

# The Fundamental Theorem of Calculus for Lebesgue Integral

*El Teorema Fundamental del Cálculo  
para la Integral de Lebesgue*

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## Abstract

In this paper we prove the Theorem announced in the title without using Vitali's Covering Lemma and have as a consequence of this approach the equivalence of this theorem with that which states that absolutely continuous functions with zero derivative almost everywhere are constant. We also prove that the decomposition of a bounded variation function is unique up to a constant.

**Key words and phrases:** Radon-Nikodym Theorem, Fundamental Theorem of Calculus, Vitali's covering Lemma.

## Resumen

En este artículo se demuestra el Teorema Fundamental del Cálculo para la integral de Lebesgue sin usar el Lema del cubrimiento de Vitali, obteniéndose como consecuencia que dicho teorema es equivalente al que afirma que toda función absolutamente continua con derivada igual a cero en casi todo punto es constante. También se prueba que la descomposición de una función de variación acotada es única a menos de una constante.

**Palabras y frases clave:** Teorema de Radon-Nikodym, Teorema Fundamental del Cálculo, Lema del cubrimiento de Vitali.

## 1 Introduction

The Fundamental Theorem of Calculus for Lebesgue Integral states that:

*A function  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous if and only if it is differentiable almost everywhere, its derivative  $f' \in L^1[a, b]$  and, for each  $t \in [a, b]$ ,*

$$f(t) = f(a) + \int_a^t f'(s)ds.$$

This theorem is extremely important in Lebesgue integration Theory and several ways of proving it are found in classical Real Analysis. One of the better known proofs relies on the non trivial Vitali Covering Lemma, perhaps influenced by Saks monography [9]; we recommend Gordon's book [5] as a recent reference. There are other approaches to the subject avoiding Vitali's Covering Lemma, such as using the Riesz Lemma: Riez-Nagy [7] is the classical reference. Another approach can be seen in Rudin ([8], chapter VIII) which treats the subject by differentiating measures and of course makes use of Lebesgue Decomposition and the Radon-Nikodym Theorem. The usual form of proving this fundamental result runs more or less as follows:

First of all, Lebesgue Differentiation Theorem is established:

*Every bounded variation function  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable almost everywhere with derivative belonging to  $L^1[a, b]$ . If the function  $f$  is non-decreasing, then*

$$\int_a^b f'(s)ds \leq f(b) - f(a).$$

In the classical proof of the above theorem Vitali's Covering Lemma (Riesz Lemma either) is used, as well as in that of the following Lemma, previous to the proof of the theorem we are interested in.

*If  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous with  $f' = 0$  almost everywhere then  $f$  is constant.*

An elementary and elegant proof of this Lemma using tagged partitions has recently been achieved by Gordon [6].

In this paper we present another approach to the subject; indeed, we start with Lebesgue Decomposition and Radon Nikodym Theorem and soon after, we derive directly the Fundamental Theorem of Calculus and get relatively simple proofs of well known results.

We start outlining the proof of the Radon Nikodym Theorem given by Bradley [4] in a slightly different way; indeed, instead of giving the proof of Radon Nikodym Theorem as in [4], we make a direct (shorter) proof of the Lebesgue Decomposition Theorem which has as a corollary the Radon Nikodym Theorem. Readers familiarized with probability theory will soon recognize the presence of martingale theory in this approach.

The needed preliminaries for this paper are all studied in a regular graduate course in Real Analysis, especially we need the Monotone and Dominated Convergence theorems, the Cauchy-Schwartz inequality (an elementary proof is provided by Apostol [1]), the fact that if  $(\Omega, \Sigma, \mu)$  is a measure space and  $g \in L^1(\mu)$ , then  $\nu : \Sigma \rightarrow \mathbb{R}$  defined for  $\nu(E) = \int_E g d\mu$  is a real measure and  $f \in L^1(\nu)$  if and only if  $fg \in L^1(\mu)$  and

$$\int_E f d\nu = \int_E fg d\mu, \quad \forall E \in \Sigma.$$

We also need the definitions of mutually orthogonal measures  $(\mu \perp \lambda)$  and measure  $\lambda$  absolutely continuous with respect to  $\mu$  ( $\lambda \ll \mu$ ). While these preliminary facts, previous to Lebesgue Decomposition Theorem and Radon Nikodym Theorem, are nicely treated in Rudin [8], a good account of distribution functions (bounded variation functions) can be found in Burrill [3]. Particularly important are the following results:

*Every bounded variation function  $f : [a, b] \rightarrow \mathbb{R}$  determines a unique Lebesgue-Stieljes measure  $\mu$ . The function  $f$  is absolutely continuous if and only if its corresponding Lebesgue-Stieljes measure  $\mu$  is absolutely continuous with respect to Lebesgue measure.*

It is also important for our purposes the following fact:

*If  $f : [a, b] \rightarrow \mathbb{R}$  is a bounded variation function with associated Lebesgue-Stieljes measure  $\mu$ , then the following statements are equivalent:*

- a)  *$f$  is differentiable at  $x$  and  $f'(x) = A$ .*
- b) *For each  $\epsilon > 0$  there is a  $\delta > 0$  such that  $|\frac{\mu(I)}{m(I)} - A| < \epsilon$ , whenever  $I$  is an open interval with Lebesgue measure  $m(I) < \delta$  and  $x \in I$ ,*

Both references Rudin [8] and Burrill [3] are worth to look for the proof of this equivalence.

## 2 Lebesgue Decomposition and Radon-Nikodym Theorem

The Radon Nikodym Theorem plays a key role in our proof of the Fundamental Theorem of Calculus, particularly the proof given by Bradley [4], so we will outline this proof but deriving it from Lebesgue Decomposition Theorem (Theorem 1 below).

**Theorem 1. (Lebesgue Decomposition Theorem).**

Let  $(\Omega, \Sigma, \mu)$  be a finite, positive measure space and  $\lambda : \Omega \rightarrow \mathbb{R}$  a bounded variation measure. Then there is a unique pair of measures  $\lambda_a$  and  $\lambda_s$  so that  $\lambda = \lambda_a + \lambda_s$  with  $\lambda_a \ll \mu$  and  $\lambda_s \perp \mu$ .

*Proof.* We first prove the uniqueness: if  $\lambda_a^1 + \lambda_s^1 = \lambda_a^2 + \lambda_s^2$  with

$$\lambda_a^i \ll \mu; \quad \lambda_s^i \perp \mu \quad i = 1, 2,$$

then

$$\lambda_a^1 - \lambda_a^2 = \lambda_s^2 - \lambda_s^1$$

with both sides of this equation simultaneously absolutely continuous and singular with respect to  $\mu$ . This quickly yields

$$\lambda_a^1 = \lambda_a^2 \quad \text{and} \quad \lambda_s^1 = \lambda_s^2.$$

*Existence:* This portion of the proof is modelled on Bradley's idea [4]. Let us denote by  $|\lambda|$  the variation of  $\lambda$  and put  $\sigma = \mu + |\lambda|$ . Then  $\mu$  and  $\lambda$  are absolutely continuous with respect to the finite positive measure  $\sigma$ . Now for any measurable finite partition  $P = \{A_1, A_2, \dots, A_n\}$  of  $\Omega$  we define

$$h_P = \sum_{i=1}^n c_i \chi_{A_i},$$

where

$$c_i = \begin{cases} 0 & \text{if } \sigma(A_i) = 0, \\ \frac{\lambda(A_i)}{\sigma(A_i)} & \text{otherwise.} \end{cases}$$

(Our proof omits some details which can be found in [4]).

Notice that the following three statements hold for each finite measurable partition  $P$ :

- a)  $0 \leq |h_P(x)| \leq 1$ .
- b)  $\lambda(\Omega) = \int_{\Omega} h_P d\sigma$ .
- c) If  $P$  and  $\Pi$  are finite measurable partitions of  $\Omega$  with  $\Pi$  a refinement of  $P$ , then

$$\begin{aligned} \int_{\Omega} h_{\Pi}^2 d\sigma &= \int_{\Omega} h_P^2 d\sigma + \int_{\Omega} (h_{\Pi} - h_P)^2 d\sigma \\ &\geq \int_{\Omega} h_P^2 d\sigma. \end{aligned}$$

If we put

$$k = \sup \int_{\Omega} h_P^2 d\sigma,$$

where the supremum is taken over all finite measurable partitions of  $\Omega$ , then, since for any finite measurable partition  $P$  of  $\Omega$  we have that

$$\begin{aligned} \int_{\Omega} h_P^2 d\sigma &= \int_{\Omega} \sum_{i=1}^n \frac{\lambda(A_i)^2}{\sigma(A_i)^2} d\sigma \\ &= \sum_{i=1}^n \frac{|\lambda(A_i)|^2}{\sigma(A_i)} \\ &\leq \sum_{i=1}^n |\lambda(A_i)| \\ &\leq |\lambda|(\Omega) < \infty, \end{aligned}$$

we conclude that  $0 \leq k < \infty$ .

For each  $n = 1, 2, \dots$  take as  $P_n$  a finite measurable partition of  $\Omega$  such that

$$k - \frac{1}{4^n} \leq \int_{\Omega} h_{P_n}^2 d\sigma$$

and let  $\Pi_n$  be the least common refinement of  $P_1, P_2, \dots, P_n$ . By (c),

$$k - \frac{1}{4^n} \leq \int_{\Omega} h_{P_n}^2 d\sigma \leq \int_{\Omega} h_{\Pi_n}^2 d\sigma \leq k.$$

So for each  $n > 1$ , we have

$$\int_{\Omega} (h_{\Pi_{n+1}} - h_{\Pi_n})^2 d\sigma = \int_{\Omega} h_{\Pi_{n+1}}^2 d\sigma - \int_{\Omega} h_{\Pi_n}^2 d\sigma \leq \frac{1}{4^n}$$

and by the Cauchy-Schwartz inequality we get

$$\int_{\Omega} |h_{\Pi_{n+1}} - h_{\Pi_n}| d\sigma \leq \frac{1}{2^n} (\sigma(\Omega))^{\frac{1}{2}} < \infty$$

and so

$$\int_{\Omega} \sum_{i=1}^n |h_{\Pi_{n+1}} - h_{\Pi_n}| d\sigma \leq (\sigma(\Omega))^{\frac{1}{2}} < \infty$$

which implies that

$$h_{\Pi_{n+1}} = h_{\Pi_1} + \sum_{i=1}^n h_{\Pi_{i+1}} - h_{\Pi_i}$$

converges  $\sigma$ -almost everywhere. Put

$$h(x) = \begin{cases} \lim_{n \rightarrow \infty} h_{\Pi_n}(x) & \text{if the limits exists,} \\ 0 & \text{otherwise.} \end{cases}$$

If  $A \in \Sigma$  and  $n \in \mathbb{N}$  take  $R_n$  as the smallest common refinement of  $\Pi_n$  and  $\{A, \Omega \setminus A\}$  to get

$$\int_{\Omega} |h_{R_n} - h_{\Pi_n}|^2 d\sigma \leq \frac{1}{4^n}.$$

By (c) and the Cauchy-Schwartz inequality, for  $n \geq 1$ ,

$$\left| \int_{\Omega} (h_{R_n} - h_{\Pi_n})^2 d\sigma \right| \leq \int_{\Omega} |h_{R_n} - h_{\Pi_n}|^2 d\sigma < \frac{1}{2^n},$$

which implies that

$$\lim_{n \rightarrow \infty} \int_A (h_{R_n} - h_{\Pi_n}) d\sigma = 0.$$

Since by (b),

$$\lambda(A) = \int_A h_{\Pi_n} d\sigma + \int_A (h_{R_n} - h_{\Pi_n}) d\sigma,$$

we conclude that

$$\lambda(A) = \lim_{n \rightarrow \infty} \int_A h_{\Pi_n} d\sigma$$

and applying the Dominated Convergence Theorem, we get

$$\lambda(A) = \int_A h d\sigma \quad \forall A \in \Sigma.$$

Summarizing we have gotten a  $\sigma$ -integrable function  $h_\lambda$ , depending on  $\lambda$ , such that

$$\lambda(A) = \int_A h_\lambda d\sigma, \quad \forall A \in \Sigma.$$

Similarly we can find a  $\sigma$ -integrable function  $h_\mu$  (depending on  $\mu$ ) such that

$$\mu(A) = \int_A h_\mu d\sigma \quad \forall A \in \Sigma.$$

Put

$$\Omega_1 = \{x : h_\mu(x) \leq 0\},$$

$$\Omega_2 = \{x : h_\mu(x) > 0\}.$$

Plainly  $\mu(\Omega_1) = 0$  and defining, for  $A \in \Sigma$ ,  $\lambda_s(A) = \lambda(A \cap \Omega_1)$  and  $\lambda_a(A) = \lambda(A \cap \Omega_2)$ , we see that  $\lambda_s \perp \mu$  and  $\lambda_a + \lambda_s = \lambda$ .

It remains to prove that  $\lambda_a \ll \mu$ . Put

$$h(x) = \begin{cases} \frac{h_\lambda(x)}{h_\mu(x)} & \text{if } x \in \Omega_2 \\ 0 & \text{if } x \in \Omega_1. \end{cases}$$

Then for any measurable set  $A \subset \Omega_2$  we have

$$\lambda_a(A) = \lambda(A) = \int_A h_\lambda d\sigma = \int_A h h_\mu d\sigma = \int_A h d\mu,$$

which implies  $\lambda_a \ll \mu$ . □

From Lebesgue Decomposition Theorem (and its proof) we get

**Corollary 2. (Radon Nikodym Theorem)**

If in the former theorem  $\lambda \ll \mu$ , then there is a unique  $h \in L^1(\mu)$  such that

$$\lambda(A) = \int_A h d\mu, \quad \forall A \in \Sigma.$$

### 3 The Fundamental Theorem of Calculus

Now we are in position to prove the theorem we are interested in.

Let  $f : [a, b] \rightarrow \mathbb{R}$  be an absolutely continuous function and  $\mu$  its corresponding Lebesgue-Stieljes measure. Then  $\mu$  is absolutely continuous with respect to the Lebesgue measure  $m$ . By the Radon-Nikodym Theorem there exists  $h \in L^1(m)$  such that

$$\mu(A) = \int_A h dm, \quad \forall A \in \Sigma.$$

*Proof.* Let us define the sequence of partitions  $P_n$  as

$$P_n = \{[x_i, x_{i+1}]\}_{i=1}^{2^n}, \quad x_i = a + \frac{i}{2^n}(b-a).$$

Now define

$$h_n(x) = \begin{cases} \sum_{i=1}^{2^n} \frac{\mu(A_i)}{m(A_i)} \chi_{A_i}(x) & \text{if } x \neq b, \\ 0 & \text{if } x = b. \end{cases}$$

and notice that

$$\lim_{n \rightarrow \infty} h_n(x) = h(x)$$

$m$ -almost everywhere. Hence  $f$  is differentiable almost everywhere and  $f'(x) = h(x)$   $m$ -a.e. Furthermore for each  $t \in [a, b]$ ,

$$f(t) - f(a) = \mu([a, t]) = \int_a^t h(s) ds = \int_a^t f'(s) ds.$$

So the Fundamental Theorem of Calculus for Lebesgue integral has been proved.  $\square$



The proof of the following corollary can't be easier.

**Corollary 3.** *If  $f$  is absolutely continuous and  $f' = 0$   $m$ -almost everywhere, then  $f$  is constant.*

Since classical proofs of the Fundamental Calculus Theorem use the above corollary we conclude the following result:

**Theorem 4.** *The Following statements are equivalent*

- a) *Every absolutely continuous function for which  $f' = 0$   $m$ -a.e. is a constant function.*
- b) *If  $f : [a, b] \rightarrow \mathbb{R}$  is absolutely continuous then  $f$  is differentiable almost everywhere with*

$$f(t) - f(a) = \int_a^t f'(s) ds, \quad \forall t \in [a, b].$$

Another old gem from Lebesgue Integration Theory is the following one.

**Corollary 5.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is Lebesgue integrable (regarding Lebesgue measure) and  $g(t) = \int_a^t f(s) ds$  ( $t \in [a, b]$ ), then  $g$  is differentiable almost everywhere and  $g'(s) = f(s)$  for almost every  $t \in [a, b]$ .*

*Proof.* Using standard techniques we establish the absolute continuity of  $g$ . So  $g$  is differentiable almost everywhere and  $g(t) = \int_a^t g'(s) ds$  for each  $t \in [a, b]$ . From this and the definition of  $g$ , we conclude that  $g'(s) = f(s)$  almost everywhere.  $\square$

Lebesgue Differentiation Theorem has been proved by Austin [2] without using Vitali's Covering Lemma. Because differentiable functions have measurable derivatives, this allows us to get the following theorem without using Vitali's Lemma. Recall that a function  $f : [a, b] \rightarrow \mathbb{R}$  is *singular* if and only if it has a zero derivative almost everywhere ([8]).

**Theorem 6.** *If  $f : [a, b] \rightarrow \mathbb{R}$  is a bounded variation function, then it is the sum of an absolutely continuous function plus a singular function. This Decomposition is unique up to a constant.*

*Proof. Uniqueness:* Suppose

$$f = f_1 + f_2 = g_1 + g_2$$

with  $f_1, g_1$  singular and  $f_2, g_2$  absolutely continuous functions. Then

$$f_1 - g_1 = g_2 - f_2$$

is both absolutely continuous and singular. These facts imply that  $(f_2 - g_2)' = 0$  almost everywhere and so  $f_2 - g_2$  is constant.

*Existence:* Since  $f$  is of bounded variation, then  $f'$  exists almost everywhere. If  $f$  is non-decreasing, it is not hard to see that for each  $t \in [a, b]$

$$f' \in L^1[a, b] \quad \text{and} \quad 0 \leq \int_0^t f'(s)ds \leq f(t) - f(a)$$

and, since any bounded variation function is the difference of two non-decreasing functions, we realize that  $f' \in L^1[a, b]$ .

If we define for  $t \in [a, b]$

$$h(t) = \int_a^t f'(s)ds + f(a),$$

then  $h$  is absolutely continuous and  $g(t) = f(t) - h(t)$  is singular with  $f = g + h$ . This ends the proof.  $\square$

**Remark.** Regarding the above Theorem, this is the only proof that we know about uniqueness.

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