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Cauchy-Schwarz and Hölder's inequalities are equivalent

Las desigualdades de Cauchy-Schwarz y Hölder son equivalentes

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Abstract

We give an elementary proof that the integral Cauchy–Schwarz and Hölder's inequalities are equivalent.
Key words and phrases: inequalities, Cauchy-Schwarz inequality, Hölder's inequality.

Resumen

En este articulo se da una demostración elemental de que las desigualdades de Cauchy–Schwarz y Hölder son equivalentes.
Palabras y frases clave: desigualdades, desigualdad de Cauchy-Schwarz, desigualdad de Hölder.

Let α, β, x, y denote positive numbers with $\alpha + \beta = 1$. At the root of this note lies the possibility of obtaining the Young inequality $x^\alpha y^\beta \leq \alpha x + \beta y$ from the Cauchy-Jensen inequality

$$\sqrt{xy} \leq \frac{x+y}{2}. \qquad (C-J)$$

The proof follows from the inequality below, deduced from $(C-S)$ by means of the mathematical induction:

$$(x_1 x_2 \cdots x_{2^n})^{1/2^n} \leq \frac{1}{2^n}(x_1 + x_2 + \cdots + x_{2^n}), \qquad (C-J)_n$$

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where n is a positive integer and all x_i's are positive numbers. (Indeed, for every integer k with $1 \leq k \leq 2^n$, putting in $(C-J)_n$: $x_i = x$ for $i = 1, 2, \ldots, k$ and $x_i = y$ for $i = k+1, k+2, \ldots, 2^n$, we get $x^{A(k,n)}y^{B(k,n)} \leq A(k,n)x + B(k,n)y$ for $A(k,n) = k/2^n$ and $B(k,n) = 1 - A(k,n)$. Hence, approximating every *real* number $\alpha \in (0,1)$ by the numbers of the above form $A(k,n)$ we obtain the Young inequality).

Basing on the above observation we give an explicit proof that the classical Hölder inequality

$$\int_\Omega f(x)g(x)d\mu(x) \leq \Big(\int_\Omega f(x)^p d\mu(x)\Big)^{1/p}\Big(\int_\Omega g(x)^q d\mu(x)\Big)^{1/q}, \qquad (*)$$

with $1/p + 1/q = 1$, $p, q > 1$, and f, g nonnegative μ-measurable functions, may be derived from the case $p = q = 2$, that is, from the Cauchy-Schwarz integral inequality.

This implication (as a result) is known, but no general direct proof was presented as yet. In the classical monograph by Hardy, Littlewood and Polya [1] the authors prove the implication for *series*, while in the proof of $(*)$ for *integrals* they use the Young inequality (see [1], Theorems 11 and 188, respectively). (Hence, by the remarks made at the begining of this paper, the Hölder inequality follows formally from $(C-J)$.) The discrete case is also addressed in the recent book [5] by Steele, in his Exercise 9.3 (page 149) with a solution on page 262.

As far as the general case is concerned, in 1972 Infantozzi [2] remarked that the mentioned implication is included in a sequence of equivalent inequalities, and in 1979 Marshall and Olkin, proving a series of inequalities ([3], pp. 457-462), showed that the Cauchy-Schwarz inequality implies the Lyapunov inequality and this gives $(*)$. These two observations can be found in the monograph by Mitrinovič, Pečarič and Fink [4] on pages 200 and 191, respectively.

A comment which seems to be useful is that iteration of an L_2-bound will often suggest an interesting L_p-bound, but usually one does better to seek an independent proof of the L_p bound after it has been conjectured. The proof given below also shows that the approach to the Hölder inequality by the use of the Young inequality is more elementary and easier indeed, but it loses the beautiful link with its origin - the Cauchy-Schwarz inequality.

Our proof (that the Cauchy-Schwarz inequality implies $(*)$) will be established by steps.

STEP 1. *Inequality* $(*)$ *holds for* f, g *abitrary nonnegative and* $p = 2^n/k$, $q = 2^n/l$, *where* n, k, l *are arbitrary positive integers with* $1 \leq k < 2^n$ *and* $k + l = 2^n$.

The proof of case $p = q = 2$ is easy and very well known. By induction, we now have

$$\int_\Omega f_1(x)\cdot\ldots\cdot f_{2^n}(x)d\mu(x) \leq \big(\int_\Omega f_1(x)^{2^n}d\mu(x)\big)^{2^{-n}}\cdot\ldots\cdot\big(\int_\Omega f_{2^n}(x)^{2^n}d\mu(x)\big)^{2^{-n}},$$

for every positive integer n, and hence (putting $f_i = f^{1/k}$ for $i = 1, \ldots, k$, and $f_i = g^{1/(2^n-k)}$ for $i = k+1, \ldots, 2^k$) we obtain that $(*)$ holds for all numbers p, q with $p = 2^n/k$, $q = 2^n/l$, and $1 \leq k < 2^n$, $k + l = 2^n$, as claimed.

STEP 2. *Inequality* $(*)$ *holds for arbitrary nonnegative step-functions* f, g *and* $p, q > 1$ *arbitrary with* $1/p + 1/q = 1$.

For a μ-measurable subset A of Ω we let χ_A to denote the characteristic function of A. Now let the step-functions $f = \sum_{i=1}^m \alpha_i\chi_{A_i}$ and $g = \sum_{j=1}^r \beta_j\chi_{B_j}$ be fixed, where $\alpha_i, \beta_j > 0$ and A_i, B_j are μ-measurable subsets of Ω, $1 \leq i \leq m$, $1 \leq j \leq r$, and m, r are positive integers. Then we have

$$\big(\int_\Omega f(x)^p d\mu(x)\big)^{1/p} = \big(\sum_{i=1}^m \alpha_i^p\mu(A_i)\big)^{1/p}, \textit{ and}$$

$$\big(\int_\Omega g(x)^q d\mu(x)\big)^{1/g} = \big(\sum_{j=1}^r \beta_j^q\mu(B_j)\big)^{1/q}, \tag{1}$$

for p, q as in Step 1.

Since the integer part $[w]$ of a real number w fulfils the inequality $0 \leq w - [w] \leq 1$, for the numbers w_t, $t = 1, 2, \ldots$, defined by the formula $w_t = [w \cdot 2^t]/2^t$ we have $0 \leq w - w_t \leq 1/2^t$, whence $\lim_{t\to\infty} w_t = w$. It follows that if $p > 1$, then there is $t_0 \geq 1$ such that, for all $t \geq t_0$, the integers $k_t := [2^t/p]$ are positive, and hence for the sequence $p_t := 2^t/k_t$, $t \geq t_0$, we have

$$\lim_{t\to\infty} p_t = 1/(1/p) = p. \tag{2}$$

Similarly, for the numbers $q_t := p_t/(p_t - 1)$, $t \geq t_0$, we have $q_t = 2^t/(2^t - k_t)$ and

$$\lim_{t\to\infty} q_t = q, \tag{3}$$

with $1/q = 1 - 1/p$. We see that the numbers p_t, q_t are as in Step 1 and so, by (1), (2), (3) and the continuity of the functions

$$(1, \infty) \ni w \mapsto \big(\sum_{i=1}^m \alpha_i^w\mu(A_i)\big)^{1/w}$$

and

$$(1, \infty) \ni w \mapsto \big(\sum_{j=1}^{r} \beta_j^w \mu(B_j)\big)^{1/w},$$

we obtain

$$\int_\Omega f(x)g(x)d\mu(x) \leq \lim_{t\to\infty} \big(\int_\Omega f(x)^{p_t} d\mu(x)\big)^{1/p_t} \big(\int_\Omega g(x)^{q_t} d\mu(x)\big)^{1/q_t} =$$

$$\big(\int_\Omega f(x)^p d\mu(x)\big)^{1/p} \big(\int_\Omega g(x)^q d\mu(x)\big)^{1/q},$$

as claimed.

STEP 3. *Inequality* $(*)$ *is true in general.*

Now let f, g be two arbitrary nonnegative μ-measurable functions, and let $(f_s)_{s=1}^\infty, (g_s)_{s=1}^\infty$ be two nondecreasing sequences of step-functions converging pointwise to f and g, respectively. It follows that for arbitrary positive numbers p, q with $1/p + 1/q = 1$ the sequences $(f_s^p)_{s=1}^\infty$ and $(g_s^q)_{s=1}^\infty$ consist of nondecreasing step-functions and converge pointwise to f^p and g^q, respectively. Without loss of generality we may assume that the both integrals, $\int_\Omega f(x)^p d\mu(x)$ and $\int_\Omega g(x)^q d\mu(x)$, are finite. Therefore, by Step 2 and Lebesgue's Monotonic Convergence Theorem, we obtain

$$\big(\int_\Omega f(x)^p d\mu(x)\big)^{1/p} \big(\int_\Omega g(x)^q d\mu(x)\big)^{1/q} =$$

$$\lim_{s\to\infty} \big(\int_\Omega f_s(x)^p d\mu(x)\big)^{1/p} \big(\int_\Omega g_s(x)^q d\mu(x)\big)^{1/q} \geq$$

$$\int_\Omega f(x)g(x)d\mu(x).$$

Thus, we have shown that the Cauchy-Schwarz inequality implies $(*)$ indeed, and the proof of our main claim is complete.

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