# Completely N-continuous Multifunctions

*Multifunciones completamente N-continuas*

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**Abstract**

In this paper, the concept of completely N-continuous multifunction is defined and their basic characterizations and properties are obtained.
**Key words and phrases**: complete N-continuity, multifunction.

**Resumen**

En este artículo se define el concepto de multifunción completamente N-continua y se obtienen su caracterización y propiedades básicas.
**Palabras y frases clave:** N-continuidad completa, multifunción.

## 1 Introduction

In recent years, various classes of weak and strong forms of continuity have been major interest among General Topologists. The aim of this paper is to present and study the notion of completely N-continuous multifunctions.

A topological space $(X, \tau)$ is called nearly compact [7] if every cover of $X$ by regular open sets has a finite subcover.

Let $(X, \tau)$ be a topological space and let $A$ be a subset of $X$. If every cover of $A$ by regular open subsets of $(X, \tau)$ has a finite subfamily whose union covers $A$, then $A$ is called N-closed (relative to $X$) [4]. Sometimes, such sets are called N-sets or $\alpha$-nearly compact. The class of N-closed sets is important in the study of functions with strongly closed graphs [5].

Basic observation about N-closed sets involve the fact that every compact set is N-closed.

On the other hand, many authors studied N-closed sets and their topologies [2, 3, etc.] in the literature.

In this paper, a multifunction $F : X \to Y$ from a topological space $X$ to a topological space $Y$ is a point to set correspondence and is assumed that $F(x) \neq \emptyset$ for all $x$ where $\emptyset$ denotes the empty set.

If $A$ is a subset of a topological space, then $cl(A)$ denote the closure of $A$ and $int(A)$ denote the interior of $A$ and $co(A)$ denote the complement of $A$.

For a multifunction $F : X \to Y$ from a topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$, we denote the upper and lower inverse of a subset $V$ of $Y$ by $F^{+}(V)$ and $F^{-}(V)$, respectively; $F^{+}(V) = \{x \in X : F(x) \subseteq V\}$ and $F^{-}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$ [1].

A multifunction $F : X \to Y$ is said to be (i) upper semi continuous at a point $x \in X$ if for each open set $V$ in $Y$ with $F(x) \subseteq V$, there exists an open set $U$ containing $x$ such that $F(U) \subseteq V$; and (ii) lower semi continuous at a point $x \in X$ if for each open set $V$ in $Y$ with $F(x) \cap V \neq \emptyset$, there exists an open set $U$ containing $x$ such that $F(u) \cap V \neq \emptyset$ for every $u \in U$ [6].

For a multifunction $F : X \to Y$, the graph multifunction $G_F : X \to X \times Y$ is defined as $G_F(x) = \{x\} \times F(x)$ for every $x \in X$ [8].

## 2    Completely N-continuous multifunctions

**Definition 1.** *Let $F : X \to Y$ be a multifunction from a topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$. $F$ is said to be*

(1) *lower completely N-continuous if for each $x \in X$ and for each open set $V$ such that $x \in F^{-}(V)$, there exists an open set $U$ containing $x$ and having N-closed complement such that $U \subseteq F^{-}(V)$,*

(2) *upper completely N-continuous if for each $x \in X$ and for each open set $V$ such that $x \in F^{+}(V)$, there exists an open set $U$ containing $x$ and having N-closed complement such that $U \subseteq F^{+}(V)$.*

We know that a net $(x_\alpha)$ in a topological space $(X, \tau)$ is called eventually in the set $U \subseteq X$ if there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$.

**Definition 2.** *Let $(X, \tau)$ be a topological space and let $(x_\alpha)$ be a net in $X$. It is said that the net $(x_\alpha)$ n-converges to $x$ (written $x_\alpha \to_n x$) if for each open*

set $G \subseteq X$ having N-closed complement, there exists an index $\alpha_0 \in I$ such that $x_\alpha \in G$ for each $\alpha \geq \alpha_0$.

**Definition 3.** *Let $(X, \tau)$ be a topological space. A set $G$ in $X$ said to be n-open if for each $x \in G$, there exists an open set $H$ having N-closed complement such that $x \in H$ and $H \subseteq G$. The complement of a n-open set is called to as a n-closed set.*

The following theorem gives us some characterizations of lower (upper) completely N-continuous multifunction.

**Theorem 4.** *Let $F : X \to Y$ be a multifunction from a topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$. Then the following statements are equivalent:*

*i-) $F$ is lower (upper) completely N-continuous,*

*ii-) For each $x \in X$ and for each open set $V$ such that $F(x) \cap V \neq \emptyset$ $(F(x) \subseteq V)$, there exists an open set $U$ containing $x$ and having N-closed complement such that if $y \in U$, then $F(y) \cap V \neq \emptyset$ $(F(y) \subseteq V)$,*

*iii-) For each $x \in X$ and for each closed set $K$ such that $x \in F^-(coK)$ $(x \in F^+(coK))$, there exists a closed N-closed set $H$ such that $x \in co(H)$ and $F^+(K) \subseteq H$ $(F^-(K) \subseteq H)$,*

*iv-) $F^-(V)$ $(F^+(V))$ is a n-open set for any open set $V \subseteq Y$,*

*v-) $F^+(K)$ $(F^-(K))$ is a n-closed set for any closed set $K \subseteq Y$,*

*vi-) For each $x \in X$ and for each open set $V$ such that $F(x) \cap V \neq \emptyset$ $(F(x) \subseteq V)$, there exists a n-open set $U$ containing $x$ such that if $y \in U$, then $F(y) \cap V \neq \emptyset$ $(F(y) \subseteq V)$,*

*vii-) For each $x \in X$ and for each net $(x_\alpha)$ which n-converges to $x$ in $X$ and for each open set $V \subseteq Y$ such that $x \in F^-(V)$ $(x \in F^+(V))$, the net $(x_\alpha)$ is eventually in $F^-(V)$ $(F^+(V))$.*

*Proof.* (i)⇔(ii): Obvious.

(i)⇔(iii): Let $x \in F^-(coK)$ and let $K$ be a closed set. From (i), there exists an open set $U$ containing $x$ and having N-closed complement such that $U \subseteq F^-(coK)$. It follows that $co(F^-(coK)) = F^+(K) \subseteq co(U)$. We take $H = co(U)$. Then $x \in co(H)$ and $F^+(K) \subseteq H$.

The converse can be shown similarly.

(i)⇔(iv): Let $V \subseteq Y$ be an open set and let $x \in F^-(V)$. From (i), there exists an open set $U$ containing $x$ and having N-closed complement such that $U \subseteq F^-(V)$. It follows that $F^-(V)$ is a n-open set.

The converse can be obtain similarly from the definition of n-open set.

(iv)⇔(v): Let $K$ be a closed set. Then $co(K)$ is an open set. From (iv), $F^-(co(K)) = co(F^+(K))$ is a n-open set. It follows that $F^+(K)$ is a n-closed set.

The converse can be shown similarly.

(ii)⇔(vi): Let $x \in X$ and let $V$ be an open set such that $F(x) \cap V \neq \emptyset$. From (ii), there exists an open set $U$ containing $x$ and having N-closed complement such that if $y \in U$, then $F(y) \cap V \neq \emptyset$. Since $U$ is a n-open set, it follows that (vi) holds.

The converse is similar.

(i)⇒(vii): Let $(x_\alpha)$ be a net which n-converges to $x$ in $X$ and let $V \subseteq Y$ be any open set such that $x \in F^-(V)$. Since $F$ is lower completely N-continuous, it follows that there exists an open set $U \subseteq X$ containing $x$ and having N-closed complement such that $U \subseteq F^-(V)$. Since $(x_\alpha)$ n-converges to $x$, it follows that there exists an index $\alpha_0 \in J$ such that $x_\alpha \in U$ for all $\alpha \geq \alpha_0$. From here, we obtain that $x_\alpha \in U \subseteq F^-(V)$ for all $\alpha \geq \alpha_0$. Thus, the net $(x_\alpha)$ is eventually in $F^-(V)$.

(vii)⇒(i): Suppose that (i) is not true. There exist a point $x$ and an open set $V$ with $x \in F^-(V)$ such that $U \nsubseteq F^-(V)$ for each open set $U \subseteq X$ containing $x$ and having N-closed complement. Let $x_U \in U$ and $x_U \notin F^-(V)$ for each open set $U \subseteq X$ containing $x$ and having N-closed complement. Then for the neighbourhood net $(x_U)$, $x_U \to_n x$, but $(x_U)$ is not eventually in $F^-(V)$. This is a contradiction. Thus, $F$ is lower completely N-continuous.

The proof of the upper completely continuity of $F$ is similar to the above. □

**Remark 5.** *For a multifunction $F : X \to Y$ from a topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$, the following implication hold:*

*$F$ is lower (upper) completely N-continuous $\Rightarrow$ $F$ is lower (upper) semi continuous.*

However the converse is not true in general by the following example.

**Example 6.** *Take the discrete topology $\tau_D$ on $\mathbb{R}$. We define the multifunction as the follows; $F : (\mathbb{R}, \tau_D) \to (\mathbb{R}, \tau_D)$, $F(x) = \{x\}$ for each $x \in X$. Then $F$ is lower (upper) semi continuous but $F$ is not lower (upper) completely N-continuous.*

**Definition 7.** *Let $(X, \tau)$ be a topological space. $(X, \tau)$ is called n-regular space if $(X, \tau)$ has a base consisting of open sets having N-closed complements.*

The following theorem gives us the condition for the converse.

**Theorem 8.** *Let $F : X \to Y$ be a multifunction from a n-regular topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$. If $F$ is upper (lower) semi continuous multifunction, then $F$ is upper (lower) completely N-continuous.*

*Proof.* Since each open set is a n-open set in a n-regular space, the proof is obvious. □

**Definition 9.** *Let $(X, \tau)$ be a topological space and let $A \subseteq X$. A point $x \in X$ is said to be a n-adherent point of $A$ if each open set containing $x$ and having N-closed complement intersects $A$. The set of all n-adherent points of $A$ is denoted by n-$cl(A)$.*

It is obvious that the set $A$ is n-closed if and only if n-$cl(A) = A$.

**Theorem 10.** *Let $F : X \to Y$ be a multifunction from a topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$. $F$ is lower completely N-continuous if and only if $F(n\text{-}cl(A)) \subset cl(F(A))$ for each $A \subset X$.*

*Proof.* Suppose that $F$ is lower completely N-continuous and $A \subset X$. Since $cl(F(A))$ is a closed set, it follows that $F^+(cl(F(A)))$ is a n-closed set in $X$ from Theorem 4. Since $A \subset F^+(cl(F(A)))$, then n-$cl(A) \subset$ n-$cl(F^+(cl(F(A)))) = F^+(cl(F(A)))$. Thus, we obtain that $F(n\text{-}cl(A)) \subset F(F^+(cl(F(A)))) \subset cl(F(A))$.

Conversely, suppose that $F(n\text{-}cl(A)) \subset cl(F(A))$ for each $A \subset X$. Let $K$ be any closed set in $Y$. Then $F(n\text{-}cl(F^+(K))) \subset cl(F(F^+(K)))$ and $cl(F(F^+(K))) \subset cl(K) = K$. Hence, n-$cl(F^+(K)) \subset F^+(K)$ which shows that $F$ is lower completely N-continuous. □

**Theorem 11.** *Let $F : X \to Y$ be a multifunction from a topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$. $F$ is lower completely N-continuous if and only if n-$cl(F^+(B)) \subset F^+(cl(B))$ for each $B \subset Y$.*

*Proof.* Suppose that $F$ is lower completely N-continuous and $B \subset Y$. Then $F^+(cl(B))$ is n-closed in $X$ and $F^+(cl(B)) =$n-$cl(F^+(cl(B)))$. Hence, n-$cl(F^+(B)) \subset F^+(cl(B))$.

Conversely, let $K$ be any closed set in $Y$. Then n-$cl(F^+(K)) \subset F^+(cl(K)) = F^+(K) \subset$n-$cl(F^+(K))$. Thus, $F^+(K) =$n-$cl(F^+(K))$ which shows that $F$ is lower completely N-continuous. □

**Theorem 12.** *Let $F : X \to Y$ be a multifunction from a topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$ and let $F(X)$ be endowed with subspace topology. If $F$ is upper completely N-continuous, then $F : X \to F(X)$ is upper completely N-continuous.*

*Proof.* Since $F$ is upper completely N-continuous, $F^+(V \cap F(X)) = F^+(V) \cap F^+(F(X)) = F^+(V)$ is n-open for each open subset $V$ of $Y$. Hence $F : X \to F(X)$ is upper completely N-continuous. $\qquad\qquad\square$

Suppose that $(X, \tau)$, $(Y, \upsilon)$ and $(Z, \omega)$ are topological spaces. It is known that if $F_1 : X \to Y$ and $F_2 : Y \to Z$ are multifunctions, then the multifunction $F_2 \circ F_1 : X \to Z$ is defined by $(F_2 \circ F_1)(x) = F_2(F_1(x))$ for each $x \in X$.

**Theorem 13.** *Let $(X, \tau)$, $(Y, \upsilon)$, $(Z, \omega)$ be topological spaces and let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. If $F : X \to Y$ is upper (lower) completely N-continuous and $G : Y \to Z$ is upper (lower) semi continuous, then $G \circ F : X \to Z$ is an upper (lower) completely N-continuous multifunction.*

*Proof.* Let $V \subseteq Z$ be any open set. From the definition of $G \circ F$, we have $(G \circ F)^+(V) = F^+(G^+(V))$ $((G \circ F)^-(V) = F^-(G^-(V)))$. Since $G$ is upper (lower) semi continuous multifunction, it follows that $G^+(V)$ $(G^-(V))$ is an open set. Since $F$ is upper (lower) completely N-continuous multifunction, it follows that $F^+(G^+(V))$ $(F^-(G^-(V)))$ is a n-open set. It shows that $G \circ F$ is an upper (lower) completely N-continuous multifunction. $\qquad\square$

**Corollary 14.** *Let $(X, \tau)$, $(Y, \upsilon)$, $(Z, \omega)$ be topological spaces and let $F : X \to Y$ and $G : Y \to Z$ be multifunctions. If $F : X \to Y$ is upper (lower) completely N-continuous and $G : Y \to Z$ is upper (lower) completely N-continuous, then $G \circ F : X \to Z$ is an upper (lower) completely N-continuous multifunction.*

**Theorem 15.** *Let $F : X \to Y$ be a multifunction from a topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$. Then the graph multifunction of $F$ is upper completely N-continuous if and only if $F$ is upper completely N-continuous and $X$ is n-regular space.*

*Proof.* $(\Rightarrow)$: Suppose that $G_F$ is upper completely N-continuous. From Theorem 13, $F = P_Y \circ G_F$ is upper completely N-continuous where $P_Y$ is the projection $X \times Y$ onto $Y$.

Let $U$ be any open set in $X$ and let $U \times Y$ be an open set containing $G_F(x)$. Since $G_F$ upper completely N-continuous, there exists an open set $V$ containing $x$ and having N-closed complement such that if $x \in V$, then $G_F(x) \subset U \times Y$. Thus, $x \in V \subset U$ which shows that $U$ is a n-open set and $X$ is a n-regular space.

($\Leftarrow$): Let $x \in X$ and let $W$ be an open set containing $G_F(x)$. There exist open sets $U \subset X$ and $V \subset Y$ such that $(x, F(x)) \subset U \times V \subset W$. Since $X$ is n-regular space, there exists an open set $G_1 \subseteq X$ containing $x$ and having N-closed complement such that $x \in G_1 \subset U$. Since $F$ is upper completely N-continuous, there exists an open set $G_2$ in $X$ containing $x$ and having N-closed complement such that if $a \in G_2$, then $F(a) \subset V$. Let $H = G_1 \cap G_2$. Then $H$ is an open set containing $x$ and having N-closed complement and $G_F(H) \subset U \times V \subset W$ which implies that $G_F$ is upper completely N-continuous. $\square$

**Theorem 16.** *Suppose that $(X, \tau)$ and $(X_\alpha, \tau_\alpha)$ are topological spaces where $\alpha \in J$. Let $F : X \to \prod_{\alpha \in J} X_\alpha$ be a multifunction from $X$ to the product space $\prod_{\alpha \in J} X_\alpha$ and let $P_\alpha : \prod_{\alpha \in J} X_\alpha \to X_\alpha$ be the projection for each $\alpha \in J$. If $F$ is upper (lower) completely N-continuous, then $P_\alpha \circ F$ is (lower) upper completely N-continuous for each $\alpha \in J$.*

*Proof.* Take any $\alpha_0 \in J$. Let $V_{\alpha_0}$ be an open set in $(X_{\alpha_0}, \tau_{\alpha_0})$. Then $(P_{\alpha_0} \circ F)^+(V_{\alpha_0}) = F^+(P_{\alpha_0}^+(V_{\alpha_0})) = F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively, $(P_{\alpha_0} \circ F)^-(V_{\alpha_0}) = F^-(P_{\alpha_0}^-(V_{\alpha_0})) = F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$). Since $F$ is upper (lower) completely N-continuous multifunction and since $V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is an open set, it follows that $F^+(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$ (respectively, $F^-(V_{\alpha_0} \times \prod_{\alpha \neq \alpha_0} X_\alpha)$) is n-open in $(X, \tau)$. It shows that $P_{\alpha_0} \circ F$ is upper (lower) completely N-continuous.

Hence, we obtain that $P_\alpha \circ F$ is upper (lower) completely N-continuous for each $\alpha \in J$. $\square$

Let $(X, \tau)$ be a topological space. It is known that the collection of all open subsets having N-closed complements of $(X, \tau)$ is a base for a topology $\tau^*$ on $X$.

**Theorem 17.** *Let $F : X \to Y$ be a multifunction from a topological space $(X, \tau)$ to a topological space $(Y, \upsilon)$. Then $F : (X, \tau) \to (Y, \upsilon)$ is upper (lower) completely N-continuous if and only if $F : (X, \tau^*) \to (Y, \upsilon)$ is upper (lower) semi continuous.*

*Proof.* Let $V \subseteq Y$ be an open set and let $x \in F^+(V)$. Since $F : (X, \tau) \to (Y, \upsilon)$ is upper completely N-continuous, it follows that there exists an open set $U$ containing $x$ and having N-closed complement such that $U \subseteq F^+(V)$. From here, $U \in \tau^*$. Thus, $F : (X, \tau^*) \to (Y, \upsilon)$ is upper semi continuous.

Converse is similar.

The proof of the lower continuity of $F$ is similar to the above.     $\square$

**Theorem 18.** *Let $(X, \tau)$ be a topological space. Then the following statements are equivalent:*

*i-) $(X, \tau)$ is a n-regular space,*

*ii-) Each upper (lower) semi continuous multifunction from $(X, \tau)$ into a topological space $(Y, \upsilon)$ is upper (lower) completely N-continuous.*

*Proof.* (i)$\Rightarrow$(ii): Let $F : X \to Y$ be an upper semi continuous multifunction. Let $V \subseteq Y$ be an open set and let $x \in F^+(V)$. Then there exists an open set $U$ containing $x$ such that $U \subseteq F^+(V)$. Since $(X, \tau)$ is a n-regular space, it follows that there exists an open set $G$ containing $x$ and having N-closed complement such that $G \subseteq U \subseteq F^+(V)$. Thus, we obtain tahat $F$ is upper completely N-continuous.

(ii)$\Rightarrow$(i): Take $(Y, \upsilon) = (X, \tau)$. Then the identity multifunction $I_X$ on $X$ is upper semi continuous and hence $I_X$ is upper completely N-continuous. From Theorem 17, $I_X : (X, \tau^*) \to (X, \tau)$ is upper semi continuous. Take $U \in \tau$. Then $I_X^+(U) = U \in \tau^*$ and it follows that $\tau \subset \tau^*$. Therefore, $\tau = \tau^*$ and we obtain that $(X, \tau)$ is a n-regular space.

The proof interested in the lower continuity of $F$ is similar to the above.     $\square$

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