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A New Method for the Explicit Integration of Lotka-Volterra Equations

Un Nuevo Método para la Integración Explícita de las Ecuaciones de Lotka-Volterra

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Abstract

In this work we study some first order nonlinear ordinary differential equations describing the time evolution (or "motion") of those hamiltonian systems provided with a first integral linking implicitly both variables to a motion constant. An application has been performed on the Lotka-Volterra predator-prey system, turning to a strongly nonlinear differential equation in the phase variables.

Our method grasps all the capabilities of modern computer algebra in order to solve (algebraic approximation) some equations of third and fourth degree with intricate forcing terms, obtaining symbolic explicit expressions osculating the solution in a neighborhood of the initial conditions.

Another approach is also developed managing a Taylor truncated series and inverting it (asymptotic approximation). After having evaluated how both approximations differ from the traditional numerical techniques, finally we accomplish the much more probatory control of the approximants' accuracy referred, through the motion constant, to the first integral of the equation itself.

Key words and phrases: Lotka-Volterra equations, Lagrange Reversion, explicit integration, implicit function theorem.

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Resumen

En este trabajo se consideran algunas ecuaciones diferenciales de primer orden que describen la evolución temporal (o “movimiento”) de algunos sistemas hamiltonianos dotados de una preintegral que relaciona implícitamente ambas variables a una constante de movimiento. Se realiza una aplicación al sistema predador-presa de Lotka-Volterra, que conduce a una ecuación diferencial fuertemente no lineal en las variables de fase. Nuestro método aprovecha todas las capacidades de la moderna álgebra computacional para resolver (aproximación algebraica) ecuaciones algebraicas de tercer y cuarto grado con términos complicados, obteniendo expresiones simbólicas explícitas de funciones que osculan la solución en un entorno de las condiciones iniciales.

Tambi’en se desarrolla otro enfoque basado en la inversión una serie de Taylor truncada (aproximación asintótica). After having evaluated how both approximations differ from the traditional numerical techniques, finally we accomplish the much more probatory control of the approximants’ accuracy referred, through the motion constant, to the first integral of the equation itself.

Después de evaluar el modo como nuestras soluciones difieren de aquellas obtenidas con las técnicas numéricas tradicionales, finalmente se realiza, hecho de mayor evidencia probatoria, un control cuidadoso de la aproximación obtenida referrida, a través de la constante de movimiento, a una preintegral de la misma ecuación.
Palabras y frases clave: Ecuaciones de Lotka-Volterra, reversión de Lagrange, integración explícita, teorema de la función implícita.

1 Introduction

This paper is devoted to some novel application of the scalar form of the Theorem of implicit functions (see [3] for a historical outline of this theorem) in order to obtain a new treatment for systems of coupled nonlinear ordinary differential equations. Let us recall the theorem’s statement:

Theorem (Ulisse Dini, 1878). *Let Ω an open set in $\mathbb{R}^2$ and $f:\Omega\to\mathbb{R}$ a $\mathcal{C}^1$ function. Suppose there exists $(\bar{x},\bar{y})\in\Omega$ such that:*

$$f(\bar{x},\bar{y})=0, \qquad \frac{\partial f}{\partial y}(\bar{x},\bar{y})>0,$$

then there must be a real interval B centered around $\overline{x}$, a real interval I centered around $\bar{y}$ and a function $\varphi : I\to\mathbb{R}$ such that:

- *$B \times I \subset \Omega$,*
- *for any $(x, y) \in B \times I$:*
$$\frac{\partial f}{\partial y}(x, y) \neq 0,$$
- *if $(x, y) \in B \times I$ then:*
$$f(x, y) = 0,$$
if and only if $y = \varphi(x)$,
- *$\bar{y} = \varphi(\bar{x})$,*
- *$\varphi \in \mathcal{C}^1(B)$ and for any $x \in B$:*
$$\varphi'(x) = -\frac{\dfrac{\partial f}{\partial x}(x, \varphi(x))}{\dfrac{\partial f}{\partial y}(x, \varphi(x))}.$$

A careful observation of this theorem's proof reveals that the most important fact is determining the radius of the interval B. Indeed if we choose $a, b \in \mathbb{R},\ a, b > 0$ such that if $B_1 = [\bar{x} - a, \bar{x} + a]$ and $I_1 = [\bar{y} - a, \bar{y} + a]$ then $\Omega_{a,b} = B_1 \times I_1 \subset \Omega$, we define:

$$m = \min_{\Omega_{a,b}} \frac{\partial f}{\partial y},$$

and:

$$M = 1 + \max_{\Omega_{a,b}} \left| \frac{\partial f}{\partial x} \right|.$$

We observe that $m > 0,\ M > 0$. Therefore the interval B in the statement of the theorem of implicit functions is any interval of the form:

$$B = [\bar{x} - \delta, \bar{x} + \delta] \tag{1.1}$$

with $\delta \in]0, a[$ and $\delta \leq \frac{mb}{2M}$. This leads to the conclusion that the solution for y of (1.6) will be defined in the neighborhood (1.1) of $(\bar{x}, \bar{y})$.
We need the implicit functions theorem for integrating separable nonlinear differential equations. For instance, we could consider the nonlinear problem:

$$\begin{cases} \dot{y}(t) = a(t) b\left(y(t)\right), \\ y\left(t_0\right) = y_0, \end{cases} \tag{1.2}$$

where $a : B \to \mathbb{R}$, $b : I \to \mathbb{R}$ are real continuous functions on the open intervals B and I with $t_0 \in B$, $y_0 \in I$ and with $b(y_0) \neq 0$. The unique solution of the problem is implicitly defined by:

$$\int_{y_0}^{y} \frac{1}{b(\xi)} d\xi = \int_{t_0}^{t} a(\eta) d\eta, \tag{1.3}$$

in such a way the function of two variables:

$$f(t, y) = \int_{y_0}^{y} \frac{1}{b(\xi)} d\xi - \int_{t_0}^{t} a(\eta) d\eta, \tag{1.4}$$

supplies the implicit solution of (1.2) .

Therefore, we see that the Cauchy problem (1.2) supplies an implicit function problem of the special form:

$$g(x) = h(y), \tag{1.5}$$

where $g : B \to \mathbb{R}, h : I \to \mathbb{R}$ are two smooth real functions defined in the open real intervals B, I and where $x_0 \in B$, $y_0 \in I$ and $g(x_0) = h(y_0)$. Moreover $h'(y_0) \neq 0$. The particular form of (1.5) in order to solve for y the transcendental equation (1.5), suggests the approach of replacing $h(y)$ by its n^{th} order Taylor polynomial evaluated at y_0. Of course this requires $h \in \mathcal{C}^n(I), n \in \mathbb{N}$, which covers the most occurrences. Thus, if $p_n(y, y_0)$ is the n^{th} order Taylor polynomial of $h(y)$ at y_0, instead of (1.5), we will consider the integration of (1.2) as mapped by solving the n^{th} order algebraic equation in y :

$$g(x) = p_n(y, y_0). \tag{1.6}$$

Please note that the degree zero coefficient (the forcing term) of (1.6) depends on the parameter $x \in B$. We can then state:

Theorem 1. *The following facts hold:*

1. *equation* (1.6) *meets all the hypotheses of the theorem of implicit functions and thus* (1.6) *will be satisfied by one and only one function* $y_n = y_n(x, x_0)$*;*

2. *if h is real analytic, then:*

$$\lim_{n\to\infty} y_n(x, x_0) = h^{-1}(g(x)) = \varphi(x),$$

where $\varphi(x)$ is the unique function, defined in a suitable subinterval of B which satisfies the implicit function problem (1.5).

Proof. The first statement follows immediately because $h(y)$ and the n^{th} order Taylor polynomial $p_n(y, y_0)$ have the same derivative at $y = y_0$. The second property comes down from the following argument. We suppose $h(y)$ real analytic with radius of convergence $\rho > 0$, i.e.:

$$h(y) = \sum_{n=0}^{\infty} \frac{h^{(n)}(y_0)}{n!} (y - y_0)^n, \quad |y - y_0| < \rho.$$

It is known (see [2] page 184 chap V. §21 n. 107) that, by means of the Lagrange's power series reversion theorem, the inverse function, $h^{-1}(z)$ is analytic too, i.e. his Taylor series converges. Moreover the series' coefficients are uniquely determined by the Lagrange recursion formula. Of course the same holds for $p_n(y, y_0)$ thought as a function of y, and if $p_n^{-1}(z, y_0)$ is the relevant inverse function, the reversion algorithm ensures that:

$$\lim_{n\to\infty} p_n^{-1}(z, y_0) = h^{-1}(z).$$

Finally, we prove that $y_n \to \varphi(x)$. In fact:

$$|y_n - y| = \left|p_n^{-1}(g(x), y_0) - h^{-1}(g(x))\right| \to 0$$

as $n \to \infty$. □

Our strategy of finding closed form solutions of (1.2), deals therefore with the classical problem of solving algebraic equations of degree greater than two. In this framework we used Mathematica® with its routines for the symbolic solution of third and fourth degree equations.[1] In the next section we shall apply our method to the the Lotka-Volterra predator-prey planar system (*sans* overcrowding).

2 Algebraic approximations

The Lotka-Volterra predator-prey system of nonlinear differential equations is:

$$\begin{cases} \dot{u}(t) = u(t)\big(-c + dv(t)\big), \\ \dot{v}(t) = v(t)\big(a - bu(t)\big), \end{cases} \tag{2.1}$$

$v(t)$ and $u(t)$ being the populations of the preys and the predators respectively at the moment t; while a, b, c and d are positive constants of units t^{-1}, where:

[1] We will compare the solutions given by our method with the highly accurate numerical solutions obtained by Mathematica® internal algorithms and specifically those suggested by Knapp and Wagon in [1].

- a is the birth rate of the preys,
- b is the proportion of preys actually eaten by the predators,
- c is the rate at which predators die if not nourished,
- d is the biomass conversion constant of the predators.

The main properties of the planar system (2.1) solutions are:

- the first quadrant is an invariant set for all the solutions of (2.1) with positive initial conditions,
- (2.1) has two equilibria: the origin and $\mathbb{E} = (\frac{a}{b}, \frac{c}{d})$,
- any solution of (2.1) shall be greater than zero, periodic in the first quadrant and their orbits shall wander around $\mathbb{E}$.

In spite of its simple formulation, no closed form solution of (2.1) is known. However, writing (2.1) as:

$$\begin{cases} \dfrac{d}{dt}\big(\ln u(t)\big) = -c + dv(t), \\ \dfrac{d}{dt}\big(\ln v(t)\big) = a - bu(t), \end{cases}$$

and making the change of variables:

$$\begin{cases} x = \ln\left(\dfrac{d}{c}v\right), \\ y = \ln\left(\dfrac{b}{a}u\right). \end{cases}$$

which will move $\mathbb{E}$ to the origin, now the system depends on two parameters alone, namely a and c:

$$\begin{cases} \dot{x}(t) = a\big(1 - \exp y(t)\big), \\ \dot{y}(t) = -c\big(1 - \exp x(t)\big), \end{cases} \tag{2.2}$$

where the x variable can be termed *logarithmic prey* and y the *logarithmic predator*, and, of course, even negative solutions have physical meaning.

The initial conditions $x(0) = x_0$, $y(0) = y_0$ are no longer necessarily positive and one can immediately obtain the orbit differential equation in the (x, y) plane:

$$\begin{cases} \dfrac{dy}{dx} = k\Big(\dfrac{1 - \exp x}{1 - \exp y}\Big), \\ y(x_0) = y_0, \\ k = -\dfrac{c}{a} < 0, \end{cases} \tag{2.3}$$

which is of the type (1.2). Therefore due to (1.3), the solution of (2.3) can be found to be:

$$y - \exp y = k\,(x - \exp x) + C, \tag{2.4}$$

where:

$$C = y_0 - \exp y_0 - k\,(x_0 - \exp x_0)\,,$$

is the constant of integration (or *motion* constant for people mechanically inclined), with $x_0 = x(0)$ and $y_0 = y\,(x(0))$. If (2.4) could be solved explicitly for y, the orbit of the Lotka-Volterra system would be known: but unfortunately the transcendental nonlinearity of (2.4) prevents this. Note that (2.4) is an implicit equation like (1.5): therefore the Dini theorem comes into play. We are going to replace the left hand side of (2.4) with its Taylor expansion: of course this bounds us to work in a neighborhood of the initial value y_0 in order to achieve a reasonably good approximation for y. The solution's periodicity allows us to take the initial data: $y(0) = y_0 > 0$ and $x(0) = x_0 = 0$ without loss of generality, the latter being the *motion* coordinates around the origin of the new logarithmic system.

2.1 Numerical reference solutions of Lotka-Volterra equations

In order to test the accuracy of the approximate explicit solution we are going to compute, and compare it with the numerical ones, we consider a sample case by fixing the coefficients a and c and the initial values x_0 and y_0. This does not affect the generality of our method, because the procedure can be easily implemented in different situations; moreover we can so test our accuracy. Let us display the Mathematica® solution first. Let $a = 2, c = 1, x_0 = 0$ (the prey's population is at the equilibrium value) and $y_0 = 1$ (predators are

e times their equilibrium value). Then (2.2) and (2.4) can be written as

$$\begin{cases} \dot{x}(t) = 2\big(1 - \exp y(t)\big), \\ \dot{y}(t) = -\big(1 - \exp x(t)\big), \\ x(0) = 0, \quad y(0) = 1, \end{cases} \tag{2.5}$$

$$y - e^y + \frac{1}{2}\left(x - e^x\right) - \frac{1}{2} + e = 0. \tag{2.6}$$

Making use of the package **VisualDSolve.m**, by D. Schwalbe and S. Wagon [4], the system's orbit on the (x, y) plane will be found, and we will be able to store the numerical solution for a later comparison. The orbit [2], which runs counterclockwise, is shown at Figure 3.
Moreover, as suggested in [1], the numerical computation's accuracy can be checked by the conservation law (2.6). First of all, let us observe that the period of system (2.5) can be numerically computed as $\tau \simeq 5.27$. Therefore let us introduce the finite subset of the interval $I = [0, 5.27]$:

$$\Delta = \left\{ \frac{527n}{5000000} : 0 \leq n \leq 50000 \right\}.$$

Thus making the time span discrete, each time interval will be $\frac{1}{50000}$ of the overall interval I. Let $(x(t), y(t))$ denote the solution found by Mathematica®. Eeplacing the left hand side of (2.6), one finds that

$$\sup_{t\in\Delta} \left| y(t) - e^{y(t)} + \frac{1}{2}\left(x(t) - e^{x(t)}\right) + e - \frac{1}{2} \right| = 1.43783 \times 10^{-5} \tag{2.7}$$

instead of zero. The time span has been also divided into 10000 steps finding the error substantially unchanged. This means that Mathematica®'s *predictor-corrector* technique provides solutions which differ from the *motion* constant by less than 10^{-4}, and therefore are surely correct to four decimal digits.

[2]Much computational work has gone into the case where $a = c$; in this case the orbit's shape is rather similar to an oblate circle.

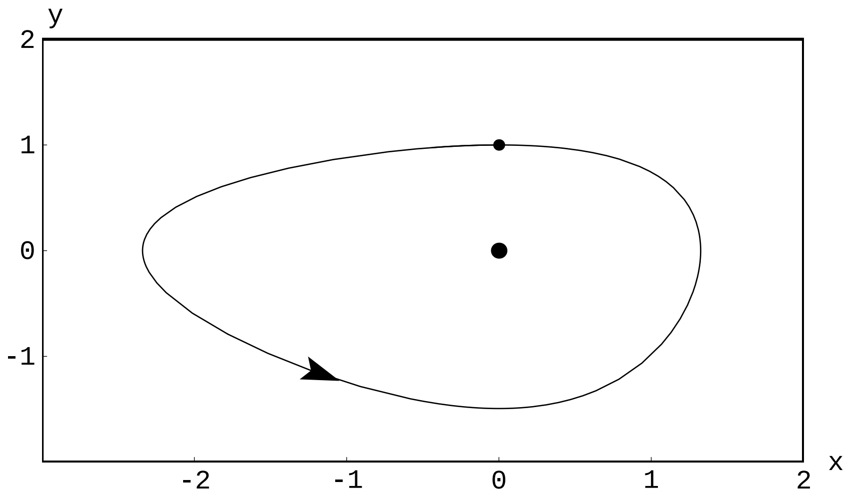

Figure 3: The Lotka-Volterra orbit.

2.2 Algebraic approximations at work

And what about our approach? First we approximate $y - e^y$ by means of its Taylor expansion around $y_0 = 1$; note that Dini's theorem does not allow the Maclaurin approximation of $y - e^y$, because its derivative vanishes for $y = 0$.

Therefore the second, third and fourth order algebraic approximations to equation (2.6) around $y_0 = 1$ will generate the following equations for y:

$$\begin{aligned} e\,y^2 - 2y + e^x - x - e + 1 &= 0, \qquad (2.8)\\ e\,y^3 + 3\,(e-2)\,y + 3\,e^x - 3\,x - 4\,e + 3 &= 0, \qquad (2.9)\\ e\,y^4 + 6\,e\,y^2 + 8\,(e-3)\,y + 3\,(4\,e^x - 4\,x - 5\,e + 4) &= 0. \qquad (2.10)\end{aligned}$$

Once again let us denote the solutions of (2.8), (2.9), (2.10) as $s_2(x)$, $s_3(x)$ and $s_4(x)$, respectively, such that the initial conditions $s_k(0) = y(0) = 1$ for $k = 2, 3, 4$ are satisfied. The existence and uniqueness of such solutions are guaranteed by Dini's theorem. For instance

$$s_2(x) = \frac{1 + \sqrt{1 - e\,(1 - e + e^x - x)}}{e},$$

whilst $s_3(x)$ is more complicated:

$$s_3(x) = \frac{36\,e - 18\,e^2 + 2^{\frac{1}{3}}\left(27\,e^{\frac{3}{2}}\sqrt{\delta(x)} + 27\,e^2\,(-3 + 4\,e - 3\,e^x + 3\,x)\right)^{\frac{2}{3}}}{3\,2^{\frac{2}{3}}\,e\left(27\,e^{\frac{3}{2}}\sqrt{\delta(x)} + 27\,e^2\,(-3 + 4\,e - 3\quad e^x + 3\,x)\right)^{\frac{1}{3}}}$$

where

$$\delta(x) = 4\,(e-2)^3 + e\,(3 - 4e + 3e^x - 3x)^2\,.$$

We shall not bother our readers by writing the full expression for $s_4(x)$: it can be obtained by the formulas of L. Ferrari (1522–1565) through some computer algebra system. The essential point is that we are indeed able to evaluate $s_4(x)$ and, later, check its accuracy.

We deem meaningful to verify how the explicit approximations are related to the invariant (2.6). For that purpose the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, which includes the origin, is discretized into 1000 subintervals. Hence we obtain the finite set:

$$\Lambda = \left\{-\frac{1}{2} + \frac{n}{1000} : 0 \le n \le 1000\right\}$$

for which we then compute:

$$\zeta_k = \sup_{x\in\Lambda}\left|s_k(x) - e^{s_k(x)} + \frac{1}{2}\,(x - e^x) + e - \frac{1}{2}\right|,\quad k = 1, 2, 3, \tag{2.11}$$

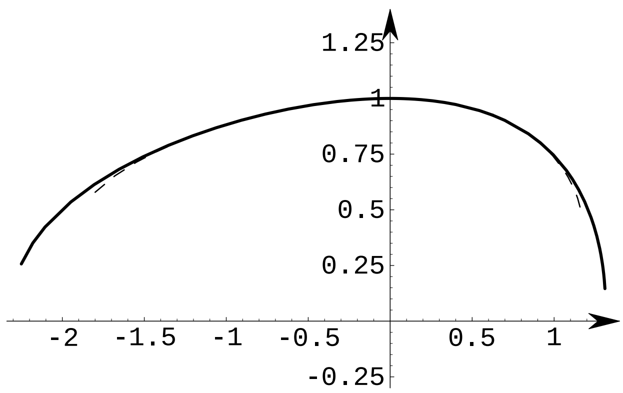

Figure 4: The second and fourth order approximations

obtaining: [3]

$$\zeta_2 = 4.04683 \times 10^{-5},\ \zeta_3 = 4.53918 \times 10^{-7},\ \zeta_4 = 4.0769 \times 10^{-9},$$

and this estimation is better than (2.7) concerning the solution $(x(t), y(t))$ obtained by means of the usual numerical methods. Figure 4 gives a pictorial view of this, showing the explicit approximation $s_2(x)$ (dotted line) and $s_4(x)$ (continuous line).

A fair objection that could be raised is that, when testing (2.7), one must sample the whole orbit, whilst our computation of ζ_k took place on a strictly local basis. Accordingly, let us perform a new computation defining Δ_1 and Δ_2 as:

$$\Delta_1 = \left\{ \frac{3n}{10000} : \ 0 \leq n \leq 500 \right\}, \qquad \Delta_2 = \left\{ \frac{128}{25} + \frac{3n}{10000} : \ 0 \leq n \leq 500 \right\}.$$

We introduce

$$\begin{aligned} \rho_1 &= \sup_{t \in \Delta_1} \left| y(t) - e^{y(t)} + \frac{1}{2}\left(x(t) - e^{x(t)} \right) + e - \frac{1}{2} \right|, \\ \rho_2 &= \sup_{t \in \Delta_2} \left| y(t) - e^{y(t)} + \frac{1}{2}\left(x(t) - e^{x(t)} \right) + e - \frac{1}{2} \right|, \end{aligned}$$

obtaining:

$$\rho_1 = 1.68752 \times 10^{-6}, \quad \rho_2 = 1.54722 \times 10^{-6}.$$

Note that the comparison is significant because the simulation shows that $x(t) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ for $t \in \Delta_1$ and $t \in \Delta_2$.

All this means that our symbolic approximate solutions $s_3(x)$ and $s_4(x)$ fit the first integral better than the numerical solutions do. The final figure gives the complete Mathematica® solution $(x(t), y(t))$ as a dotted line versus the $s_4(x)$ plotted with a continuous line for $x \in [-2.30, 1.309]$.

[3] It should be taken into account that the maximum could be evaluated setting to zero the derivative of the function inside the absolute value in (2.11), and searching the relevant solution. Of course this will allow, for $k = 2$, to solve a quite difficult equation, becoming prohibitive for growing k. Then this *sup* has been searched computing (2.11), and so detecting its highest values displayed as $\zeta_2, \zeta_3, \zeta_4$.

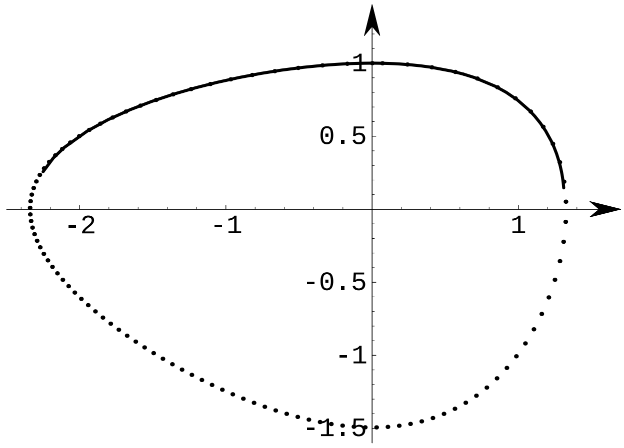

Figure 5: s_4 versus the Mathematica® numerical solution.

3 Asymptotic approximations

It should be clear up to now that, even if the theoretical ground for approximating the explicit solution of the Lotka-Volterra (or other hamiltonian) equations is the Dini theorem, nevertheless only the algebraic track has been led for doing the explicitation. Beyond its remarkable outsets, it has a severe boundary in its own nature, being possible to write it only till the fourth order.

In this section we present a meaningful improvement to obtain once again a closed form, but more accurate approximations of the Lotka-Volterra problem on the phase plane. Following closely the proof of Theorem 1, we will obtain the explicit solution of the equation $h(y) = g(x)$ through the series reversion method applied to a Taylor polynomial of the function $h(y)$.

The loss of accuracy due to the fact we cannot write for $n \geq 5$ the exact solution of the approximate equation:

$$p_n\left(y, y_0\right) = \sum_{k=0}^{n} \frac{h^{(n)}(y_0)}{n!}\left(y - y_0\right)^k = g(x),$$

is balanced by the better accuracy for $h(y)$ that one can achieve increasing the Taylor's polynomial degree. E.g. in the test case previously seen $h(y) = y - e^y$ with $y \in [0, 1]$, putting:

$$\Theta = \left\{\frac{n}{2000} : 0 \leq n \leq 2000\right\},$$

we have:

$$
\begin{aligned}
m_4 &= \max_{y\in\Theta} |h(y) - p_4(y, y_0)| = \frac{3}{8}e - 1 \simeq 0.0193557. \\
m_8 &= \max_{y\in\Theta} |h(y) - p_8(y, y_0)| = \frac{2119}{5760}e - 1 \simeq 6.8046 \times 10^{-6}.
\end{aligned}
$$

Of course the reversion doesn't settle the function that, according to the previous sections, we call $s_n(x)$—which, generally speaking, will be a transcendental function of x—but the n^{th} order Taylor polynomial relevant to such a function, and we note it as $r_n(x)$. The reverse series shall then be computed for $g(x)$, which, in the test case is $g(x) = \frac{1}{2} - e - \frac{1}{2}\left(x - e^x\right).$

E.g., for $n = 8$ the elapsed time for inverting by Mathematica® $p_8\left(y, y_0\right)$ by a computer Macintosh G4 with clock rated at 733 MHz, bus of 133 MHz and 640 MB RAM, has been of 0.7 seconds:

$$
\begin{aligned}
r_8(x) \quad = \quad & 1 - \frac{e^x - x - 1}{2\,(e-1)} - \frac{e\,(e^x - x - 1)^2}{8\,(e-1)^3} - \frac{e\,(1 + 2\,e)\,(e^x - x - 1)^3}{48\,(e-1)^5} \\
& - \frac{e\,\left(1 + 8\,e + 6\,e^2\right)\,(e^x - x - 1)^4}{384\,(e-1)^7} \\
& - \frac{e\,\left(1 + 22\,e + 58\,e^2 + 24\,e^3\right)\,(e^x - x - 1)^5}{3840\,(e-1)^9} \\
& - \frac{e\,\left(1 + 52\,e + 328\,e^2 + 444\,e^3 + 120\,e^4\right)\,(e^x - x - 1)^6}{46080\,(e-1)^{11}} \\
& - \frac{e\,\left(1 + 114\,e + 1452\,e^2 + 4400\,e^3 + 3708\,e^4 + 720\,e^5\right)}{645120\,\,(e-1)^{13}} \times\,(e^x - x - 1)^7 \\
& - \frac{e\,\left(1 + 240\,e + 5610\,e^2 + 32120\,e^3 + 58140\,e^4 + 33984\,e^5 + 5040\,e^6\right)}{10321920\,(e-1)^{15}} \\
& \qquad \times\,(e^x - x - 1)^8.
\end{aligned}
$$

Up to this point we can compute ζ_8, see (2.11), namely the figure appreciating the accuracy of the asymptotic solution versus the first integral:

$$
\zeta_8 = \max_{x\in\Lambda} \left| r_8(x) - e^{r_8(x)} + \frac{1}{2}\,(x - e^x) + e - \frac{1}{2} \right|,
$$

where the discrete set Λ has the same meaning of the previous section. With a time machine of 0,05 seconds we will appreciate an accuracy improvement:

$$\zeta_8 = 3.10769 \times 10^{-11}.$$

We put our reversion till to $r_{22}(x)$, even if in such a case the elapsed time was 14617 seconds, but the accuracy is:

$$\zeta_{22} = 8.88178 \times 10^{-16},$$

the same even doubling the discretization step.

Last, the range of x by $\left[-\frac{1}{2}, \frac{1}{2}\right]$ has been broadened to $[-1, 1]$ (for the test case we have $-2,35 < x < 1,33$). Then the finite sets have been introduced:

$$\Omega_a = \left\{-1 + \frac{n}{500} : 0 \le n \le 1000\right\}, \quad \Omega_b = \left\{-1 + \frac{n}{1000} : 0 \le n \le 2000\right\},$$

in which it has been checked that

$$\begin{aligned} &\max_{x\in\Omega_a}\left|r_{22}(x) - e^{r_{22}(x)} + \frac{1}{2}\left(x - e^x\right) + e - \frac{1}{2}\right| \\ &= \max_{x\in\Omega_b}\left|r_{22}(x) - e^{r_{22}(x)} + \frac{1}{2}\left(x - e^x\right) + e - \frac{1}{2}\right| \\ &= 7.70113 \times 10^{-10}. \end{aligned}$$

Then, even if the broadening of the sample range makes worse the accuracy, such an approximation keeps its validity till to the tenth digit for the 54% of the solution existence range.

Last we compare the highest algebraic approximation $s_4(x)$ with the asymptotic approximation $r_4(x)$. The asymptotic solution's accuracy is obviously less, but not immensely less:

$$\max_{x\in\Lambda}\left|r_4(x) - e^{r_4(x)} + \frac{1}{2}\left(x - e^x\right) + e - \frac{1}{2}\right| = 6.64716 \times 10^{-7},$$

while

$$\max_{x\in\Lambda}\left|s_4(x) - e^{s_4(x)} + \frac{1}{2}\left(x - e^x\right) + e - \frac{1}{2}\right| = 4.08 \times 10^{-9}$$

and the distance beween them will be not greater than:

$$\max_{x\in\Lambda}\left|s_4(x) - r_4(x)\right| = 4.18236 \times 10^{-7}.$$

4 Conclusions

We obtained an explicit solution of Lotka-Volterra equations founded upon the existence of its first integral holding a couple of phase variables (x, y) inextricably tied.

We started with a Taylor's power expansion in the variable y. Truncating the expansion to the fourth term, we can use the machinery of classical algebra in order to solve for y. This algebraic approximation has been proved to be better than the highly accurate predictor/corrector numerical procedures. Our benchmark is in any case based on the approximate solutions capability to satisfy the first integral.

The above approximation can be improved (asymptotic approximation) taking more terms beyond the fourth in the Taylor expansion, and inverting for y. This approximation is by far more accurate than the algebraic one.

Nothing has been told hitherto about the approximate solution's domain. This problem is too involved for being solvable in its generality: therefore we will provide an overview of it treating some concrete cases. In order to do this the hamiltonian of the original system, depending on y, is replaced by some Taylor approximation. Doing so, we succeed in calculating the (approximate) explicit equation of its integral curve. Its range of existence is then determined by the crossings of the relevant orbit with the horizontal axis. E. g. the closed curve of equation (2.6) has been pictured by the line of equation

$$\frac{1}{2} + \frac{x - e^x}{2} + (1 - e)\ (y - 1) - \frac{e\,(y-1)^2}{2} - \frac{e\,(y-1)^3}{6} - \frac{e\,(y-1)^4}{24} = 0,$$

which we use for calculating $s_4(x)$, and then capable of approximating the solution for $y > 0$.

In such a way, with a numerical approach, we can see that putting $y = 0$, (2.6) gives $x \simeq -2.34026$ and $x \simeq 1.32477$, while the approximate hamiltonian cuts the x axis at $x \simeq -2.29732$ and $x \simeq 1.31062$.

It should be observed that the highest absolute difference between the hamiltonians: the exact one, and the approximate, referred to the positive half-orbit only $(y > 0)$, can be numerically appreciated as $\Delta \simeq 0.0193557$.

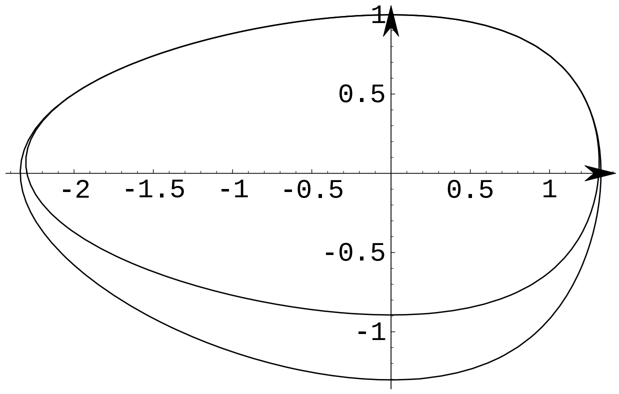

Figure 6: The exact (outer) and the approximate (inner) orbits.

Of course, increasing the n^{th}-approximation order, the approximant orbit becomes closer and closer to the original one: in such a way for $n = 8$ the approximate curve crosses at $x \simeq -2.34025$ and $x \simeq 1.32477$, almost the same as the original orbit. The highest absolute difference between the hamiltonians, the exact one and the 8^{th} order approximation, for $y > 0$ is 6.8046×10^{-6}.

Both approaches to simulate the first integrals by means of Taylor polynomials seem quite viable but, as far as we know, they have not been attempted before, probably due to prohibitive computations. In any case, our method's effectiveness is by large extent due to the computer algebra systems.

The algebraic approximations, by which we succeeded in calculating long arcs of the orbit on the population plane (x, y) for the Lotka-Volterra system, leads to very complicated integrals, as a right hand side of the so called *time equation.* Let us take, for example, the first equation from system (2.5):

$$\dot{x}(t) = 2\,(1 - \exp y(t))\,,$$

and set the initial condition $x(0) = 0$. Replacing $y(t)$ with one of the approximation $s_k(x)$, in the easiest case $k = 2$ we obtain:

$$t = \frac{1}{2}\int_0^x \frac{d\xi}{1 - \exp\left(\dfrac{1 + \sqrt{1 - e}\;(1 - e + \exp\xi - \xi)}{e}\right)}.$$

All told, it isn't even worth investing any effort in such endeavour: even an exact solution of the above integral would lead only to a very poor approximation of time versus x, which would then have to be inverted, of course.

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