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Weyl's type theorems and perturbations

Teoremas de tipo Weyl y perturbaciones

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Abstract

Weyl's theorem for a bounded linear operator T on complex Banach spaces, as well as its variants, a-Weyl's theorem and property (w), in general is not transmitted to the perturbation $T + K$, even when K is a "good" operator, as a commuting finite rank operator or compact operator. Weyl's theorems do not survive also if K is a commuting quasi-nilpotent operator. In this paper we discuss some sufficient conditions for which Weyl's theorem, a-Weyl's theorem as well as property (w) is transmitted under such kinds of perturbations.
Key words and phrases: Local spectral theory, Fredholm theory, Weyl's theorem.

Resumen

El Teorema de Weyl para un operador lineal acotado T sobre un espacio de Banach complejo, como también sus variantes, el teorema a-Weyl y la propiedad (w), en general no se transmite a la perturbación $T + K$, ni aún cuando K es un "buen" operador, como operadores

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conmutantes de rango finito u operadores compactos. Los teoremas de Weyl no sobreviven tampoco si K es un operador conmutante cuasi-nilpotente. En este artículo discutimos algunas condiciones suficientes para las cuales el teorema de Weyl, el teorema a-Weyl y la propiedad (w) se transmite bajo tales tipos de perturbaciones.
Palabras y frases clave: Teoría espectral local, teoría de Fredholm, teorema de Weyl.

1 Introduction and preliminaries

Weyl [35] examined the spectra of all compact perturbations of a hermitian operator T on a Hilbert space and proved that their intersection coincides with the isolated point of the spectrum $\sigma(T)$ which are eigenvalues of finite multiplicity. Weyl's theorem has been extended to several classes of Hilbert spaces operators and Banach spaces operators. More recently, two variants of Weyl's theorem have been introduced by Rakočević [32], [31], the so called a-Weyl's theorem and the property (w) studied also in [10].

In general, Weyl's theorem is not sufficient for Weyl's theorem for $T+K$, where $K \in L(X)$. Weyl's theorems are liable to fail also under "small" perturbations if "small" is interpreted in the sense of compact or quasi-nilpotent operators. In this note, we study sufficient conditions for which we have the stability of Weyl's theorems and property (w), under perturbations by finite rank operators, compact operators, or quasi-nilpotent operator commuting with T. We shall see that the stability of Weyl's theorems requires some special conditions on the isolated points of the spectrum (or on the isolated points of the approximate point spectrum).

We begin with some standard notations on Fredholm theory. Throughout this note, let $L(X)$ denote the algebra of all bounded linear operators acting on an infinite dimensional complex Banach space X. If $T \in L(X)$ write $\alpha(T)$ for the dimension of the kernel ker T and $\beta(T)$ for the codimension of the range $T(X)$. Denote by

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$$

the class of all *upper semi-Fredholm* operators, and by

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$$

the class of all *lower semi-Fredholm* operators. The class of all semi-Fredholm operators is defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, while the class of all

Fredholm operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. The *index* of a semi-Fredholm operator is defined by $\text{ind } T := \alpha(T) - \beta(T)$. Recall that the *ascent* $p := p(T)$ of a linear operator T is defined to be the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}(X)$. If such an integer does not exist we put $p(T) = \infty$. Analogously, the *descent* $q := q(T)$ of an operator T is the smallest non-negative integer q such that $T^q(X) = T^{q+1}(X)$, and if such an integer does not exist we put $q(T) = \infty$. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, [1, Theorem 3.3]. Two important classes of operators are the class of all *upper semi-Browder operators*

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

and the class of all *lower semi-Browder operators*

$$\mathcal{B}_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

The class of all *Browder operators* is defined by $\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. Recall that a bounded operator $T \in L(X)$ is said to be a *Weyl operator* if $T \in \Phi(X)$ and $\text{ind } T = 0$. Clearly, if T is Browder then T is Weyl, since the finiteness of $p(T)$ and $q(T)$ implies, for a Fredholm operator, that T has index 0, see [1, Theorem 3.4].

These classes of operators motivate the definition of several spectra. The *upper semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\rm ub}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_+(X)\},$$

the *lower semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\rm lb}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_-(X)\},$$

while the *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\rm b}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}(X)\}.$$

The *Weyl spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\rm w}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{is not Weyl}\}.$$

We have that $\sigma_{\rm w}(T) = \sigma_{\rm w}(T^*)$, where T^* denotes the dual of T, while

$$\sigma_{\rm ub}(T) = \sigma_{\rm lb}(T^*), \quad \sigma_{\rm lb}(T) = \sigma_{\rm ub}(T^*).$$

Evidently,

$$\sigma_{\rm w}(T) \subseteq \sigma_{\rm b}(T) = \sigma_{\rm w}(T) \cup \text{acc } \sigma(T),$$

where we write acc K for the accumulation points of $K \subseteq \mathbb{C}$, see [1, Chapter 3]. The *Weyl (or essential) approximate point spectrum* $\sigma_{\mathrm{wa}}(T)$ of a bounded operator $T \in L(X)$ is the complement of those $\lambda \in \mathbf{C}$ for which $\lambda I - T \in \Phi_+(X)$ and ind $(\lambda I - T) \leq 0$. Note that $\sigma_{\mathrm{wa}}(T)$ is the intersection of all approximate point spectra $\sigma_a(T+K)$ of compact perturbations K of T, see [32]. The *Weyl surjectivity spectrum* $\sigma_{\mathrm{ws}}(T)$ is the complement of those $\lambda \in \mathbf{C}$ for which $\lambda I - T \in \Phi_-(X)$ and ind $(\lambda I - T) \geq 0$. The spectrum $\sigma_{\mathrm{wa}}(T)$ coincides with the intersection of all surjectivity spectra $\sigma_{\mathrm{s}}(T+K)$ of compact perturbations K of T, see [32] or [1, p.151]. Clearly, the last two spectra are dual each other, i.e.,

$$\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{ws}}(T^\star) \quad \text{and} \quad \sigma_{\mathrm{ws}}(T) = \sigma_{\mathrm{wa}}(T^\star).$$

Furthermore, $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{wa}}(T) \cup \sigma_{\mathrm{ws}}(T)$. Since $p(T) < \infty$ (respectively, $q(T) < \infty$) entails that $\operatorname{ind} T \leq 0$ (respectively, $\operatorname{ind} T \geq 0$), we have $\sigma_{\mathrm{wa}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{ws}}(T) \subseteq \sigma_{\mathrm{lb}}(T)$, and the precise relationship between these spectra is given by the following equalities:

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{wa}}(T) \cup \text{acc } \sigma_{\mathrm{a}}(T), \tag{1}$$

and

$$\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ws}}(T) \cup \text{acc } \sigma_{\mathrm{s}}(T), \tag{2}$$

see [33]. This article also deals with the single valued extension property. This property has a basic role in the local spectral theory, see the recent monograph of Laursen and Neumann [24] or Aiena [1]. In this article we shall consider a localized version of this property, recently studied by several authors [8], [5], [9], and previously by Finch [20], and Mbekhta [26].

Definition 1.1. *Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have* the single valued extension property *at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, and $f : U \to X$ satisfying the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ we have $f \equiv 0$ on U. An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbb{C}$.*

An operator $T \in L(X)$ has the SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$. The identity theorem for analytic function ensures that for every $T \in L(X)$, both T and T^* have the SVEP at the points of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, that both T and T^* have SVEP at every isolated point of $\sigma(T) = \sigma(T^*)$. The SVEP is inherited by the restrictions to closed invariant subspaces, i.e. if $T \in L(X)$ has the

SVEP at λ_0 and M is a closed T-invariant subspace then $T\,|M$ has SVEP at λ_0.

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Let $\sigma_{\rm a}(T)$ denote the classical *approximate point spectrum* of T, i. e. the set

$$\sigma_{\rm a}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

and let

$$\sigma_{\rm s}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}$$

denote the *surjectivity spectrum* of T. Note that

$$p(\lambda I - T) < \infty \Rightarrow T \text{ has SVEP at } \lambda, \tag{3}$$

and dually

$$q(\lambda I - T) < \infty \Rightarrow T^* \text{ has SVEP at } \lambda, \tag{4}$$

see [1, Theorem 3.8]. From the definition of SVEP we also have

$$\sigma_{\rm a}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda, \tag{5}$$

and dually

$$\sigma_{\rm s}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda. \tag{6}$$

The *quasi-nilpotent part* of T is defined by

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is easily seen that $H_0(T)$ is a linear T-invariant subspace and $\ker\,(T^m) \subseteq H_0(T)$ for every $m \in \mathbb{N}$. The subspace $H_0(T)$ may be not closed and coincides with the local spectral subspace $X_T(\{0\}$ in the case where T has SVEP, see Theorem 2.20 of [1]. The following implication holds for every operator $T \in L(X)$,

$$H_0(\lambda I - T) \text{ is closed} \Rightarrow T \text{ has SVEP at } \lambda, \tag{7}$$

see [5].

Remark 1.2. There is a fundamental relationship between SVEP and the Fredholm theory: all the implications (3)-(7) become equivalences if we assume that $\lambda I - T \in \Phi_\pm(X)$, see [5], [9].

2 Weyl's theorem under perturbations

If $T \in L(X)$ let us consider the complement in $\sigma(T)$ of the Browder spectrum:

$$p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \{\lambda \in \sigma(T) : \lambda I - T \text{ is Browder}\}.$$

Write iso K for the set of all isolated points of $K \subseteq \mathbb{C}$, and set

$$\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\},$$

the set of isolated eigenvalues of finite multiplicities. Obviously,

$$p_{00}(T) \subseteq \pi_{00}(T) \quad \text{for every } T \in L(X). \tag{8}$$

Following Coburn [15], we say that *Weyl's theorem holds* for $T \in L(X)$ if

$$\Delta(T) := \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T), \tag{9}$$

while T is said to satisfy *Browder's theorem* if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = p_{00}(T),$$

or equivalently, $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$. Note that

$$\text{Weyl's theorem} \Rightarrow \text{Browder's theorem},$$

see, for instance [1, p. 166]. Browder's theorem may be characterized by means of SVEP:

$$T \text{ satisfies Browder's theorem} \Leftrightarrow T \text{ has SVEP at all } \lambda \notin \sigma_{\mathrm{w}}(T).$$

Browder's theorem corresponds to the half of Weyl's theorem, in the following sense:

Theorem 2.1. *[2] If $T \in L(X)$ then the following assertions are equivalent:*

(i) *Weyl's theorem holds for T;*

(ii) *T satisfies Browder's theorem and $\pi_{00}(T) = p_{00}(T)$;*

The conditions $p_{00}(T) = \pi_{00}(T)$ is equivalent to several other conditions, see [1, Theorem 3.84]. The equality $p_{00}(T) = \pi_{00}(T)$ may be described in another way. To see this, let $\mathcal{P}_0(X)$, where X is a Banach space, denote the class of all operators $T \in L(X)$ such that there exists $p := p(\lambda) \in \mathbb{N}$ for which

$$H_0(\lambda I - T) = \ker (\lambda I - T)^p \quad \text{for all } \lambda \in \pi_{00}(T). \tag{10}$$

Then we have

Theorem 2.2. *[6] With the notation above, $T \in \mathcal{P}_0(X)$ if and only if $p_{00}(T) = \pi_{00}(T)$. In particular, if T has SVEP then Weyl's theorem holds for T if and only if $T \in \mathcal{P}_0(X)$.*

The following classes of operators has been introduced by Oudghiri [28], see also [11].

Definition 2.3. *A bounded operator $T \in L(X)$ is said to satisfy property $H(p)$ if*

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \textit{for all } \lambda \in \mathbb{C}. \tag{11}$$

for some $p = p(\lambda) \in \mathbb{N}$, .

From the implication in (7) it is clear that property $H(p)$ entails SVEP.

Many of the commonly considered operators on Banach spaces and Hilbert spaces have property $H(p)$, or have SVEP and belong to the class $\mathcal{P}_0(X)$. We list now some classes of operators for which property $H(p)$ holds. In the sequel by X we shall denote a Banach space, while a Hilbert space will be denoted by H.

(A) A bounded operator $T \in L(X)$ is said to be *paranormal* if

$$\|Tx\|^2 \leq \|T^2x\|\|x\| \quad \text{for all } x \in X.$$

The class of paranormal operators properly contains a relevant number of Hilbert space operators, among them p-hyponormal operators, log-hyponormal operators, quasi-hyponormal operators (see later for definitions). Every paranormal operator $T \in L(H)$ has SVEP. This may be easily seen as follows: if $\lambda \neq 0$ and $\lambda \neq \mu$ then, by Theorem 2.6 of [14], we have $\|x + y\| \geq \|y\|$ whenever $x \in \ker(\mu I - T)$ and $y \in \ker(\lambda I - T)$. It then follows that if U is an open disc and $f : U \to X$ is an analyitic function such that $0 \neq f(z) \in \ker(zI - T)$ for all $z \in U$, then f fails to be continuous at every $0 \neq \lambda \in U$.

(B) An operator $T \in L(X)$ is called *totally paranormal* if $\lambda I - T$ is paranormal for all $\lambda \in \mathbb{C}$. Every totally paranormal operator satisfies property $H(1)$, see [11]. In the sequel we shall denote by T' the Hilbert adjoint of $T \in L(H)$. An operator $T \in L(H)$ is said to be **-paranormal* if

$$\|T'x\|^2 \leq \|T^2x\|$$

holds for all unit vectors $x \in H$. $T \in L(H)$ is said to be *totally *-paranormal* if $\lambda I - T$ is *-paranormal for all $\lambda \in \mathbb{C}$. Every totally $*$-paranormal operator satisfies property $H(1)$, see [23]. The class of totally paranormal operators includes all hyponormal operators on Hilbert spaces H. Recall that the operator

$T \in L(H)$ is said to be *hyponormal* if

$$\|T'x\| \leq \|Tx\| \quad \text{for all } x \in X.$$

Hyponormal operators are p-hyponormal with $p = 1$, where $T \in L(H)$ is said to be *p-hyponormal*, with $0 < p \leq 1$, if

$$(T'T)^p \geq (TT')^p.$$

Every injective p-hyponormal operator satisfies property $H(1)$, see [11]. The concept of hyponormality may be relaxed as follows: $T \in L(H)$ is said to be *quasi-hyponormal* if

$$(T'T)^2 \leq {T'}^2 T^2$$

Also quasi-normal operators are totally paranormal, since these operators are hyponormal, see Conway [16] for details.

(C) An operator $T \in L(H)$ is said to be *log-hyponormal* if T is invertible and satisfies

$$\log\,(T'T) \geq \log\,(TT').$$

Every log-hyponormal operator satisfies the condition $H(1)$, see [11]. In fact, log-hyponormal operator are similar to hyponormal operators and property $H(1)$ is preserved by similarity [11, Theorem 2.4].

(D) Let A be a commutative Banach algebra. A linear map $T : A \to A$ is said to be a *multiplier* if $(Tx)y = x(Ty)$ holds for all $x, y \in \mathcal{A}$. If A is a commutative semi-simple Banach algebra, then every multiplier satisfies the condition $H(1)$ see [5]. Moreover, every multiplier of a commutative semi-simple Banach algebra has SVEP, [1, Chapter 4]. In particular, the condition $H(1)$ holds for every convolution operator on the group algebra $L^1(G)$, where G is a locally compact abelian group.

(E) An important class of operators which satisfy property $H(p)$ is given by the class of *subscalar* operators [28]. Recall that $T \in L(X)$ is said to be *generalized scalar* if there exists a continuous algebra homomorphism $\Psi : \mathcal{C}^\infty(\mathbb{C}) \to L(X)$ such that

$$\Psi(1) = I \quad \text{and} \quad \Psi(Z) = T,$$

where $\mathcal{C}^\infty(\mathbb{C})$ denote the Fréchet algebra of all infinitely differentiable complex-valued functions on $\mathbb{C}$, and Z denotes the identity function on $\mathbb{C}$. Every generalized scalar operator has SVEP, see [24]. An operator is subscalar whenever is similar to the restriction of a generalized scalar operator to one of its closed invariant subspaces. The interested reader can find a well organized study of these operators in the book by Laursen and Neumann [24]. Property $H(p)$ is then satisfied by p-hyponormal operators and log-hyponormal operators, M-hyponormal operators [24, Proposition 2.4.9], and totally paranormal operators [11], since all these operators are subscalar.

(F) An operator $T \in L(X)$ for which there exists a complex non-constant polynomial h such that $h(T)$ is paranormal is said to be *algebraically paranormal*. If $T \in L(H)$ is algebraically paranormal then $T \in \mathcal{P}_0(H)$, so it satisfies the condition (10), see [2], but in general the condition $H(p)$ is not satisfied by paranormal operators, (for an example see [6, Example 2.3]). Since every paranormal operator has SVEP then also every algebraically paranormal operator has SVEP, see Theorem 2.40 of [1] or [18], so that Weyl's theorem holds for every algebraically paranormal operator, see also [21].

Weyl's theorem is transmitted by some special perturbations. The following result has been first proved by Oberai [27].

Theorem 2.4. *Weyl's theorem is transmitted from $T \in L(X)$ to $T+N$ when N is a nilpotent operator commuting with T.*

The following example shows that Oberai's result does not hold if we do not assume that the nilpotent operator N commutes with T

Example 2.5. Let $X := \ell^2(\mathbb{N})$ and T and N be defined by

$$T(x_1, x_2, \dots,) := (0, \frac{x_1}{2}, \frac{x_2}{3}, \dots), \quad (x_n) \in \ell^2(\mathbb{N})$$

and

$$N(x_1, x_2, \dots,) := (0, -\frac{x_1}{2}, 0, 0, \dots), \quad (x_n) \in \ell^2(\mathbb{N})$$

Clearly, N is a nilpotent operator, and T is a quasi-nilpotent operator satisfying Weyl's theorem. On the other hand, it is easily seen that $0 \in \pi_{00}(T+N)$ and $0 \notin \sigma(T+N) \setminus \sigma_{\mathrm{w}}(T+N)$, so that $T+N$ does not satisfies Weyl's theorem.

Note that the operator N in Example 2.5 is also a finite rank operator not commuting with T. In general, Weyl's theorem is also not transmitted under commuting finite rank perturbation.

Example 2.6. Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be an injective quasi-nilpotent operator, and let $U : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be defined :

$$U(x_1, x_2, \dots,) := (-x_1, 0, 0, \dots), \quad \text{with } (x_n) \in \ell^2(\mathbb{N}).$$

Define on $X := \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})$ the operators T and K by

$$T := I \oplus S \quad \text{and} \quad K := U \oplus 0$$

Clearly, K is a finite rank operator and $KT = TK$. It is easy to check that

$$\sigma(T) = \sigma_{\rm w}(T) = \sigma_{\rm a}(T) = \{0, 1\}.$$

Now, both T and T^* have SVEP, since $\sigma(T) = \sigma(T^*)$ is finite. Moreover, $\pi_{00}(T) = \sigma(T) \setminus \sigma_{\rm w}(T) = \emptyset$, so T satisfies Weyl's theorem.
On the other hand,

$$\sigma(T + K) = \sigma_{\rm w}(T + K) = \{0, 1\},$$

and $\pi_{00}(T + K) = \{0\}$, so that Weyl's theorem does not hold for $T + K$.

Recall that $T \in L(X)$ is said to be a *Riesz operator* if $\lambda I - T \in \Phi(X)$ for all $\lambda \neq 0$. Evidently, quasi-nilpotent operators and compact operators are Riesz operators. A bounded operator $T \in L(X)$ is said to be *isoloid* if every isolated point of the spectrum is an eigenvalue. $T \in L(X)$ is said to be *finite-isoloid* if every isolated spectral point is an eigenvalue having finite multiplicity.

A result of W. Y. Lee and S. H. Lee [25] shows that Weyl's theorem for an isoloid operator is also preserved by perturbations of commuting finite rank operators. This result has been generalized by Oudghiri [29] as follows:

Theorem 2.7. *If $T \in L(X)$ is an isoloid operator which satisfies Weyl's theorem, if $ST = TS$, $S \in L(X)$, and there exists $n \in \mathbb{N}$ such that S^n is finite-dimensional, then $T + S$ satisfies Weyl's theorem.*

More recently, Y. M. Han and W. Y. Lee in [22] have shown that in the case of Hilbert spaces if T is a finite-isoloid operator which satisfies Weyl's theorem and if S a compact operator commuting with T then also $T + S$ satisfies Weyl's theorem. Again, Oudghiri [29] has shown that we have much more:

Theorem 2.8. *If $T \in L(X)$ is a finite-isoloid operator which satisfies Weyl's theorem and if K is a Riesz operator commuting with T then also $T + K$ satisfies Weyl's theorem.*

Recall that a bounded operator T is said to be *algebraic* if there exists a non-trivial polynomial h such that $h(T) = 0$. The following two results show that Weyl's theorem survives under commuting algebraic perturbations in some special cases.

Theorem 2.9. *[29] Suppose that $T \in L(X)$ has property $H(p)$, K algebraic and $TK = KT$. Then Weyl's theorem holds for $T + K$*

As observed in (G) the property $H(p)$ may fail for paranormal operators, in particular fails for quasi-hyponormal operators [6]. However, since quasi-hyponormal operators are paranormal we can apply next result to this case.

Theorem 2.10. *[6] Suppose that $T \in L(H)$ is paranormal, K algebraic and $TK = KT$. Then Weyl's theorem holds for $T + K$.*

It is well-known that if $S \in L(X)$, and there exists $n \in \mathbb{N}$ such that S^n is finite-dimensional then S algebraic. If T satisfies $H(p)$, or is paranormal, then T is isoloid (see [2]), so that the result of Theorem 2.7 in these special cases also follows from Theorem 2.9 and Theorem 2.10.

In the perturbation theory the "commutative" condition is rather rigid. On the other hand, it is known that without the commutativity, the spectrum can however undergo a large change under even rank one perturbations. However, we have the following result due to Y. M. Han and W. Y. Lee [22].

Theorem 2.11. *Suppose that $T \in L(H)$ is a finite-isoloid operator which satisfies Weyl's theorem. If $\sigma(T)$ has no holes (bounded components of the complement) and has at most finitely many isolated points then Weyl's theorem holds for $T + K$, where $K \in L(H)$ is either a compact or quasi-nilpotent operator commuting with T modulo the compact operators.*

The result below follows immediately from Theorem 2.11

Corollary 2.12. *Suppose that $T \in L(H)$ satisfies Weyl's theorem. If $\sigma(T)$ has no holes and has at most finitely many isolated points then Weyl's theorem holds for $T + K$ for every compact operator K.*

Corollary 2.12 applies to Toeplitz operators and not quasi-nilpotent unilateral weighted shifts, see [22].

3 a-Weyl's theorem and perturbations

For a bounded operator $T \in L(X)$ on a Banach space X let us denote

$$p^a_{00}(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = \{\lambda \in \sigma_{\mathrm{a}}(T) : \lambda I - T \in \mathcal{B}_+(X)\}.$$

We have,

$$p^a_{00}(T) \subseteq \pi^a_{00}(T) \quad \text{for every } T \in L(X).$$

In fact, if $\lambda \in p_{00}^a(T)$ then is $\lambda I - T \in \Phi_+(X)$ and $p(\lambda I - T) < \infty$. By Remark 1.2 then λ is isolated in $\sigma_{\rm a}(T)$. Furthermore, $0 < \alpha(\lambda I - T) < \infty$ since $(\lambda I - T)(X)$ is closed and $\lambda \in \sigma_{\rm a}(T)$.

A bounded operator $T \in L(X)$ is said to satisfy *a-Browder's theorem* if

$$\sigma_{\rm wa}(T) = \sigma_{\rm ub}(T).$$

Also a-Browder's theorem may be characterized by localized SVEP:

$$T \text{ satisfies } a\text{-Browder's theorem} \Leftrightarrow T \text{ has SVEP at all } \lambda \notin \sigma_{\rm wa}(T).$$

An approximate point variant of Weyl's theorem is a-Weyl's theorem: according to Rakočević [32] an operator $T \in L(X)$ is said to satisfy *a-Weyl's theorem* if

$$\Delta_a(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm wa}(T) = \pi_{00}^a(T).$$

It should be noted that the set $\Delta_a(T)$ may be empty. This is, for instance, the case of a right shift on $\ell^2(\mathbb{N})$, see [4]. Furthermore,

$$a\text{-Weyl's theorem holds for } T \Rightarrow \text{Weyl's theorem holds for } T,$$

while the converse does not hold, in general. Note that a-Weyl's theorem entails also a-Browder's theorem. Precisely, we have:

Theorem 3.1. *[2] For $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies a-Weyl's theorem;*

(ii) *T satisfies a-Browder's theorem and $p_{00}^a(T) = \pi_{00}^a(T)$.*

The following two results show that a-Weyl's theorem is satisfied by a considerable number of operators. In the sequel we shall denote by $\mathcal{H}(\sigma(T))$ the set of all analytic functions defined on a neighborhood of $\sigma(T)$. If $T \in L(X)$ then for every $f \in \mathcal{H}(\sigma(T))$ the operator $f(T)$ is defined by the classical functional calculus.

Theorem 3.2. *[2] If $T \in L(X)$ has property (H_p) then a-Weyl's holds for $f(T^*)$ for every $f \in \mathcal{H}(\sigma(T))$. Analogously, if T^* has property (H_p) then a-Weyl's holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Theorem 3.3. *[2] If $T' \in L(H)$ has property (H_p) or T' is paranormal, then a-Weyls theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

It is easy to find an example of an operator such that a-Weyl's theorem holds for T while there is a commuting finite rank operator K such that a-Weyl's theorem fails for $T + K$.

Example 3.4. Let Q be any injective quasi-nilpotent operator on a Banach space X. Define $T := Q \oplus I$ on $X \oplus X$. Clearly, T satisfies a-Weyl's theorem. Take $P \in L(X)$ any finite rank projection and set $K := 0 \oplus (-P)$. Then $TK = KT$ and $0 \in \pi_{00}^a(T+K) \cap \sigma_{\mathrm{wa}}(T * K)$. Clearly, $0 \notin \sigma_{\mathrm{a}}(T+K) \setminus \sigma_{\mathrm{wa}}(T+K)$, so that $\sigma_{\mathrm{a}}(T+K) \setminus \sigma_{\mathrm{wa}}(T+k) \neq \pi_{00}^a(T)$ and hence a-Weyl's theorem does not holds for $T+K$.

To see when a-Weyls theorem is transmitted under perturbation we need to introduce some definitions. A bounded operator $T \in L(X)$ is said to be *a-isoloid* if every isolated point of the approximate point spectrum $\sigma_{\mathrm{a}}(T)$ is an eigenvalue. $T \in L(X)$ is said to be *finite a-isoloid* if every isolated point $\sigma_{\mathrm{a}}(T)$ is an eigenvalue having finite multiplicity. The following two results are due to Oudghiri [30].

Theorem 3.5. *[30] If $T \in L(X)$ is an a-isoloid operator which satisfies a-Weyl's theorem, if $ST = TS$, $S \in L(X)$, and there exists $n \in \mathbb{N}$ such that S^n is finite-dimensional, then $T + S$ satisfies a-Weyl's theorem.*

Theorem 3.5 extends a result of D. S. Djordjević [19], see also Theorem 2.3 of [12], where a-Weyl's theorem was proved for $T + S$ when S is a finite rank operator commuting with T.

Theorem 3.6. *[30] If $T \in L(X)$ is a finite a-isoloid operator which satisfies a-Weyl's theorem and if K a Riesz operator commuting with T then also $T+K$ satisfies a-Weyl's theorem.*

In particular, Theorem 3.6 applies to compact perturbations $T + K$.

Theorem 3.7. *[12] Suppose that $T \in L(X)$ and Q is an injective quasi-nilpotent operator commuting with T. If T satisfies a-Weyl's theorem then also $T + Q$ satisfies a-Weyl's theorem.*

It is easily seen that quasi-nilpotent operators do not satisfy a-Weyl's theorem, in general. For instance, if

$$T(x_1, x_2, \dots,) := (0, \frac{x_2}{2}, \frac{x_3}{3}, \dots), \quad (x_n) \in \ell^2(\mathbb{N})$$

then T is quasi-nilpotent but a-Weyl's theorem fails for T.

Theorem 3.8. *[12] Suppose that Q is quasi-nilpotent operator satisfying a-Weyl's theorem. If K is a finite rank operator commuting with Q then $Q+K$ satisfies a-Weyl's theorem.*

4 Property (ω) and perturbations

Weyl's theorem admits another interesting variant, introduced by Rakočević in [31] and studied in the recent paper [10].

Definition 4.1. *A bounded operator $T \in L(X)$ satisfies property (w) if*

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \pi_{00}(T).$$

As observed in [10], we also have:

$$T \text{ satisfies property } (w) \Rightarrow \text{Weyl's theorem holds for } T, \qquad (12)$$

and examples of operators satisfying Weyl's theorem but not property (w) may be found in [10]. Property (w) is fulfilled by a relevant number of Hilbert space operators, see [10], and this property is equivalent to Weyl's theorem for T whenever T^* satisfies SVEP ([10, Theorem 2.16]). For instance, property (w) is satisfied by generalized scalar operator, or if the Hilbert adjoint T' has property $H(p)$ [10, Corollary 2.20]. Note that

$$T \text{ satisfies property } (w) \Rightarrow a\text{-Browder's theorem holds for } T,$$

and precisely we have:

Theorem 4.2. *[10] If $T \in L(X)$ then the following statements are equivalent:*

(i) *T satisfies property (w);*
(ii) *a-Browder's theorem holds for T and $p_{00}^a(T) = \pi_{00}(T)$.*

Note that property (w) is not intermediate between Weyl's theorem and a-Weyl's theorem, see [10] for examples. Property (w) is preserved by commuting finite-dimensional perturbations in the case that T is a-isoloid.

Theorem 4.3. *[3] Suppose that $T \in L(X)$ is a-isoloid and satisfies property (w), K is a finite rank operator that commutes with T. Then $T + K$ satisfies property (w).*

A similar result holds for nilpotent perturbations.

Theorem 4.4. *[3] Suppose that $T \in L(X)$ is a-isoloid. If T satisfies property (w) and N is nilpotent operator that commutes with T then $T + N$ satisfies property (w).*

Example 4.5. Both Theorem 4.4 and Theorem 4.3 fail if we assume that the nilpotent operator N, and the finite rank operator K do not commute with T. For instance, let T and N be as in Example 2.5. Then T satisfies Weyl's theorem and is decomposable, since T is a quasi-nilpotent operator. This implies that T satisfies property (w), by Corollary 2.10 of [10].

On the other hand, $T + N$ does not satisfy Weyl's theorem, and consequently, by the implication (12), $T + N$ does not satisfy property (w).

Example 4.6. The following example shows that Theorem 4.3 fails if we do not assume that T is a-isoloid. Let T and K be defined as in Example 2.6. Clearly, $\alpha(T) = 0$ so T is not a-isoloid. Now, T satisfies Weyl's theorem, and hence by Theorem 2.16 of [10] T satisfies property (w).

On the other hand, Weyl's theorem does not hold for $T + K$ and this implies (see (12)) that property (w) fails for $T + K$.

Example 4.7. In general, property (w) is not transmitted from T to a quasi-nilpotent perturbation $T + Q$. For instance, take $T = 0$, and $Q \in L(\ell^2(\mathbb{N}))$ defined by

$$Q(x_1, x_2, \dots) = (\frac{x_2}{2}, \frac{x_3}{3}, \dots) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N},$$

Then Q is quasi-nilpotent and $\{0\} = \pi_{00}(Q) \neq \sigma_{\mathrm{a}}(Q) \setminus \sigma_{\mathrm{aw}}(Q) = \emptyset$. Hence T satisfies property (w) but $T + Q = Q$ fails this property.

However, property (w) is preserved under injective quasi-nilpotent perturbations.

Theorem 4.8. *[3] Suppose that $T \in L(X)$ and Q is an injective quasi-nilpotent operator commuting with T. If T satisfies property (w) then also $T + Q$ satisfies property (w).*

Property (w) is preserved under finite rank perturbations or nilpotent perturbations commuting with T in the case that $T^* \in L(X)$ has property $H(p)$.

Theorem 4.9. *[3] Suppose that $T^* \in L(X)$ has property $H(p)$, K is either a finite rank operator or a nilpotent operator commuting with T. Then property (w) holds for $T + K$.*

An analogous result holds for paranormal operators acting on Hilbert spaces.

Theorem 4.10. *[3] Suppose that $T' \in L(H)$ is paranormal and K is either a finite rank operator or a nilpotent operator commuting with T. Then property (w) holds for $T + K$.*

Recall that every generalized scalar operator satisfies property (w) [10, Corollary 2.11].

Theorem 4.11. *[3] Suppose that $T \in L(X)$ is generalized scalar operator, K an algebraic operator commuting with T. Then property (w) holds for $T + K$.*

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