Higher-order Lucas Numbers

Números de Lucas de Orden Superior

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Abstract

We consider a generalization of the Lucas numbers that follows from an application of the higher order Hosoya Z indices. The Hosoya Z index is defined by the count of disjoint edges in a graph. The higher order Hosoya indices are based on the count of disjoint longer paths in a graph. Hosoya has demostrated that the counts of Z for cyclic graphs lead to the Lucas numbers. By extension, we call the numbers obtained by the count of higher order Z numbers for cyclic graphs the higher order Lucas numbers. Some properties of the newly derived higher order Lucas numbers are discussed.

Key words and phrases: Hosoya index, Fibonacci numbers, Lucas numbers.

Resumen

Presentamos una generalización de los números de Lucas como consecuencia de la aplicación del índice superior de Hosoya. El índice Z de Hosoya es definido por el conteo de las aristas disjuntas en un grafo. Los índices superiores de Hosoya se determinan por el conteo de caminos disjuntos en un grafo. Hosoya ha demostrado que el conteo de Z para grafos cíclicos conduce a los números de Lucas. Por analogía, los números obtenidos por el conteo de los índices superiores de Hosoya para ciclos son denominados números de Lucas de orden superior. Se

presentan algunas propiedades de esos números.

Palabras y frases clave: Índice de Hosoya, números de Fibonacci, números de Lucas.

1 Introduction

The Lucas numbers

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, \ldots$$

with the recursion: $L_n = L_{n-1} + L_{n-2}$ represent one of the first generalizations of the Fibonacci numbers [4]. The sequence differs from the Fibonacci sequence only by the choice of the initial two terms that define the sequence. In this paper we will outline a generalization of the Lucas numbers which we believe deserves some attention. Our generalization follows naturally from the relationship between the Hosoya Z topological index and the Lucas numbers for cyclic graphs [5,6]. The Hosoya topological index, which can be calculated for any graph, is a graph theoretical invariant defined as [5]

$$Z = \sum_{k=0}^{\lfloor n/2 \rfloor} p(G, k) \tag{1}$$

where $p(G, k)$ represents the number of different ways of choosing k non adjacent edges in graph G. By definition $p(G, 0) = 1$. The summation extends over all possible combinations of edges. When the Z index is calculated for paths, as illustrated below, the Fibonacci numbers are obtained.

<div align="center">

Table 1   Paths

</div>

Paths	$p(G, k)$	Fibonacci numbers
P_1	$Z = 1 + 1$	$F_2 = 2$
P_2	$Z = 1 + 2$	$F_3 = 3$
P_3	$Z = 1 + 3 + 1$	$F_4 = 5$
P_4	$Z = 1 + 4 + 3$	$F_5 = 8$
$\cdots$	$\cdots$	$\cdots$

By definition $F_0 = 1$, $F_1 = 1$. When Z is calculated for cycles, the Lucas numbers are obtained.

Recently, Hosoya has found several new sequences of graphs whose Z-values are either Fibonacci or Lucas numbers, or their multiples [7]. These new graphs form the golden family of graphs.

Lucas numbers

In the following table we give the Lucas numbers and their partitions (in terms of $p(G, k)$) for small cycles, as outlined by Hosoya [6].

	Table 2		Cycles
Cycles	$p(G, k)$		Lucas numbers
C_3	$Z = 1 + 3$		$L_3 = 4$
C_4	$Z = 1 + 4 + 2$		$L_4 = 7$
C_5	$Z = 1 + 5 + 5$		$L_5 = 11$
C_6	$Z = 1 + 6 + 9 + 2$		$L_6 = 18$
C_7	$Z = 1 + 7 + 14 + 7$		$L_7 = 29$
C_8	$Z = 1 + 8 + 20 + 16 + 2$		$L_8 = 47$
C_9	$Z = 1 + 9 + 27 + 30 + 9$		$L_9 = 76$
C_{10}	$Z = 1 + 10 + 35 + 50 + 25 + 2$		$L_{10} = 123$

By definition $L_1 = 1$ and $L_2 = 3$. There are some interesting relationships between the Fibonacci and the Lucas numbers such as
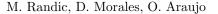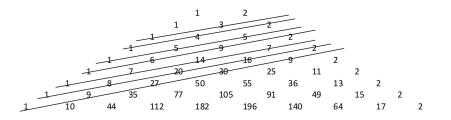
$$L_n = F_n + F_{n-2}. \tag{2}$$

We would like to turn attention to the numbers $p(G, k)$ in Table 2. As we will see, we may refer to $p(G, k)$ as a natural partition of the Lucas numbers, just as the $p(G, k)$ numbers of Table 1 represent natural partition of the Fibonacci numbers. As has been outlined elsewhere [3,6,10,12], the partition numbers $p(G, k)$ are "hidden" in the Pascal triangle and can be recovered by using parallel slanted lines connecting the corresponding entries which produce the respective Fibonacci numbers. What about the $p(G, k)$ numbers making the natural partition of the Lucas numbers? Can they be recovered from a Pascal triangle in an orderly fashion?

The answer does not look promising since as we see from Table 2 several entries as $14, 16, 20, 27$ do not appear in the early parts of the Pascal triangle. On the other hand a close look at Table 2 reveals a simple regularity

$$p(C_n, k) = p(C_{n-1}, k) + p(C_{n-2}, k - 1) \tag{3}$$

This relationship allows to construct $p(G, k)$ entries for larger cycles without exhaustive count of all combinatorial possibilities. On the other hand, the same regularity may be exposed by constructing a Pascal-like triangle:

```
                          1        2
                     1        3        2
                 1       4        5        2
              1      5       9        7       2
           1      6      14       16       9       2
         1     7      20       30      25      11      2
       1    8     27      50       55      36      13      2
     1    9    35     77      105      91      49      15     2
    1   10    44    112     182      196     140     64     17     2
```

If now we consider parallel slanted diagonal lines, similar to those that give Fibonacci numbers in the case of Pascal triangle, we obtain Lucas numbers. Hence, Lucas numbers are also "hidden" in a triangle like the Fibonacci numbers, the difference is that now instead of the Pascal triangle we have a Pascaloid triangle shown above in which one of the edges is like in the Pascal triangle and the other is defined by the number two.

Second order Lucas numbers

We will define the second order Lucas numbers in terms of the second order Hosoya ^2Z numbers that are given by [2,10]

$$^2Z = \sum_{k=0} p_2(G, k), \tag{4}$$

where $p_2(G, k)$ represents the number of different ways of choosing k non adjacent paths of length two in graph G. By definition $p_2(G, 0) = 1$. The summation extends over all possible combinations of edges. When the ^2Z index is calculated for paths, the second order Fibonacci numbers ^2F are obtained [10].

$$1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, \ldots$$

which satisfy the recursion: $^2F_n = {}^2F_{n-1} + {}^2F_{n-3}$.

This recursion differs from that for the Fibonacci numbers only in the last term. The last term of the Fibonacci recursion $^2F_{n-2}$ is here replaced by $^2F_{n-3}$. The second order Fibonacci numbers were obtained when the second order Hosoya index ^2Z was calculated for the path graph.

If we calculate ^2Z for cycles C_n we obtain

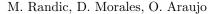|	Table 3	Cycles
Cycles	$p_2(G,k)$	Second order Lucas numbers
C_3	$Z = 1 + 3$	$^2L_3 = 4$
C_4	$Z = 1 + 4$	$^2L_4 = 5$
C_5	$Z = 1 + 5$	$^2L_5 = 6$
C_6	$Z = 1 + 6 + 3$	$^2L_6 = 10$
C_7	$Z = 1 + 7 + 7$	$^2L_7 = 15$
C_8	$Z = 1 + 8 + 12$	$^2L_8 = 21$
C_9	$Z = 1 + 9 + 18 + 3$	$^2L_9 = 31$
C_{10}	$Z = 1 + 10 + 25 + 10$	$^2L_{10} = 46$
C_{11}	$Z = 1 + 11 + 33 + 22$	$^2L_{11} = 67$
C_{12}	$Z = 1 + 12 + 42 + 40 + 3$	$^2L_{12} = 98$

Again, by definition, we take $^2L_1 = 1$ and $^2Z_2 = 3$ (which are the same as the initial Lucas numbers L_1 and L_2). We see that the recursion for the second Lucas number is very simple and is identical to that for the second order Fibonacci numbers: $^2L_n = {}^2L_{n-1} + {}^2L_{n-3}$. Hence, there is no problem to generate larger second order Lucas numbers using the recursion.
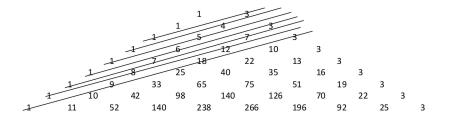
Let us again focus attention on the $p_2(G,k)$ numbers which represent the partitioning of the second order Lucas numbers. Are these numbers related to a Pascaloid triangle, as was the case with the Lucas numbers? By a close look at the numbers, and having in mind the recursion for the second order Lucas numbers which prescribes addition of rows $n-1$ and $n-3$ to obtain the value in the row n, we can see that indeed entries in rows separated by a single row, when diagonally added, give the entries in the row n, i.e.,

$$p_2(C_n, k) = p_2(C_{n-1}, k) + p_2(C_{n-3}, k - 1). \tag{5}$$

This expression is similar to the expression (3) for the Lucas numbers, the difference is only in the first index in the last term (which we emphasized). The relationship allows to construct $p(G,k)$ entries for larger cycles without exhaustive count of all combinatorial possibilities.

The same regularity may be exposed by constructing a new Pascal-like triangle

```
                          1           3
                      1       4       3
                  1       5       7       3
              1       6      12      10       3
          1       7      18      22      13       3
      1       8      25      40      35      16       3
  1       9      33      65      75      51      19       3
1      10      42      98     140     126      70      22       3
1      11      52     140     238     266     196      92      25       3
```

If now we consider parallel slanted diagonal lines, similar to those that give the second order Fibonacci numbers in the case of Pascal triangle, we obtain the second order Lucas numbers. Hence, second order Lucas numbers are also "hidden" in a triangle like the second order Fibonacci numbers, the difference is that now instead of the Pascal triangle we have a new Pascaloid triangle shown above in which one of the edge is like in the Pascal triangle and the other is defined by the number three.

Higher order Lucas numbers

The higher order Lucas numbers are defined by calculating the higher order Hosoya numbers for cycles. The higher order Hosoya numbers are given by

$$^h Z = \sum_{h=0} p_h(G, k), \tag{6}$$

where $p_h(G, k)$ represents the number of different ways of choosing k non adjacent paths of length h in graph G. By definition $p_h(G, 0) = 1$. The summation extends over all possible combinations of non adjacent paths. The derived higher order Lucas numbers satisfy the recursion

$$^h L_n = {}^h L_{n-1} + {}^h L_{n-1-h}.$$

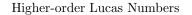The regularities for the partition contributions also hold, i.e.,

$$p_h(C_n, k) = p_h(C_{n-1}, k) + p_h(C_n - 1 - h, k - 1). \tag{7}$$

The relationship allows one to construct $p_h(G, k)$ entries for large cycles without exhaustive count of all combinatorial possibilities.

The same regularity may be exposed by constructing new Pascal-like triangles:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & h{+}1 & & & \\
 & & & & & 1 & & h{+}2 & & & h{+}1 & & \\
 & & & & 1 & & h{+}3 & & 2h{+}3 & & & h{+}1 & \\
 & & & 1 & & h{+}4 & & 3h{+}6 & & 3h{+}4 & & h{+}1 & \\
 & & 1 & & h{+}5 & & 4h{+}10 & & 6h{+}10 & & 4h{+}5 & & h{+}1 \\
 & 1 & & h{+}6 & & 5h{+}15 & & 10h{+}20 & & 10h{+}15 & & 5h{+}6 & & h{+}1
\end{array}
$$

The above triangle in fact is constructed by a superposition of two Pascal triangles slightly displaced, one for the constant entries and one for the variable entries. Because of this simple structure we can write down a general expression for $p_h(G,k)$ entry (m,n) of the generalized Pascal triangle, where m is the number of row and n is the number of the column:

$$
\begin{aligned}
p_h(G,k) &= \binom{n-k}{k} h + \binom{n-k+1}{k+1} \\
&= \binom{m-1}{n-1} h + \binom{m}{n}.
\end{aligned}
$$

We have found that the higher Lucas numbers can be expressed in closed form by the sum

$$
{}^h L_n = 1 + n \sum_{i=1}^{\lfloor n/(h+1) \rfloor} \frac{1}{i}\binom{n-1-hi}{i-1}, \quad n \geq h+1.
$$

This sum can be written, in turn, in terms of a generalized hypergeometric function ${}_{h+1}F_h$ [11]

$$
\begin{aligned}
{}^h L_n &= {}_{h+1}F_h\left(\frac{1-n}{h+1}, \frac{2-n}{h+1}, \ldots, \frac{h-n}{h+1}, -\frac{n}{h+1}; \frac{1-n}{h}, \frac{2-n}{h}, \ldots, \right. \\
& \qquad \left. \frac{h-n}{h}; -\frac{(h+1)^{h+1}}{h^h}\right),
\end{aligned}
$$

which will allow one to obtain generalized recursion relations and other explicit formulations using the extense data base for these functions, especially for those hypergeometric functions of the form ${}_{n+1}F_n$ [9].

An application

We have derived the higher-order Lucas numbers as a natural application of the higher-order Hosoya Z index to cyclic graphs. However, these numbers

have also appeared recently in statistical mechanics problems related with lattices. Indeed, our higher-order Lucas numbers ${}^{h}L_n$ give the number of ways to cover (without overlapping) a ring lattice (or necklace) of n sites with molecules that are $h + 1$ sites wide [1,8].

Conclusions

In this article we have derived a generalization of the Lucas numbers as a natural application of the higher-order Hosoya index to cyclic graphs. We have derived some important properties of these numbers and have shown that they can be found in Pascal like triangles. Finally, we showed that these numbers have a useful combinatorial interpretation and have found a close-form expression in terms of hypergeometric functions of the form $_{n+1}F_n$. Some additional properties of these numbers wait for further exploration. For instance, can the higher-order Lucas numbers be expressed in terms of higher-order Fibonacci numbers? On the other hand, it is known that for other graphs besides the cycles the Hosoya index generates the sequence of Lucas numbers. Indeed, the graphs that represent the molecules 2-methyl alkanes, the cyclic alkanes and the 1-methyl bicyclo $[X.1.0]$ alkanes possess identical values of the Hosoya index represented by the Lucas numbers. This result suggests the following question: Are there other type of graphs whose Hosoya index yields the higher-order Lucas numbers?

Acknowledgment

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