$ such that $(i+1)(a+b+c) \geq c+2\}$.*
    $m = 1 \Rightarrow d_\beta(1) = .a-2, 2a+b-2, 2a+2b+c-2, 2a+2b+2c-1, (2a+2b+2c)^\omega$
    $m = 2 \Rightarrow d_\beta(1) = .a-2, 2a+b-3, 3a+2b+c-3, 3a+3b+2c-2, 3a+3b+3c-1, (3a+3b+3c)^\omega$
    $m \geq 3 \Rightarrow d_\beta(1) = .a-2, 2a+b-3, 3a+2b+c-4, d_4, \cdots, d_{m+3}, (d_{m+4})^\omega$
    with $d_i = d_{i-1} + a + b + c$ for $4 \leq i \leq m$ and $\begin{cases} d_{m+1} = d_m + a + b + c + 1, \\ d_{m+2} = d_{m+1} + b + c + 1, \\ d_{m+3} = d_{m+2} + c + 1, \\ d_{m+4} = (m+1)(a+b+c). \end{cases}$
    - ∗ *If $(k-1)(a+b+c) \leq c+1$ and $k(a+b+c) = a-2$ we have*
    $k = 2 \Rightarrow d_\beta(1) = .a-2, 2a+b-2, 2a+2b+c-2, a-2, 0, 2c+b+2, c+2, 1,$
    $k \geq 3 \Rightarrow d_\beta(1) = .a-2, 2a+b-3, d_3, \cdots, d_{2k+1}, 2c+b+3, c+2, 1$ *such that* $d_i =$

$ia+(i-1)b+(i-2)c-4$ *for* $3 \le i \le k-1$, *(these terms do not appear for* $k=3$*)*
$d_k = ka+(k-1)b+(k-2)c-3$, $d_{k+1} = ka+kb+(k-1)c-2$,
$d_{k+2} = a-2$, $d_{k+3} = 0$, $d_{k+4} = 1-b-a$,
$d_{2k+2-i} = ia+(i+1)b+(i+2)c+4$ *for* $1 \le i \le k-3$, *(these terms do not appear for* $k=3$*).*

$*$ *If* $(k-1)(a+b+c) \le c+1$ *and* $k(a+b+c) > a-2$ *we have*
$k=2 \Rightarrow d_\beta(1) = .a-2, 2a+b-2, 2a+2b+c-1, a+2b+2c+1, 2c+b+2, c+2, 1$, $k \ge 3 \Rightarrow$ $d_\beta(1) = .a-2, 2a+b-3, d_3, \cdots, d_{2k}, 2c+b+3, c+2, 1$ *such that*
$d_i = ia+(i-1)b+(i-2)c-4$ *for* $3 \le i \le k-1$, *(these terms do not appear for* $k=3$*)*
$d_k = ka+(k-1)b+(k-2)c-3$, $d_{k+1} = ka+kb+(k-1)c-1$,
$d_{k+2} = (k-1)a+kb+kc+1$, $d_{k+3} = (k-2)a+(k-1)b+kc+3$,
$d_{2k+1-i} = ia+(i+1)b+(i+2)c+4$ *for* $1 \le i \le k-3$, *(these terms do not appear for* $k=3$*),*

$\diamond$ *for* $b \le -a$ *and* $c = a-3$ *we have*
$d_\beta(1) = .a-2, 2a+b-2, (3a+2b-4, 3a+2b-5, 2a+b-3, 0, 1-a-b, 2-a-b, 0, 2a+b-3)^\omega$,

$\diamond$ *for* $-a < b \le -1$ *and* $c \le a-3$ *we have*

$$d_\beta(1) = \begin{cases} .a-1, a+b, b+c, c+1, 1, & \text{for } b+c \ge 0; \\ .a-1, a+b-1, a-1, 0, c+1, 1, & \text{for } b+c = -1; \\ .a-1, a+b-1, a+b+c-1, (a+b+c)^\omega & \text{for } b+c \le -2; \end{cases}$$

$\diamond$ *for* $-a < b \le -1$ *and* $c = a-2$ *we have* $d_\beta(1) = .a-1, a+b, a+b-2, a-1, 1$,

$\diamond$ *for* $-a < b \le -1$ *and* $c = a-1$ *we have* $d_\beta(1) = .a-1, a+b, (a+b, 0, -b, 0, a+b-1)^\omega$,

$\diamond$ *for* $0 \le b \le a$ *we get* $d_\beta(1) = .a, b, c, 1$,

$\diamond$ *for* $b \ge a+1$ *we get* $d_\beta(1) = .a+1, b-a-1, c+a-b, b-c+1, c-1, 1$.

- *When* $a+1 \le c \le a+3$ *we have*

$\diamond$ *for* $c = a+1$, $d_\beta(1) = \begin{cases} .a, b+1, (0, a-b, b, b, a-b+1, 0, b)^\omega, & \text{for } b \le a-1; \\ .a+1, 0, 0, (0, a, 0, 0, a, a, 1)^\omega, & \text{for } b = a; \\ .a+1, b-a-1, 2a-b+1, b-a, a, 1, & \text{for } b \ge a+1; \end{cases}$

$\diamond$ *for* $c = a+2$ *we have* $d_\beta(1) = .a+1, b-a-1, 2a-b+2, b-a-1, a+1, 1$,

$\diamond$ *for* $c = a+3$ *we have*
$d_\beta(1) = .a+1, b-a-1, (2a-b+3, b-a-1, 0, 2a-b+3, 2b-3a-5, 4a-2b+6, 2b-3a-4, 2a-b+3, 0, b-a-2)^\omega$.

Example 3 *Here we want to show that, from a class of Pisot units of degree* 4 *which are roots of the polynomial* $x^4 - ax^3 - bx^2 - cx - 1 = 0$*, we can obtain an arbitrarily long* β*-expansion of* 1 *. For* $n \ge 3$*,* $a = n+2$*, and* $c = a-4 = n-2$ $b = 1-a-c = 1-2n$ *we have that* $\lfloor \beta \rfloor = a-2 = n$ *and the* β*-expansion of* 1 *is*

$$d_\beta(1) = .n, 2, \underbrace{2, 3, \cdots, n-3, n-2}_{n-3 \text{ elements}}, n, 0, n, 0, n-2, \underbrace{n-2, n-3, \cdots, 3, 2}_{n-3 \text{ elements}}, 0, n, 0, 1.$$

Hence the length of the β*-expansion of* 1 *is* $2n+4$.

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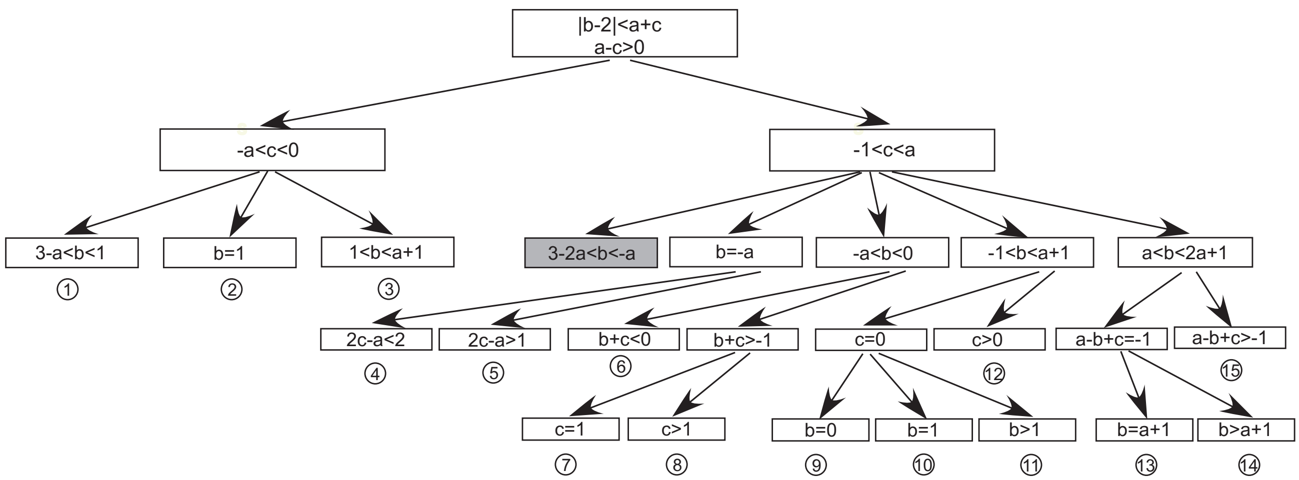
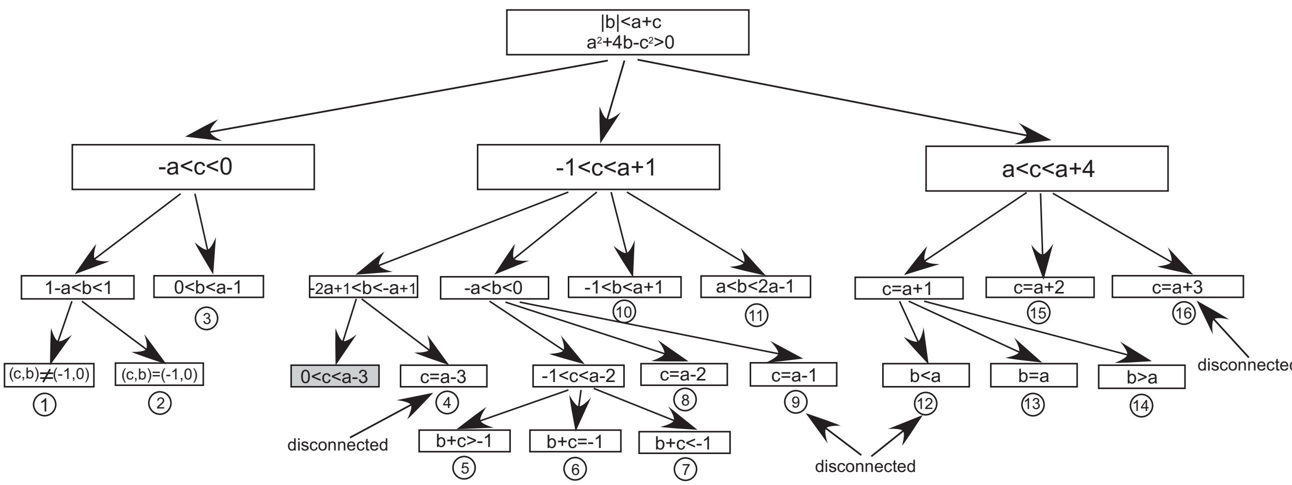


1. $d_\beta(1) = .a-1,\ a+b-1,\ a+b+c-1,\ (a+b+c-2)^\omega$
2. $d_\beta(1) = .a,\ 0,\ a+c-1,\ a-1,\ a+c,\ (a+c-1)^\omega$
3. $d_\beta(1) = .a,\ b-1,\ (c+a,\ b-2)^\omega$
4. $d_\beta(1) = .a-2,\ a-2,\ c-2,\ 2c-3,\ (2c-4)^\omega$
5. $d_\beta(1) = .a-2,\ (a-2,\ c-1,\ 2c-a-1,\ 2c-a-2,\ c-3,\ a-3)^\omega$
6. $d_\beta(1) = .a-1,\ a+b-1,\ a+b+c-1,\ (a+b+c-2)^\omega$
7. $d_\beta(1) = .a-1,\ a-2,\ a-1,\ (a-2)^\omega$
8. $d_\beta(1) = .a-1,\ (a+b,\ b+c,\ c-2,\ a-2)^\omega$
9. $d_\beta(1) = .a-1,\ a-1,\ a-1,\ (a-2)^\omega$
10. $d_\beta(1) = .a,\ 0,\ a-1,\ (a,\ 0,\ a-2)^\omega$
11. $d_\beta(1) = .a,\ b-1,\ (a,\ b-2)^\omega$
12. $d_\beta(1) = .a,\ (b,\ c-1,\ a-1)^\omega$
13. $d_\beta(1) = .a,\ a,\ (a,\ a-1)^\omega$
14. $d_\beta(1) = .a+1,\ (b-a-2,\ a+1,\ b-a-2,\ 0,\ a-1,\ b-a,\ a-1,\ b-a-1,\ a)^\omega$
15. $d_\beta(1) = .a+1,\ (b-a-1,\ a-b+c,\ b-c-1,\ c,\ a)^\omega$

Fig. 2: β-expansion of 1 for $x^4 - ax^3 - bx^2 - cx + 1 = 0$. The length is not fixed in the shaded box.



1 $d_\beta(1)$=.a-1, a+b-1, a+b+c-1, (a+b+c)$^\omega$
2 $d_\beta(1)$=.a-1, a-1, a-1, 0, 0, 1
3 $d_\beta(1)$=.a, b-1, (c+a, b)$^\omega$
4 $d_\beta(1)$=.a-2, b+2a-2, (3a+2b-4, 3a+2b-5, 2a+b-3, 0, 1-a-b, 2-a-b, 0, b+2a-3)$^\omega$
5 $d_\beta(1)$=.a-1, a+b, b+c, c+1, 1
6 $d_\beta(1)$=.a-1, a+b-1, a-1, 0, c+1, 1
7 $d_\beta(1)$=.a-1, a+b-1, a+b+c-1, (a+b+c)$^\omega$
8 $d_\beta(1)$=.a-1, a+b, a+b-2, a-1, 1
9 $d_\beta(1)$=.a-1, a+b, (a+b, 0, -b, 0, a+b-1)$^\omega$
10 $d_\beta(1)$=.a, b, c, 1
11 $d_\beta(1)$=.a+1, b-a-1, c+a-b, b-c+1, c-1, 1
12 $d_\beta(1)$=.a, b+1, (0, a-b, b, b, a-b+1, 0, b)$^\omega$
13 $d_\beta(1)$=.a+1, 0, 0, (0, a, 0, 0, a, a, 1)$^\omega$
14 $d_\beta(1)$=.a+1, b-a-1, 2a-b+1, b-a, a, 1
15 $d_\beta(1)$=.a+1, b-a-1, 2a-b+2, b-a-1, a+1, 1
16 $d_\beta(1)$=.a+1, b-a-1, (2a-b+3, b-a-1, 0, 2a-b+3, 2b-3a-5, 4a-2b+6, 2b-3a-4, 2a-b+3, 0, b-a-2)$^\omega$

Fig. 3: β-expansion of 1 for $x^4 - ax^3 - bx^2 - cx - 1 = 0$. The length is not fixed in the shaded box. Four disconnected cases are indicated.

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