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Existence and Uniqueness for a Nonlinear Dispersive Equation

Existencia y Unicidad para una Ecuación Dispersiva No Lineal

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A Lorena Conde Baquedano, con admiración, mucha admiración.

Abstract

In this paper we study the existence and uniqueness properties of solutions of some nonlinear dispersive equations of evolution. We consider the equation

$$(1) = \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[f(u)] = \epsilon \frac{\partial}{\partial x}[g(\frac{\partial u}{\partial x})] - \delta \frac{\partial^3 u}{\partial x^3} \\ u(x,0) = \varphi(x) \end{cases}$$

with $x \in \mathbb{R}$, T an arbitrary positive time and $t \in [0,T]$. The flux $f = f(u)$ and the (degenerate) viscosity $g = g(\lambda)$ are given smooth functions satisfying certain assumptions. This work presents a result *a priori* that permits to obtain gain of regularity for equation (1), motivated by the results obtained by Craig, Kappeler and Strauss [3].
Key words and phrases: Evolution equations, Lions-Aubin Theorem, Weighted Sobolev Space.

Resumen

En este artículo estudiamos las propiedades de existencia y unicidad de las soluciones de algunas ecuaciones de evolución dispersivas no lineales. Consideramos la ecuación

$$(1) = \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[f(u)] = \epsilon \frac{\partial}{\partial x}[g(\frac{\partial u}{\partial x})] - \delta \frac{\partial^3 u}{\partial x^3} \\ u(x,0) = \varphi(x) \end{cases}$$

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con $x \in \mathbb{R}$, $t \in [0,T]$ y T un tiempo positivo arbitrario. El flujo $f = f(u)$ y la viscosidad (degenerada) $g = g(\lambda)$ son funciones suaves dadas que satisfacen ciertas condiciones. En este trabajo se presenta un resultado *a priori* que permite obtener una ganancia de regularidad para la ecuación (1), motivado en los resultados obtenidos por Craig, Kappeler y Strauss [3].
Palabras y frases clave: Ecuaciones de evolución, teorema de Lions-Aubin, Espacios pesados de Sobolev.

1 Introduction

In 1976, J. C. Saut and R. Temam [23] remarked that a solution u of an equation of Korteweg-de Vries type cannot gain or lose regularity: they showed that if $u(x,0) = \varphi(x) \in H^s(\mathbb{R})$ for $s \geq 2$, then $u(\cdot,t) \in H^s(\mathbb{R})$ for all $t > 0$. The same results were obtained independently by J. Bona and R. Scott [2], by different methods. For the Korteweg-de Vries (KdV) equation on the line, T. Kato [16], motivated by work of A. Cohen [6], showed that if $u(x,0) = \varphi(x) \in L^2_b \equiv H^2(\mathbb{R}) \bigcap L^2(e^{bx}dx)$ $(b > 0)$ then the solution $u(x,t)$ of the KdV equation becomes C^∞ for all $t > 0$. A main ingredient in the proof was the fact that formally the semigroup $S(t) = e^{-t\partial_x^3}$ in L^2_b is equivalent to $S_b(t) = e^{-t(\partial_x - b)^3}$ in L^2 when $t > 0$. One would be inclined to believe this was a special property of the KdV equation. This is not, however, the case. The effect is due to the dispersive nature of the linear part of the equation. S. N. Kruzkov and A. V. Faminskii [20], for $u(x,0) = \varphi(x) \in L^2$ such that $x^\alpha \varphi(x) \in L^2((0,+\infty))$, proved that the weak solution of the KdV equation constructed there has l-continuous space derivatives for all $t > 0$ if $l < 2\alpha$. The proof of this result is based on the asymptotic behavior of the Airy function and its derivatives, and on the smoothing effect of the KdV equation found in [16, 20]. Corresponding work for some special nonlinear Schrödinger equations was done by Hayashi et al. [12, 13] and G. Ponce [22]. While the proof of T. Kato seems to depend on special *a priori* estimates, some of its mystery has been resolved by results of local gain of finite regularity for various other linear and nonlinear dispersive equations due to P. Constantin and J. C. Saut [10], P. Sjolin [24], J. Ginibre and G. Velo [11] and others. However, all of them require growth conditions on the nonlinear term.

All the physically significant dispersive equations and systems known to us have linear parts displaying this local smoothing property. To mention only a few, the KdV, Benjamin-Ono, intermediate long wave, various Boussinesq, and Schrödinger equations are included.

Continuing with the idea of W. Craig, T. Kappeler and W. Strauss [9] we study existence and uniqueness properties of solutions of some nonlinear dispersive equations of evolution. We consider the nonlinear dispersive equation

$$(1) = \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[f(u)] = \epsilon \frac{\partial}{\partial x}[g(\frac{\partial u}{\partial x})] - \delta \frac{\partial^3 u}{\partial x^3} \\ u(x,0) = \varphi(x) \end{cases}$$

with $x \in \mathbb{R}$, T an arbitrary positive time and $t \in [0,T]$. The flux $f = f(u)$ and the (degenerate) viscosity $g = g(\lambda)$ are given smooth functions satisfying certain assumptions to be listed shortly.

In section 3 we prove an important *a priori* estimate.

In section 4 we prove basic local-in-time existence and uniqueness results for (1). Specifically, we show that for initial $\varphi(x) \in H^N(\mathbb{R})$, for $N \geq 3$, there exists a unique $u \in L^\infty([0,T]; H^N(\mathbb{R}))$ where the time of existence depends of the norm of $\varphi(x) \in H^3(\mathbb{R})$.

In section 5 we develop a series of estimates for solutions of equation (1) in weighted Sobolev norms. We show that a solution u of (1) also satisfies a persistence property. Indeed, we prove that if the initial data φ lies in a certain weighted Sobolev space, then the unique solution u of the nonlinear equation (1) lies in the same Sobolev space.

2 Preliminaries

We consider the nonlinear dispersive equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[f(u)] = \epsilon \frac{\partial}{\partial x}\left[g\left(\frac{\partial u}{\partial x}\right)\right] - \delta \frac{\partial^3 u}{\partial x^3} \tag{2.1}$$

with $x \in \mathbb{R}$, $t \in [0,T]$ and T is an arbitrary positive time. The flux $f = f(u)$ and the (degenerate) viscosity $g = g(\lambda)$ are given smooth functions satisfying certain assumptions. $\epsilon, \delta > 0$.

Notation 1. We write $\partial = \frac{\partial}{\partial x}$, $\partial_t u = \frac{\partial u}{\partial t} = u_t$ and we abbreviate $u_j = \partial^j u = \frac{\partial^j u}{\partial x^j}$; $\partial_j = \frac{\partial}{\partial u_j}$.

Example. If $\partial u/\partial x = u_1$ then

$$\frac{\partial}{\partial x}\left[g\left(\frac{\partial u}{\partial x}\right)\right] = \frac{\partial}{\partial x}[g(u_1)] = \frac{\partial}{\partial u_1}[g(u_1)]\frac{\partial}{\partial x}[u_1] = \frac{\partial}{\partial u_1}[g(u_1)]\,u_2 = (\partial_1 g)u_2$$

The assumptions on f are the following:

A.1 $f\colon \mathbb{R}^2 \times [0,T] \mapsto \mathbb{R}$ is C^∞ in all its variables.

A.2 All the derivatives of $f = f(u,x,t)$ are bounded for $x \in \mathbb{R}$ for $t \in [0,T]$ and $u \in \mathbb{R}$ in a bounded set.

A.3 $x^N \partial_x^j f(0,x,t)$ is bounded for all $N \geq 0, j \geq 0$, and $x \in \mathbb{R}, t \in (0,T]$. Indeed, $\forall N \geq 0$, $\forall j \geq 0$, $x \in \mathbb{R}$, $t \in (0,T]$, there exists $c > 0$ such that $|x^N \partial_x^j f(0,x,t)| \leq c$.

The assumptions on g are the following:

B.1 $g\colon \mathbb{R}^2 \times [0,T] \mapsto \mathbb{R}$ is C^∞ in all its variables.

B.2 All the derivatives of $g(y,x,t)$ are bounded for $x \in \mathbb{R}, t \in [0,T]$ and y in a bounded set.

B.3 $x^N \partial_x^j g(0,x,t)$ is bounded for all $N \geq 0$, $j \geq 0$ and $x \in \mathbb{R}$, $t \in (0,T]$.

B.4 There exists $c > 0$ such that $\partial_1 g(u_1,x,t) \geq c > 0$, for all $u_1 \in \mathbb{R}$, $x \in \mathbb{R}$ and $t \in [0,T]$.

Lemma 1. These assumptions imply that f has the form $f = u_0 f_0 + h \equiv u f_0 + h$ where $f_0 = f_0(u_0,x,t) \equiv f_0(u,x,t)$ and $h = h(x,t)$. f_0 and h are C^∞ and each of their derivatives is bounded for u bounded, $x \in \mathbb{R}$ and $t \in [0,T]$.
Proof. Indeed, we define

$$f_0 = \begin{cases} \frac{f(u_0,x,t) - f(0,x,t)}{u_0} & \text{for } u_0 \neq 0 \\ \partial_0 f(0,x,t) & \text{for } u_0 = 0 \end{cases}$$

and $h(x,t) = f(0,x,t)$.

Remark 1. The same for g.

Definition 2.1. An evolution equation enjoys a *gain of regularity* if its solutions are smoother for all $t > 0$ than its initial data.

Definition 2.2. A function $\xi(x,t)$ belong to the weight class $W_{\sigma i k}$ if it is a positive C^∞ function on $\mathbb{R} \times [0,T]$, $\xi_x > 0$ and there exists a constant $c_j, 0 \leq j \leq 5$ such that

$$0 < c_1 \leq t^{-k} e^{-\sigma x} \xi(x,t) \leq c_2 \qquad \text{for} \quad x < -1, \quad 0 < t < T. \tag{2.2}$$

$$0 < c_3 \leq t^{-k} x^{-i} \xi(x,t) \leq c_4 \qquad \text{for} \quad x > 1, \quad 0 < t < T. \tag{2.3}$$

$$\big(t \mid \xi_t \mid + \mid \partial^j \xi \mid\big) / \xi \leq c_5 \tag{2.4}$$

in $\mathbb{R} \times [0,T]$, for all $j \in \mathbb{N}$.

Remark 2. We shall always take $\sigma \geq 0$, $i \geq 1$ and $k \geq 0$.

Example 1. Let

$$\xi(x) = \begin{cases} 1 + e^{-1/x} & \text{for } x > 0 \\ 1 & \text{for } x \leq 0 \end{cases}$$

then $\xi \in W_{0i0}$.

Definition 2.3. Fixed $\xi \in W_{\sigma ik}$ define the space (for s a positive integer)

$$H^s(W_{\sigma ik}) = \Big\{ v \colon \mathbb{R} \to \mathbb{R}; \text{ such that the distributional derivatives}$$

$$\frac{\partial^j v}{\partial x^j} \text{ for } 0 \leq j \leq s \text{ satisfy } \|v\|^2 = \sum_{j=0}^{s} \int_{-\infty}^{+\infty} |\partial^j v(x)|^2 \xi(x,t) dx < +\infty \Big\}$$

Remark 3. $H^s(W_{\sigma ik})$ depend t (because $\xi = \xi(x,t)$).

Lemma 2. For $\xi \in W_{\sigma i0}$ and $\sigma \geq 0, i \geq 0$ there exists a constant c such that, for $u \in H^1(W_{\sigma i0})$

$$\sup_{x \in \mathbb{R}} | \xi u^2 | \leq c \int_{-\infty}^{+\infty} \left(| u |^2 + | \partial u |^2 \right) \xi dx.$$

Proof. See Lemma 7.3 in [9].

Definition 2.4. Fixed $\xi \in W_{\sigma ik}$ define the space

$$\begin{aligned} L^2([0,T]; H^s(W_{\sigma ik})) &= \{ v = v(x,t), v(\cdot,t) \in H^s(W_{\sigma ik}) \text{ such that} \\ & \quad ||| \, v \, |||^2 = \int_0^T \|v(\cdot,t)\|^2 dt < +\infty \} \\ L^\infty([0,T]; H^s(W_{\sigma ik})) &= \{ v = v(x,t), v(\cdot,t) \in H^s(W_{\sigma ik}) \text{ such that} \\ & \quad ||| \, v \, |||_\infty = \text{ess} \sup_{t \in [0,T]} \|v(\cdot,t)\| < +\infty \} \end{aligned}$$

Remark 4. The usual Sobolev space is $H^s(\mathbb{R}) = H^s(W_{000})$ without a weight.

Remark 5. We shall derive the *a priori* estimates assuming that the solution is C^∞, bounded as $x \to -\infty$, and rapidly decreasing as $x \to +\infty$, together with all of its derivatives.
According to notation 1, for equation (1) we obtain

$$u_t + \delta u_3 - \epsilon(\partial_1 g) u_2 + (\partial_0 f) u_1 = 0. \tag{2.5}$$

The equation is considered for $-\infty < x < +\infty, t \in [0,T]$ and T is an arbitrary positive time.

3 An important a priori estimate

In this section we show a fundamental *a priori* estimate to demonstrate basic local-in-time existence theorem. We need to construct a mapping $T : L^\infty([0,T]; H^s(\mathbb{R})) \to L^\infty([0,T]; H^s(\mathbb{R}))$ with the following property: Given $u^{(n)} = T(u^{(n-1)})$ and $\|u^{(n-1)}\|_s \le c_0$ then $\|u^{(n)}\|_s \le c_0$, where s and $c_0 > 0$ are constants. This property tells us, in fact, that $T : \mathbb{B}_{c_0}(0) \to \mathbb{B}_{c_0}(0)$ where $\mathbb{B}_{c_0}(0) = \{v(x,t); \|v(x,t)\|_s \le c_0\}$ is a ball in space $L^\infty([0,T]; H^s(\mathbb{R}))$. To guarantee this property, we will appeal to an a priori estimate which is the main object of this section.
Differentiating the equation (2.5) two times leads to

$$\partial_t u_2 + \delta u_5 - \epsilon(\partial_1 g)u_4 + (\partial_0 f)u_3 - 2\epsilon\partial(\partial_1 g)u_3 + 2\partial(\partial_0 f)u_2 - \epsilon\partial^2(\partial_1 g)u_2 + \partial^2(\partial_0 f)u_1 = 0 \quad (3.1)$$

Let $u = \wedge v$ where $\wedge = (I - \partial^2)^{-1}$. Then $\partial_t u_2 = -v_t + u_t$. Replacing in (3.1) we have

$$-v_t + \delta\wedge v_5 - \epsilon(\partial_1 g)\wedge v_4 + (\partial_0 f)\wedge v_3 - 2\epsilon\partial(\partial_1 g)\wedge v_3 + 2\partial(\partial_0 f)\wedge v_2 - \epsilon\partial^2(\partial_1 g)\wedge v_2 + \partial^2(\partial_0 f)\wedge v_1 - [\delta\wedge v_3 - \epsilon(\partial_1 g)\wedge v_2 + (\partial_0 f)\wedge v_1] = 0 \quad (3.2)$$

where $g = g(\wedge v_1)$ and $f = f(\wedge v)$.
The equation (3.2) is linearized by substituting a new variable w in each coefficient;

$$-v_t + \delta\wedge v_5 - \epsilon\partial_1 g(\wedge w_1)\wedge v_4 + \partial_0 f(\wedge w)\wedge v_3 - 2\epsilon\partial(\partial_1 g(\wedge w_1))\wedge v_3 + 2\partial(\partial_0 f(\wedge w))\wedge v_2 - \epsilon\partial^2(\partial_1 g(\wedge w_1))\wedge v_2 + \partial^2(\partial_0 f(\wedge w))\wedge v_1 - [\delta\wedge v_3 - \epsilon\partial_1 g(\wedge w_1)\wedge v_2 - \partial_0 f(\wedge w)\wedge v_1] = 0. \quad (3.3)$$

Lemma 3.1. Let $v, w \in C^k([0,+\infty); H^N(\mathbb{R}))$ for all k, N which satisfy (3.3). Let
$\xi \ge c_1 > 0$. For each integer α there exist positive nondecreasing functions E,F and G such that for all $t \ge 0$

$$\partial_t \int_{\mathbb{R}} \xi v_\alpha^2 dx \le G(\|w\|_\lambda)\|v\|_\alpha^2 + E(\|w\|_\lambda)\|w\|_\alpha^2 + F(\|w\|_\alpha) \quad (3.4)$$

where $\|\cdot\|_\alpha$ is the norm in $H^\alpha(\mathbb{R})$ and $\lambda = \max\{1, \alpha\}$.
Proof. Differentiating α-times the equation (3.3) for some $\alpha \ge 0$

$$-\partial_t v_\alpha + \delta\wedge v_{\alpha+5} - \epsilon(\partial_1 g)\wedge v_{\alpha+4} + ((\partial_0 f) - (\alpha+2)\epsilon\partial(\partial_1 g))\wedge v_{\alpha+3} + \sum_{j=2}^{\alpha+2} h^{(j)}\wedge v_j + q(\wedge w)\wedge w_{\alpha+2} + p(\wedge w_{\alpha+1}, \ldots) = 0 \quad (3.5)$$

where $h^{(j)}$ is a smooth function depending on $\wedge w_{i+3}, \wedge w_{i+2}, \ldots$ with $i = 2+\alpha-j$.
We multiply equation (3.5) by $2\xi v_\alpha$, integrate over $x \in \mathbb{R}$

$$\begin{aligned}
&-2\int_{\mathbb{R}} \xi v_\alpha \partial_t v_\alpha dx + 2\delta \int_{\mathbb{R}} \xi v_\alpha \wedge v_{\alpha+5} dx - 2\epsilon \int_{\mathbb{R}} \xi(\partial_1 g) v_\alpha \wedge v_{\alpha+4} dx \\
&+2\int_{\mathbb{R}} \xi(\partial_0 f) v_\alpha \wedge v_{\alpha+3} dx - 2(\alpha+2)\epsilon \int_{\mathbb{R}} \xi \partial(\partial_1 g) v_\alpha \wedge v_{\alpha+3} dx \\
&+2\sum_{j=2}^{\alpha+2} \int_{\mathbb{R}} \xi h^{(j)} v_\alpha \wedge v_j dx + 2\int_{\mathbb{R}} \xi q(\wedge w) v_\alpha \wedge w_{\alpha+2} dx \\
&+2\int_{\mathbb{R}} \xi v_\alpha p(\wedge w_{\alpha+1}, \ldots) dx = 0. \qquad (3.6)
\end{aligned}$$

Each term in (3.6) is treated separately. The first two terms yield

$$-2\int_{\mathbb{R}} \xi v_\alpha \partial_t v_\alpha dx = -\partial_t \int_{\mathbb{R}} \xi v_\alpha^2 dx + \int_{\mathbb{R}} \xi_t v_\alpha^2 dx$$

and

$$\begin{aligned}
&2\delta \int_{\mathbb{R}} \xi v_\alpha \wedge v_{\alpha+5} dx = 2\delta \int_{\mathbb{R}} \xi \wedge (I-\partial^2) v_\alpha \wedge v_{\alpha+5} dx \\
&= 2\delta \int_{\mathbb{R}} \xi \wedge v_\alpha \wedge v_{\alpha+5} dx - 2\delta \int_{\mathbb{R}} \xi \wedge v_{\alpha+2} \wedge v_{\alpha+5} dx \\
&= -\delta \int_{\mathbb{R}} \partial^5 \xi (\wedge v_\alpha)^2 dx + 5\delta \int_{\mathbb{R}} \partial^3 \xi (\wedge v_{\alpha+1})^2 dx \\
&- 5\delta \int_{\mathbb{R}} \partial \xi (\wedge v_{\alpha+2})^2 dx + \delta \int_{\mathbb{R}} \partial^3 \xi (\wedge v_{\alpha+2})^2 dx - 3\delta \int_{\mathbb{R}} \partial \xi (\wedge v_{\alpha+3})^2 dx.
\end{aligned}$$

The other terms are treated similarly, integrating by parts once again. Replacing over (3.6) we have

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} \xi v_\alpha^2 dx = \int_{\mathbb{R}} \xi_t v_\alpha^2 dx - \delta \int_{\mathbb{R}} \partial^5 \xi (\wedge v_\alpha)^2 dx + 5\delta \int_{\mathbb{R}} \partial^3 \xi (\wedge v_{\alpha+1})^2 dx \\
&- 5\delta \int_{\mathbb{R}} \partial \xi (\wedge v_{\alpha+2})^2 dx + \delta \int_{\mathbb{R}} \partial^3 \xi (\wedge v_{\alpha+2})^2 dx - 3\delta \int_{\mathbb{R}} \partial \xi (\wedge v_{\alpha+3})^2 dx \\
&- \int_{\mathbb{R}} \partial^3 (\xi \partial_0 f)(\wedge v_\alpha)^2 dx - \epsilon \int_{\mathbb{R}} \partial^4 (\xi \partial_1 g)(\wedge v_\alpha)^2 dx + 4\epsilon \int_{\mathbb{R}} \partial^2 (\xi \partial_1 g)(\wedge v_{\alpha+1})^2 dx \\
&- 2\epsilon \int_{\mathbb{R}} \xi(\partial_1 g)(\wedge v_{\alpha+2})^2 dx + \epsilon \int_{\mathbb{R}} \partial^2 (\xi \partial_1 g)(\wedge v_{\alpha+2})^2 dx \\
&- 2\epsilon \int_{\mathbb{R}} \xi(\partial_1 g)(\wedge v_{\alpha+3})^2 dx + 3\int_{\mathbb{R}} \partial(\xi \partial_0 f)(\wedge v_{\alpha+1})^2 dx + \int_{\mathbb{R}} \partial(\xi \partial_0 f)(\wedge v_{\alpha+2})^2 dx
\end{aligned}$$

$$+ (\alpha+2)\epsilon\int_{\mathbb{R}}\partial^3(\xi\partial(\partial_1 g))(\wedge v_\alpha)^2dx - 3(\alpha+2)\epsilon\int_{\mathbb{R}}\partial(\xi\partial(\partial_1 g))(\wedge v_{\alpha+1})^2dx$$
$$- (\alpha+2)\epsilon\int_{\mathbb{R}}\partial(\xi\partial(\partial_1 g))(\wedge v_{\alpha+2})^2dx + \int_{\mathbb{R}}\partial^2(\xi h^{(\alpha+2)})(\wedge v_\alpha)^2dx$$
$$- 2\int_{\mathbb{R}}\xi h^{(\alpha+2)}(\wedge v_{\alpha+1})^2dx - 2\int_{\mathbb{R}}\xi h^{(\alpha+2)}(\wedge v_{\alpha+2})^2dx + 2\sum_{j=2}^{\alpha+1}\int_{\mathbb{R}}\xi h^{(j)}v_\alpha\wedge v_j dx$$
$$+ 2\int_{\mathbb{R}}\xi q(\wedge w)v_\alpha\wedge w_{\alpha+2}dx + 2\int_{\mathbb{R}}\xi v_\alpha p(\wedge w_{\alpha+1},\ldots)dx,$$

then we have

$$\partial_t\int_{\mathbb{R}}\xi v_\alpha^2dx = -\int_{\mathbb{R}}(3\delta\partial\xi + 2\epsilon\xi(\partial_1 g))(\wedge v_{\alpha+3})^2dx + \int_{\mathbb{R}}\xi_t v_\alpha^2dx$$
$$+ \delta\int_{\mathbb{R}}\partial^3\xi(\wedge v_{\alpha+2})^2dx - 5\delta\int_{\mathbb{R}}\partial\xi(\wedge v_{\alpha+2})^2dx + \epsilon\int_{\mathbb{R}}\partial^2(\xi(\partial_1 g))(\wedge v_{\alpha+2})^2dx$$
$$- 2\epsilon\int_{\mathbb{R}}\xi(\partial_1 g)(\wedge v_{\alpha+2})^2dx + \int_{\mathbb{R}}\partial(\xi(\partial_0 f))(\wedge v_{\alpha+2})^2dx$$
$$- (\alpha+2)\epsilon\int_{\mathbb{R}}\partial(\xi\partial(\partial_1 g))(\wedge v_{\alpha+2})^2dx - 2\int_{\mathbb{R}}\xi h^{(\alpha+2)}(\wedge v_{\alpha+2})^2dx$$
$$+ 5\delta\int_{\mathbb{R}}\partial^3\xi(\wedge v_{\alpha+1})^2dx + 4\epsilon\int_{\mathbb{R}}\partial^2(\xi\partial_1 g)(\wedge v_{\alpha+1})^2dx$$
$$+ 3\int_{\mathbb{R}}\partial(\xi\partial_0 f)(\wedge v_{\alpha+1})^2dx - 3(\alpha+2)\epsilon\int_{\mathbb{R}}\partial(\xi\partial(\partial_1 g))(\wedge v_{\alpha+1})^2dx$$
$$- 2\int_{\mathbb{R}}\xi h^{(\alpha+2)}(\wedge v_{\alpha+1})^2dx - \delta\int_{\mathbb{R}}\partial^5\xi(\wedge v_\alpha)^2dx - \epsilon\int_{\mathbb{R}}\partial^4(\xi\partial_1 g)(\wedge v_\alpha)^2dx$$
$$- \int_{\mathbb{R}}\partial^3(\xi\partial_0 f)(\wedge v_\alpha)^2dx + (\alpha+2)\epsilon\int_{\mathbb{R}}\partial^3(\xi\partial(\partial_1 g))(\wedge v_\alpha)^2dx$$
$$+ \int_{\mathbb{R}}\partial^2(\xi h^{(\alpha+2)})(\wedge v_\alpha)^2dx + 2\sum_{j=2}^{\alpha+1}\int_{\mathbb{R}}\xi h^{(j)}v_\alpha\wedge v_j dx$$
$$+ 2\int_{\mathbb{R}}\xi q(\wedge w)v_\alpha\wedge w_{\alpha+2}dx + 2\int_{\mathbb{R}}\xi v_\alpha p(\wedge w_{\alpha+1},\ldots)dx.$$

The first term in the righthand side is nonpositive, hence

$$\partial_t \int_{\mathbb{R}} \xi v_\alpha^2 dx \leq \int_{\mathbb{R}} \xi_t v_\alpha^2 dx + \delta \int_{\mathbb{R}} \partial^3 \xi (\wedge v_{\alpha+2})^2 dx - 5\delta \int_{\mathbb{R}} \partial \xi (\wedge v_{\alpha+2})^2 dx$$
$$+ \epsilon \int_{\mathbb{R}} \partial^2 (\xi \partial_1 g)(\wedge v_{\alpha+2})^2 dx - 2\epsilon \int_{\mathbb{R}} \xi (\partial_1 g)(\wedge v_{\alpha+2})^2 dx$$
$$+ \int_{\mathbb{R}} \partial(\xi \partial_0 f)(\wedge v_{\alpha+2})^2 dx - (\alpha+2)\epsilon \int_{\mathbb{R}} \partial(\xi \partial(\partial_1 g))(\wedge v_{\alpha+2})^2 dx$$
$$- 2 \int_{\mathbb{R}} \xi h^{(\alpha+2)} (\wedge v_{\alpha+2})^2 dx + 5\delta \int_{\mathbb{R}} \partial^3 \xi (\wedge v_{\alpha+1})^2 dx$$
$$+ 4\epsilon \int_{\mathbb{R}} \partial^2 (\xi \partial_1 g)(\wedge v_{\alpha+1})^2 dx + 3 \int_{\mathbb{R}} \partial(\xi \partial_0 f)(\wedge v_{\alpha+1})^2 dx$$
$$- 3(\alpha+2)\epsilon \int_{\mathbb{R}} \partial(\xi \partial(\partial_1 g))(\wedge v_{\alpha+1})^2 dx - 2 \int_{\mathbb{R}} \xi h^{(\alpha+2)} (\wedge v_{\alpha+1})^2 dx$$
$$- \delta \int_{\mathbb{R}} \partial^5 \xi (\wedge v_\alpha)^2 dx - \epsilon \int_{\mathbb{R}} \partial^4 (\xi \partial_1 g)(\wedge v_\alpha)^2 dx$$
$$- \int_{\mathbb{R}} \partial^3 (\xi \partial_0 f)(\wedge v_\alpha)^2 dx + (\alpha+2)\epsilon \int_{\mathbb{R}} \partial^3 (\xi \partial(\partial_1 g))(\wedge v_\alpha)^2 dx$$
$$+ \int_{\mathbb{R}} \partial^2 (\xi h^{(\alpha+2)})(\wedge v_\alpha)^2 dx + 2 \sum_{j=2}^{\alpha+1} \int_{\mathbb{R}} \xi h^{(j)} v_\alpha \wedge v_j dx$$
$$+ 2 \int_{\mathbb{R}} \xi q(\wedge w) v_\alpha \wedge w_{\alpha+2} dx + 2 \int_{\mathbb{R}} \xi v_\alpha p(\wedge w_{\alpha+1}, \ldots) dx.$$

In the last term we have

$$\left| 2 \int_{\mathbb{R}} \xi (q \wedge w_{\alpha+2} + p) v_\alpha dx \right| \leq 2 \left| \int_{\mathbb{R}} \xi q \wedge w_{\alpha+2} v_\alpha dx \right| + 2 \left| \int_{\mathbb{R}} \xi p v_\alpha dx \right|,$$

but $\wedge w_{\alpha+2} = \wedge w_\alpha - w_\alpha$ then

$$\left| 2 \int_{\mathbb{R}} \xi (q \wedge w_{\alpha+2} + p) v_\alpha dx \right| \leq 2 \left| \int_{\mathbb{R}} \xi q v_\alpha (\wedge w_\alpha - w_\alpha) dx \right| + 2 \left| \int_{\mathbb{R}} \xi p v_\alpha dx \right|$$
$$\leq 2 \left| \int_{\mathbb{R}} \xi q \wedge w_\alpha v_\alpha dx \right| + 2 \left| \int_{\mathbb{R}} \xi q w_\alpha v_\alpha dx \right| + 2 \left(\int_{\mathbb{R}} \xi p^2 dx \right)^{1/2} \left(\int_{\mathbb{R}} \xi v_\alpha^2 dx \right)^{1/2}$$
$$\leq 2 \left| \int_{\mathbb{R}} \xi q \wedge w_\alpha v_\alpha dx \right| + 2 \left| \int_{\mathbb{R}} \xi q w_\alpha v_\alpha dx \right| + \int_{\mathbb{R}} \xi p^2 dx + \int_{\mathbb{R}} \xi v_\alpha^2 dx$$

and since $p = p(\wedge w_{\alpha+1}, \ldots)$ then

$$\left| 2 \int_{\mathbb{R}} \xi \left(q \wedge w_{\alpha+2} + p \right) v_\alpha dx \right| \leq r(\|w\|_\alpha)[\|w\|_\alpha^2 + \|v\|_\alpha^2] + \|v\|_\alpha + s(\|w\|_\alpha)$$

in the same form, using that $\wedge v_n = \wedge v_{n-2} - v_{n-2}$. By standard estimates, the Lemma now follows.
We define a sequence of approximations to equation (3.3) as

$$-v_t^{(n)} + \delta\wedge v_5^{(n)} - \epsilon(\partial_1 g)\wedge v_4^{(n)} + (\partial_0 f)\wedge v_3^{(n)} - 2\epsilon\partial(\partial_1 g)\wedge v_3^{(n)}$$
$$-\delta\wedge v_3^{(n)} + O(\wedge v_2^{(n-1)}, \wedge v_1^{(n-1)}, \ldots) = 0 \quad (3.7)$$

where $g = g(\wedge v_1^{(n-1)})$, $f = f(\wedge v^{(n-1)})$ and where the initial condition is given by $v^{(n)}(x,0) = \varphi(x) - \partial^2\varphi(x)$. The first approximation is given by $v^{(0)}(x,0) = \varphi(x) - \partial^2\varphi(x)$. Equation (3.7) is a linear equation at each iteration which can be solved in any interval of time in which the coefficients are defined. This equation has the form

$$\partial_t v = \delta\wedge v_5 - \epsilon\wedge v_4 + b^{(1)}\wedge v_3 + b^{(0)} \quad (3.8)$$

Lemma 3.2. Given initial data in $\varphi \in H^\infty(\mathbb{R}) = \bigcap_{N\geq 0} H^N(\mathbb{R})$ there exists a unique solution of (3.8). The solution is defined in any time interval in which the coefficients are defined.
Proof. See [26].

4 Uniqueness and existence theorem

In this section, we study uniqueness and local existence of strong solutions for problem (2.5). Specifically, we show that for initial $\varphi(x) \in H^N(\mathbb{R})$, for $N \geq 3$, there exists a unique $u \in L^\infty([0,T]; H^N(\mathbb{R}))$ where the time of existence depends of the norm of $\varphi(x) \in H^3(\mathbb{R})$. First we address the question of uniqueness.

Theorem 4.1. (Uniqueness). Let $\varphi \in H^3(\mathbb{R})$ and $0 < T < +\infty$. Assume f satisfies A.1-A.3. and g satisfies B.1-B.4, then there is at most one solution $u \in L^\infty([0,T]; H^3(\mathbb{R}))$ of (2.5) with initial data $u(x,0) = \varphi(x)$.
Proof. Assume $u, v \in L^\infty([0,T]; H^3(\mathbb{R}))$ are two solutions of (2.5) with $u_t, v_t \in L^\infty([0,T]; L^2(\mathbb{R}))$ and with the same initial data. Then

$$(u-v)_t + \delta(u-v)_3 - \epsilon[g(u_1) - g(v_1)]_1 + [f(u) - f(v)]_1 = 0 \quad (4.1)$$

with $(u-v)(x,0) = 0$. Using the Mean Value Theorem there are smooth functions $d^{(1)}$ and $d^{(2)}$ depending smoothly on $u_1, x, t; v_1, x, t$ and $u, x, t; v, x, t$ respectively such that (4.1) has the form

$$\begin{aligned}&(u-v)_t + \delta(u-v)_3 - \epsilon[d^{(1)}]_1(u-v)_1 - \epsilon d^{(1)}(u-v)_2\\&+ [d^{(2)}]_1(u-v) + d^{(2)}(u-v)_1 = 0\end{aligned} \quad (4.2)$$

We multiply equation (4.2) by $2\xi(u-v)$, integrate over $x \in \mathbb{R}$

$$\begin{aligned} &2\int_{\mathbb{R}} \xi(u-v)(u-v)_t dx + 2\delta \int_{\mathbb{R}} \xi(u-v)(u-v)_3 dx \\ &- 2\epsilon \int_{\mathbb{R}} \xi[d^{(1)}]_1 (u-v)(u-v)_1 dx - 2\epsilon \int_{\mathbb{R}} \xi d^{(1)} (u-v)(u-v)_2 dx \\ &+ 2\int_{\mathbb{R}} \xi[d^{(2)}]_1 (u-v)^2 dx + 2\int_{\mathbb{R}} \xi d^{(2)} (u-v)(u-v)_1 dx = 0. \end{aligned} \tag{4.3}$$

Each term is treated separately. In the first term we have

$$2\int_{\mathbb{R}} \xi(u-v)(u-v)_t dx = \partial_t \int_{\mathbb{R}} \xi(u-v)^2 dx - \int_{\mathbb{R}} \xi_t (u-v)^2 dx.$$

The second term, integrating by parts, yields

$$\begin{aligned} &2\delta \int_{\mathbb{R}} \xi(u-v)(u-v)_3 dx \\ &= -2\delta \int_{\mathbb{R}} \partial\xi(u-v)(u-v)_2 dx - 2\delta \int_{\mathbb{R}} \xi(u-v)_1 (u-v)_2 dx \\ &= 2\delta \int_{\mathbb{R}} \partial^2\xi(u-v)(u-v)_1 dx + 2\delta \int_{\mathbb{R}} \partial\xi(u-v)_1^2 dx + \delta \int_{\mathbb{R}} \partial\xi(u-v)_1^2 dx \\ &= -\delta \int_{\mathbb{R}} \partial^3\xi(u-v)^2 dx + 3\delta \int_{\mathbb{R}} \partial\xi(u-v)_1^2 dx. \end{aligned} \tag{4.4}$$

Replacing over (4.3) we have

$$\begin{aligned} &\partial_t \int_{\mathbb{R}} \xi(u-v)^2 dx - \int_{\mathbb{R}} \xi_t (u-v)^2 dx - \delta \int_{\mathbb{R}} \partial^3\xi(u-v)^2 dx \\ &+ 3\delta \int_{\mathbb{R}} \partial\xi(u-v)_1^2 dx + \epsilon \int_{\mathbb{R}} \partial(\xi[d^{(1)}]_1)(u-v)^2 dx \\ &- \epsilon \int_{\mathbb{R}} \partial^2(\xi d^{(1)})(u-v)^2 dx + 2\epsilon \int_{\mathbb{R}} \xi d^{(1)} (u-v)_1^2 dx \\ &+ 2\int_{\mathbb{R}} \xi[d^{(2)}]_1 (u-v)^2 dx - \int_{\mathbb{R}} \partial(\xi d^{(2)})(u-v)^2 dx = 0. \end{aligned}$$

Then

$$\begin{aligned} &\partial_t \int_{\mathbb{R}} \xi(u-v)^2 dx + \int_{\mathbb{R}} (3\delta\partial\xi + 2\epsilon\xi d^{(1)})(u-v)_1^2 dx \\ &= \int_{\mathbb{R}} \xi_t (u-v)^2 dx + \delta \int_{\mathbb{R}} \partial^3\xi(u-v)^2 dx - \epsilon \int_{\mathbb{R}} \partial(\xi[d^{(1)}]_1)(u-v)^2 dx \\ &+ \epsilon \int_{\mathbb{R}} \partial^2(\xi d^{(1)})(u-v)^2 dx - 2\int_{\mathbb{R}} \xi[d^{(2)}]_1 (u-v)^2 dx + \int_{\mathbb{R}} \partial(\xi d^{(2)})(u-v)^2 dx. \end{aligned}$$

Using B-4, the assumptions on f and g and for a suitably chosen constant c, we have

$$\partial_t \int_{\mathbb{R}} \xi(u-v)^2 dx + \int_{\mathbb{R}} (3\delta\partial\xi + 2\epsilon\xi d^{(1)})(u-v)_1^2 dx \leq c \int_{\mathbb{R}} \xi(u-v)^2 dx.$$

By Gronwall's inequality and the fact that $(u-v)$ vanishes at $t=0$ it follows that $u=v$. This proves uniqueness.
We construct the mapping $T : L^\infty([0,T]; H^s(\mathbb{R})) \to L^\infty([0,T]; H^s(\mathbb{R}))$ by defining that

$$\begin{aligned} u^{(0)} &= \varphi(x) \\ u^{(n)} &= T(u^{(n-1)}) \qquad n \geq 1 \end{aligned}$$

where $u^{(n-1)}$ is in the position of w in equation (3.3) and $u^{(n)}$ is in the position of v which is the solution of equation (3.3). By to Lemma 3.2., $u^{(n)}$ exists and is unique in $C((0,+\infty); H^N(\mathbb{R}))$. A choice of c_0 and the use of the *a priori* estimate in §3 show that $T : \mathbb{B}_{c_0}(0) \to \mathbb{B}_{c_0}(0)$ with $\mathbb{B}_{c_0}(0)$ a bounded ball in $L^\infty([0,T]; H^s(\mathbb{R}))$.

Theorem 4.2. (Local existence). Assume f satisfies A.1 - A.4, and g satisfies B.1 - B.4. Let N be an integers ≥ 3. If $\varphi \in H^N(\mathbb{R})$, then there is $T > 0$ and u such that u is a strong solution of (2.5). $u \in L^\infty([0,T]; H^N(\mathbb{R}))$ with initial data $u(x,0) = \varphi(x)$.

Proof. We prove that for $\varphi \in H^\infty(\mathbb{R}) = \bigcap_{k\geq 0} H^k(\mathbb{R})$ there exists a solution $u \in L^\infty([0,T]; H^N(\mathbb{R}))$ with initial data $u(x,0) = \varphi(x)$ and which a time of existence $T > 0$ which only depends φ.
We define a sequence of approximations to equation (3.2) as

$$\begin{aligned} &- {v_t}^{(n)} + \delta\wedge {v_5}^{(n)} - \epsilon(\partial_1 g)\wedge {v_4}^{(n)} + (\partial_0 f)\wedge {v_3}^{(n)} - 2\epsilon\partial(\partial_1 g)\wedge {v_3}^{(n)} \\ &+ 2\partial(\partial_0 f)\wedge {v_2}^{(n)} - \epsilon\partial^2(\partial_1 g)\wedge {v_2}^{(n)} + \partial^2(\partial_0 f)\wedge {v_1}^{(n)} \\ &- [\delta\wedge v_3 - \epsilon(\partial_1 g)\wedge v_2^{(n)} + (\partial_0 f)\wedge v_1^{(n)}] = 0 \end{aligned} \tag{4.5}$$

where $g = g(\wedge v_1^{(n-1)})$ and $f = f(\wedge v^{(n-1)})$ and whith initial data $v^{(n)}(x,0) = \varphi(x) - \partial^2\varphi(x)$.
The first approximation is given by $v^{(0)}(x,0) = \varphi(x) - \partial^2\varphi(x)$. Equation (4.4) is a linear equation at each iteration which can be solved in any interval of time in which the coefficients are defined.
By Lemma 3.1. it follows that

$$\begin{aligned} \partial_t \int_{\mathbb{R}} \xi[v_\alpha^{(n)}]^2 dx \quad \leq \quad & G(\|v^{(n-1)}\|_\lambda)\|v^{(n)}\|_\alpha^2 + E(\|v^{(n-1)}\|_\lambda)\|v^{(n-1)}\|_\alpha^2 \\ & + F(\|v^{(n-1)}\|_\alpha). \end{aligned} \tag{4.6}$$

Choose $\alpha = 1$. Let $c_0 \geq \|\varphi - \partial^2\varphi\|_1 \geq \|\varphi\|_3$. For each iterate n, $\|v^{(n)}(\cdot,t)\|$ is continuous in $t \in [0,T]$, and $\|v^{(n)}(\cdot,0)\|_1 \leq c_0$. Define $c_3 = \frac{c_2}{2c_1}c_0^2 + 1$. Let $T^{(n)}$ be the maximum time such that $\|v^{(k)}(\cdot,t)\|_2 \leq c_3$ for $0 \leq t \leq T^{(n)}$, $0 \leq k \leq n$. Integrating (4.5) over $[0,t]$, we have for $0 \leq t \leq T^{(n)}$ and $j = 0, 1, \ldots$

$$\int_0^t \left(\partial_s \int_{\mathbb{R}} \xi [v_j^{(n)}]^2 dx\right) ds \leq \int_0^t G(\|v^{(n-1)}\|_1)\|v^{(n)}\|_j^2 ds$$
$$+ \int_0^t E(\|v^{(n-1)}\|_1)\|v^{(n-1)}\|_j^2 ds + \int_0^t F(\|v^{(n-1)}\|_j) ds$$

and it follows that

$$\int_{\mathbb{R}} \xi(x,t)[v_j^{(n)}(x,t)]^2 dx \leq \int_{\mathbb{R}} \xi(x,0)[v_j^{(n)}(x,0)]^2 dx + \int_0^t G(\|v^{(n-1)}\|_1)\|v^{(n)}\|_j^2 ds$$
$$+ \int_0^t E(\|v^{(n-1)}\|_1)\|v^{(n-1)}\|_j^2 ds + \int_0^t F(\|v^{(n-1)}\|_j) ds,$$

hence

$$\begin{aligned} c_1 \int_{\mathbb{R}} [v_j^{(n)}(x,t)]^2 dx &\leq \int_{\mathbb{R}} \xi(x,t)[v_j^{(n)}(x,t)]^2 dx \\ &\leq \int_{\mathbb{R}} \xi(x,0)[v_j^{(n)}(x,0)]^2 dx + \int_0^t G(\|v^{(n-1)}\|_1)\|v^{(n)}\|_j^2 ds \\ &\quad + \int_0^t E(\|v^{(n-1)}\|_1)\|v^{(n-1)}\|_j^2 ds + \int_0^t F(\|v^{(n-1)}\|_j) ds. \end{aligned}$$

In this way,

$$\int_{\mathbb{R}} [v_j^{(n)}]^2 dx \leq \frac{c_2}{c_1} \int_{\mathbb{R}} [v_j^{(n)}(x,0)]^2 dx + \frac{G(c_3)}{c_1} c_3^2 t + \frac{E(c_3)}{c_1} c_3^2 t + \frac{F(c_3)}{c_1} t$$

and we obtain for $j = 0, 1$.

$$\|v^{(n)}\|_1 \leq \frac{c_2}{c_1} c_0^2 + \frac{G(c_3)}{c_1} c_3^2 t + \frac{E(c_3)}{c_1} c_3^2 t + \frac{F(c_3)}{c_1} t.$$

Claim. $T^{(n)} \not\to 0$.

Proof. We suppose $T^{(n)} \to 0$. Since $\|v^{(n)}(\cdot,t)\|$ is continuous in $t > 0$, there exists $\tau \in [0,T]$ such that $\|v^{(k)}(\cdot,\tau)\|_1 = c_3$ for $0 \leq \tau \leq T^{(n)}$, $0 \leq k \leq n$. Then

$$c_3^2 \leq \frac{c_2}{c_1} c_0^2 + \frac{G(c_3)}{c_1} c_3^2 T^{(n)} + \frac{E(c_3)}{c_1} c_3^2 T^{(n)} + \frac{F(c_3)}{c_1} T^{(n)}$$

we do $n \to +\infty$ follows

$$\left(\frac{c_2}{2c_1}c_0^2+1\right)^2 \leq \frac{c_2}{c_1}c_0^2$$

thus

$$\frac{c_2^2}{4c_1^2}c_0^2+1 \leq 0 \qquad \text{(contradiction)}$$

this way $T^{(n)} \not\to 0$. Choosing $T = T(c_0)$ sufficiently small, but T not depending on n, one concludes that

$$\|v^{(n)}\|_1 \leq c \tag{4.7}$$

for $0 \leq t \leq T$. This shows that $T^{(n)} \geq T$.
Hence of (4.6) there exists a subsequence $v^{(n_j)} \stackrel{\text{def}}{=} v^{(n)}$ such that

$$v^{(n)} \stackrel{*}{\rightharpoonup} v \quad \text{weakly in} \quad L^\infty([0,T];H^1(\mathbb{R})) \tag{4.8}$$

Claim. $u = \wedge v$ is the solution we are looking for.
Proof. In the linearized equation (4.4) we have

$$\begin{aligned} \wedge v_5^{(n)} &= \wedge(I-(I-\partial^2))v_3^{(n)} \\ &= \wedge v_3^{(n)} - v_3^{(n)} \\ &= \partial^2(\underbrace{\wedge v_1^{(n)}}_{\in L^2(\mathbb{R})}) - \underbrace{\partial^2(v_1^{(n)})}_{\in H^{-2}(\mathbb{R})} \end{aligned}$$

since $\wedge = (I-\partial^2)^{-1}$ is bounded in $H^1(\mathbb{R})$ then $\wedge v_5^{(n)}$ belong to $H^{-2}(\mathbb{R})$, so still $v^{(n)}$ is bounded in $L^\infty([0,T];H^1(\mathbb{R})) \hookrightarrow L^2([0,T];H^1(\mathbb{R}))$ and since $\wedge : L^2(\mathbb{R}) \longrightarrow H^2(\mathbb{R})$ is a bounded operator, $\|\wedge v_1^{(n)}\|_{H^2(\mathbb{R})} \leq c\|v_1^{(n)}\|_{L^2(\mathbb{R})} \leq c\|v_1^{(n)}\|_{H^1(\mathbb{R})}$ hence $\wedge v_1^{(n)}$ is bounded in $L^2([0,T];H^2(\mathbb{R})) \hookrightarrow L^2([0,T];L^2(\mathbb{R}))$, follows $\partial^2(\wedge v_1^{(n)})$ is bounded in $L^2([0,T];H^{-2}(\mathbb{R}))$. This way

$$\wedge v_5^{(n)} \quad \text{is bounded in} \quad L^2([0,T];H^{-2}(\mathbb{R})) \tag{4.9}$$

Similarly all other terms are bounded. By equation (4.4), $v_t^{(n)}$ is a sum of terms each of which is the product of a coefficient, bounded uniformly in n and a function in $L^2([0,T];H^{-2}(\mathbb{R}))$ bounded uniformly n such that $v_t^{(n)}$ is bounded in $L^2([0,T];H^{-2}(\mathbb{R}))$ for on the other hand $H^1_{loc}(\mathbb{R}) \overset{c}{\hookrightarrow} H^{1/2}_{loc}(\mathbb{R}) \hookrightarrow H^{-2}(\mathbb{R})$. By Lions-Aubin's compactness Theorem there is a subsequence $v^{(n_j)} \overset{\text{def}}{=} v^{(n)}$ such that $v^{(n)} \longrightarrow v$ strongly in $L^2([0,T];H^{1/2}_{loc}(\mathbb{R}))$. Hence for a subsequence $v^{(n_j)} \overset{\text{def}}{=} v^{(n)}$ we have $v^{(n)} \longrightarrow v$ a. e. in $L^2([0,T];H^{1/2}_{loc}(\mathbb{R}))$. Moreover from (4.8) $\wedge v_5^{(n)} \rightharpoonup \wedge v_5$ weakly in $L^2([0,T];H^{-2}(\mathbb{R}))$. Similarly $\wedge v_4^{(n)} \rightharpoonup \wedge v_4$ weakly in $L^2([0,T];H^{-2}(\mathbb{R}))$. Since $\|\wedge v^{(n)}\|_{H^4(\mathbb{R})} \leq c\|v^{(n)}\|_{H^1(\mathbb{R})}$ $\leq c^{\cdot}\|v^{(n)}\|_{H^{1/2}(\mathbb{R})}$ and ${v_1}^{(n)} \longrightarrow v_1$ strongly in $L^2([0,T];H^{1/2}_{loc}(\mathbb{R}))$, then $\wedge v^{(n)}$ $\longrightarrow \wedge v$ strongly in $L^2([0,T];H^4_{loc}(\mathbb{R}))$, thus $\partial(\wedge v^{(n)}) \longrightarrow \partial(\wedge v)$ strongly in $L^2([0,T];H^3_{loc}(\mathbb{R})) \hookrightarrow L^2([0,T];H^2_{loc}(\mathbb{R}))$, then $\wedge v_1^{(n)} \longrightarrow \wedge v_1$ strongly in $L^2([0,T];H^3_{loc}(\mathbb{R})) \hookrightarrow L^2([0,T];H^2_{loc}(\mathbb{R}))$. In this way $\partial_1 g(\wedge v_1^{(n)}) \longrightarrow$ $\partial_1 g(\wedge v_1)$ strongly in $L^2([0,T];H^3_{loc}(\mathbb{R})) \hookrightarrow L^2([0,T];H^2_{loc}(\mathbb{R}))$. Thus the third term on the right hand side of (4.4), $\partial g(\wedge v_1^{(n-1)})\wedge v_4^{(n)} \rightharpoonup \partial g(\wedge v_1)\wedge v_4$ weakly in $L^2([0,T];L^1_{loc}(\mathbb{R}))$ as $\wedge v_4^{(n)} \rightharpoonup \wedge v_4$ weakly in $L^2([0,T];H^{-2}(\mathbb{R}))$, and $\partial g(\wedge v_1^{(n-1)}) \longrightarrow \partial g(\wedge v_1)$ strongly in $L^2([0,T];H^2_{loc}(\mathbb{R}))$. Similarly all other terms in (4.4) converge to their correct limits, implying $v_t^{(n)} \rightharpoonup v_t$ weakly in $L^2([0,T];L^1_{loc}(\mathbb{R}))$. Passing to limits,

$$\begin{aligned} v_t &= \delta\wedge v_5 - \epsilon\partial_1 g(\wedge v_1)\wedge v_4 + \partial_0 f(\wedge v)\wedge v_3 - 2\epsilon\partial(\partial_1 g(\wedge v_1))\wedge v_3 \\ &\quad + O(\wedge v_2, \wedge v_1, \ldots) - [\delta\wedge v_3 - \epsilon\partial_1 g(\wedge v_1)\wedge v_2 + \partial_0 f(\wedge v)\wedge v_1], \end{aligned}$$

then

$$\begin{aligned} v_t &= \partial^2(\delta\wedge v_3 - \epsilon\partial_1 g(\wedge v_1)\wedge v_2 + \partial_0 f(\wedge v)\wedge v_1) \\ &\quad - (\delta\wedge v_3 - \epsilon\partial_1 g(\wedge v_1)\wedge v_2 + \partial_0 f(\wedge v)\wedge v_1) \\ &= -(I-\partial^2)(\delta\wedge v_3 - \epsilon\partial_1 g(\wedge v_1)\wedge v_2 + \partial_0 f(\wedge v)\wedge v_1). \end{aligned}$$

Thus $v_t + (I-\partial^2)(\delta\wedge v_3 - \epsilon\partial_1 g(\wedge v_1)\wedge v_2 + \partial_0 f(\wedge v)\wedge v_1) = 0$. This way we have (2.5) for $u = \wedge v$.

We prove that there exists a solution of equation (2.5), $u \in L^\infty([0,T];H^N(\mathbb{R}))$, with $N \geq 4$, where T depends only on φ. We already know that there is a solution (previously) $u \in L^\infty([0,T];H^3(\mathbb{R}))$. It suffices to prove that the approximating sequence $v^{(n)}$ is bounded in $L^\infty([0,T];H^{N-2}(\mathbb{R}))$. Take $\alpha = N-2$ and consider (4.5) for $\alpha \geq 2$. By the same arguments as for $\alpha = 1$ we conclude that there exists $T^{(\alpha)}$ depending on the norm of φ but independent n such that $\|v^{(n)}\|_\alpha \leq c$ for all $0 \leq t \leq T^{(\alpha)}$. Thus $v \in L^\infty([0,T^{(\alpha)}];H^\alpha(\mathbb{R}))$. Now denote by $0 \leq T^{*(\alpha)} \leq +\infty$ the maximal number such that for all $0 < T \leq T^{*(\alpha)}$, $u = \wedge v \in L^\infty([0,T];H^N(\mathbb{R}))$ with $T^{(1)} \leq T^{*(\alpha)}$ for all $\alpha \geq 2$. Thus T can be chosen depending only on norm of φ. Approximating φ by

$\{\varphi_j\} \in C_0^\infty(\mathbb{R})$ such that $\|\varphi - \varphi_j\|_{H^N(\mathbb{R})} \overset{j\longrightarrow+\infty}{\longrightarrow} 0$. Let u_j be a solution of (2.5) with $u_j(x,0) = \varphi_j(x)$. According to the above argument, there exists T which is independent of n but depending only $\sup_j \|\varphi_j\|$ such that u_j exists on $[0,T]$ and a subsequence $u_j \overset{j\longrightarrow+\infty}{\longrightarrow} u$ in $L^\infty([0,T];H^N(\mathbb{R}))$. As a consequence of Theorem 4.1 and Theorem 4.2 and its proof one gets

Corollary 4.3. Let $\varphi \in H^N(\mathbb{R})$ with $N \geq 3$ such that $\varphi^{(\gamma)} \longrightarrow \varphi$ in $H^N(\mathbb{R})$. Let u and $u^{(\gamma)}$ be the corresponding unique solutions given by Theorems 4.1 and 4.2 in $L^\infty([0,T];H^N(\mathbb{R}))$ with T depending only on $\sup_\gamma \|\varphi^{(\gamma)}\|_{H^3(\mathbb{R})}$ then

$$u^{(\gamma)} \overset{*}{\rightharpoonup} u \quad \text{weakly in} \quad L^\infty([0,T];H^N(\mathbb{R}))$$

and

$$u^{(\gamma)} \longrightarrow u \quad \text{strongly in} \quad L^2([0,T];H^{N+1}(\mathbb{R})).$$

Theorem 4.4. (Persistence) Let $i \geq 1$ and $L \geq 3$ be non-negative integers, $0 < T < +\infty$. Assume that u is the solution to (2.5) in $L^\infty([0,T];H^3(\mathbb{R}))$ with initial data $\varphi(x) = u(x,0) \in H^3(\mathbb{R})$. If $\varphi(x) \in H^L(W_{0i0})$ then

$$u \in L^\infty([0,T];H^3(\mathbb{R}) \bigcap H^L(W_{0i0})) \tag{4.10}$$

where σ is arbitrary, $\eta \in W_{\sigma,i-1,0}$ for $i \geq 1$.
Proof. Similar to Theorem 4.2.

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