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INCREASING REGULARLY VARYING SOLUTIONS OF THE EQUATIONS OF THOMAS-FERMI TYPE

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A b s t r a c t. Thomas-Fermi type differential equation $x'' = q(t)\phi(x)$, where $q(t)$ and $\phi(x)$ are regularly varying functions, is studied in the framework of regular variation. The aim of this paper is to establish necessary and sufficient conditions for the existence of increasing regularly varying solutions of (A) as well as to acquire the precise information about the asymptotic behavior at infinity of these solutions.

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1. *The locally conformal Kähler manifolds*

The present paper is devoted to the existence and the asymptotic analysis of increasing positive solutions of nonlinear ordinary differential equations of the type

$$x'' = \alpha\, q(t)\phi(x), \qquad \alpha = \pm 1 \tag{A}$$

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in the framework of regular variation. The following assumptions are always required for (A):

(a) $q : [a,\infty) \to (0,\infty)$, $a > 0$, is a continuous function which is regularly varying of index σ;

(b) $\phi : (0,\infty) \to (0,\infty)$ is a continuous function which is regularly varying of index γ.

We will separately consider cases:

(b-1) $\gamma > 1$ while we assume that $\phi(x)/x$ is eventually increasing;

(b-2) $0 < \gamma < 1$ while we assume that $\phi(x)/x$ is eventually decreasing.

More specifically, $q(t)$ and $\phi(x)$ are expressed as

$$q(t) = t^{\sigma} l(t), \quad l(t) \in \mathrm{SV}, \qquad \phi(x) = x^{\gamma} L(x), \quad L(x) \in \mathrm{SV}. \tag{1.1}$$

Here SV denotes the set of slowly varying functions introduced in 1930 by J. Karamata by the following:

Definition 1.1. *A positive measurable function L defined on $[a,\infty)$ for some $a > 0$ is called slowly varying at infinity if for each $\lambda > 0$*

$$\lim_{t\to\infty} \frac{L(\lambda t)}{L(t)} = 1.$$

Comprehensive treatises on regular variation are given in N. H. Bingham et al. [3] and by E. Seneta [11].

Equation (A) is said to be *superlinear* if $\gamma > 1$ and *sublinear* if $0 < \gamma < 1$. It is also said to be of *Thomas-Fermi type* if $\alpha = 1$ and of *Emden-Fowler type* if $\alpha = -1$.

Our purpose here is to show that effective use of theory of regular variation (in the sense of Karamata) makes it possible to provide accurate information about the existence and asymptotic behavior of increasing regularly varying solutions of equation (A) in the Thomas-Fermi case, i.e. when $\alpha = 1$. One can verify that all possible increasing regularly varying solutions of (A) have one and the same index ρ determined by σ and γ and the asymptotic behavior at infinity of any such solution is governed by a unique formula depending only on $q(t)$, $\phi(x)$ and ρ. Thus, the set of all increasing regularly varying solutions of (A), if non-empty, is shown to have a surprisingly clear and definite structure.

The study of nonlinear differential equations in the framework of regular variation was initiated by Avakumović [2] and followed by Marić and Tomić [8, 9, 10]. See also Marić [7, Chapter 3]. These papers and some closely related [12, 13] are concerned exclusively with *decreasing* positive solutions of Thomas-Fermi type equations. No analysis from the viewpoint of regular variation seems to have been made of *increasing* positive solutions of such equations. The reason is that the original Thomas-Fermi singular boundary problem reads

$$x''(t) = t^{-1/2} x(t)^{3/2}, \quad x(0) = 1,\ x(\infty) = 0\,.$$

Our main results are presented in Sections 3 and 4. In Section 2 we depict an explicit picture of the structure of increasing regularly varying solutions of equation (A) based on an existence theorem of regularly varying solutions for linear differential equations. The construction of regularly varying solutions of (A) with the desired asymptotic behavior is carried out in Sections 3 and 4 concerned, respectively, with the regularity indices $\rho > 1$ and $\rho = 1$. The main tool employed in both sections is the Schauder-Tychonoff fixed point theorem in locally convex spaces.

In 2007 V.M. Evtukhov and V.M. Kharkov in a remarkable paper [4] studied simultaneously both Thomas-Fermi and Emden-Fowler type of equation (A) and gave sharp conditions for the existence of solutions (which may decrease and increase) belonging to a certain class and possessing certain asymptotic behavior. The condition imposed in [4] on function $q(t)$ means, due to Karamata theorem [3, Theorem 1.6.1], that it is of regular variation. The condition imposed in [4] on function $\phi(x)$ means, due to Lemma 3.2 and 3.3 in [7] that it is either regularly or rapidly varying. These facts are neither used (nor mentioned) by Evtukhov and Kharkov which makes their method of proof different from ours and the statements on solutions somewhat weaker than ours (of course, for the Thomas-Fermi case which we consider here).

2. *Structure of regularly varying solutions*

It is easy to see that if $0 < \gamma < 1$, all positive solutions of (A) can be extended to $t = \infty$, whereas if $\gamma > 1$, (A) always has a *singular* positive solution which cannot be extended to $t = \infty$, that is, blows up at a finite point. If $\gamma > 1$ it may happen that all positive solutions are singular. This is the case if, for example, $\phi(x)/x^\gamma$ is nondecreasing and the function $q(t)$, which need not be regularly varying, satisfies

$$\liminf_{t\to\infty} t^{\gamma+1} q(t) > 0.$$

See e.g. I. T. Kiguradze and T. A. Chanturiya [6].

Let us restrict our attention to increasing regularly varying solutions of equation (A) with regularly varying $q(t)$. We introduce the following notation:

$\mathcal{R}_+$ denotes the totality of increasing regularly varying solutions of (A),

$\mathcal{R}(\rho)$ denotes the set of regularly varying solutions of index ρ of (A).

It is clear that for any positive increasing solution $x(t)$ of (A) existing on $[t_0,\infty)$, $x'(t)$ is positive and increasing, so it tends either to ∞ or to some positive constant as $t\to\infty$. In both cases, $x'(t)\geq k$ for some positive constant k and for $t\geq t_1\geq t_0$. Accordingly, by integration we get $x(t)\geq x(t_1)+k(t-t_1)$ which implies that $x(t)\to\infty$ as $t\to\infty$. Thus, all possible positive increasing solutions of (A) fall into the following two types:

$$\lim_{t\to\infty} x(t)=\infty, \quad \lim_{t\to\infty}\frac{x(t)}{t}=\text{const}>0, \tag{2.1}$$

$$\lim_{t\to\infty} x(t)=\infty, \quad \lim_{t\to\infty}\frac{x(t)}{t}=\infty\,. \tag{2.2}$$

It follows that any positive proper solution $x(t)$ of (A) satisfies

$$x(t)\geq k\,t \quad \text{for} \;\; t\geq t_0\,,$$

for some positive constant k and accordingly

$$\mathcal{R}_+=\bigcup_{\varrho\geq\infty}\mathcal{R}(\varrho)\,.$$

For a sublinear equation (A) the set $\mathcal{R}_+$ is always non-empty, while in the superlinear case it may happen that $\mathcal{R}_+=\emptyset$. Actually the structure of $\mathcal{R}_+$ is particularly simple as is shown in the following theorem.

Theorem 2.1*Suppose that $q(t)$ is a regularly varying function. Then, the structure of $\mathcal{R}_+$, is as follows:*

$$0<\gamma<1 \quad\Longrightarrow\quad \mathcal{R}_+=\mathcal{R}(\rho_+) \quad \textit{for a single} \;\; \rho_+\in[1,\infty). \tag{2.3}$$

$$\gamma > 1 \quad \Longrightarrow \quad \mathcal{R}_+ = \mathcal{R}(1) \cup \mathcal{R}(\rho_+) \quad \textit{for a single} \;\; \rho_+ \in (1, \infty). \tag{2.4}$$

P r o o f. Let $x(t)$ be an RV(λ)-solution of (A). Then, it is regarded as a solution of the linear differential equaion

$$x''(t) = q_x(t)x(t), \quad q_x(t) = q(t)\frac{\phi(x(t))}{x(t)},$$

and so applying a result of Howard and Marić [5] (cf. Marić [7, Theorem 1.11]) for linear equations, we have

$$\lim_{t\to\infty} t \int_t^\infty q(s)\frac{\phi(x(s))}{x(s)} ds = \lambda(\lambda - 1). \tag{2.5}$$

Now suppose that (A) has two solutions $x(t) \in \mathrm{RV}(\lambda)$ and $y(t) \in \mathrm{RV}(\mu)$, where λ and μ are positive constants such that $\lambda > \mu > 1$. Letting $x(t) = t^\lambda \xi(t)$, $y(t) = t^\mu \eta(t)$, $\xi(t), \eta(t) \in \mathrm{SV}$, we obtain in view of (2.5)

$$\lim_{t\to\infty} \frac{t\int_t^\infty q(s)\Big[\phi(s^\lambda \xi(s))/s^\lambda \xi(s)\Big] ds}{t\int_t^\infty q(s)\Big[\phi(s^\mu \eta(s))/s^\mu \eta(s)\Big] ds} = \frac{\lambda(\lambda-1)}{\mu(\mu-1)}. \tag{2.6}$$

Using l'Hospitals's rule, we see that the left-hand side of (2.6) equals

$$\lim_{t\to\infty} t^{(\lambda-\mu)(\gamma-1)} \left(\frac{\xi(t)}{\eta(t)}\right)^{\gamma-1} \frac{L(t^\lambda \xi(t))}{L(t^\mu \eta(t))} = \begin{cases} 0 & \text{if } \; 0 < \gamma < 1 \\ \infty & \text{if } \; \gamma > 1 \end{cases} \tag{2.7}$$

where use is made of the fact that $L(t^\lambda \xi(t))/L(t^\mu \eta(t))$ is slowly varying (cf. Bingham et al. [3, Proposition 1.5.7]).

It is clear that (2.6) and (2.7) are incompatible, which means that no two different sets $\mathcal{R}(\lambda)$, $\mathcal{R}(\mu)$, $\lambda > \mu > 1$, can be contained in $\mathcal{R}_+$. Note that if $0 < \gamma < 1$ the same is true of the border case $\mu = 1$, which means that $\mathcal{R}_+ = \mathcal{R}(\rho_+)$ for some $\rho_+ \geq 1$. However, if $\gamma > 1$, (2.6) and (2.7) are compatible for $\mu = 1$, which implies that $\mathcal{R}_+$ may consists of two elements $\mathcal{R}(1)$ and $\mathcal{R}(\rho_+)$ for some $\rho_+ > 1$. This completes the proof. ⊠

3. *Increasing regularly varying solutions of index $\rho > 1$*

Theorem 3.1. *Equation* (A) *possesses a regularly varying solution of index $\rho > 1$ if and only if*

$$\gamma > 1 \;\; \textit{and} \;\; \sigma < -\gamma - 1, \qquad \textit{or} \qquad 0 < \gamma < 1 \;\; \textit{and} \;\; \sigma > -\gamma - 1, \tag{3.1}$$

in which case the regularity index of any such solution $x(t)$ is a unique constant given by

$$\rho = \frac{\sigma + 2}{1 - \gamma}, \tag{3.2}$$

and the asymptotic behavior of $x(t)$ is governed by one and the same formula

$$\frac{\phi(x(t))}{x(t)} \sim \rho(\rho - 1)\Big(t^2 q(t)\Big)^{-1}, \quad t \to \infty, \tag{3.3}$$

or

$$x(t) \sim \psi\Big[\rho(\rho - 1)\Big(t^2 q(t)\Big)^{-1}\Big], \quad t \to \infty, \tag{3.4}$$

where ψ denotes the inverse function of $\phi(x)/x$ defined for all sufficiently large $x > 0$.

P r o o f. (The "only if" part): Suppose that (A) has a solution $x(t) \in \mathrm{RV}(\rho)$, $\rho > 1$, on $[t_0, \infty)$. Let $x(t) = t^\rho \xi(t)$, $\xi(t) \in \mathrm{SV}$. Integrating (A) on $[t_0, t]$, we have due to (1.1)

$$x'(t) \sim \int_{t_0}^{t} q(s)\phi(x(s))ds = \int_{t_0}^{t} s^{\sigma+\rho\gamma} l(s)\xi(s)^\gamma L(s^\rho \xi(s))ds, \quad t \to \infty. \tag{3.5}$$

By (2.1) and (2.2), either $x'(t) \to c > 0$ or $x'(t) \to \infty$ as $t \to \infty$. However, the former case would give $x(t) \sim ct$, $t \to \infty$ contradicting $\varrho > 1$. Therefore, $x'(t) \to \infty$ as $t \to \infty$ and using that $l(t)\xi(t)^\gamma L(t^\rho \xi(t)) \in \mathrm{SV}$, (3.5) implies that $\sigma + \rho\gamma \geq -1$. But the possibility $\sigma + \rho\gamma = -1$ should be excluded. In fact, if this would hold, then we would see from (3.5) that

$$x'(t) \sim \int_{t_0}^{t} s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))ds \in \mathrm{SV}, \quad t \to \infty$$

and hence that

$$x(t) \sim t \int_{t_0}^{t} s^{-1} l(s)\xi(s)^\gamma L(s^\rho \xi(s))ds \in \mathrm{RV}(1), \quad t \to \infty$$

which is impossible. Thus we must have $\sigma + \rho\gamma > -1$, in which case Karamata's integration theorem [3, Theorem 1.5.11] applied to the last integral in (3.5) shows that

$$x'(t) \sim \frac{t^{\sigma+\rho\gamma+1} l(t)\xi(t)^\gamma L(t^\rho \xi(t))}{\sigma + \rho\gamma + 1}, \quad t \to \infty.$$

Integrating the above from t_0 to t and using Karamata's theorem again, we obtain

$$x(t) \sim \frac{t^{\sigma+\rho\gamma+2} l(t) \xi(t)^{\gamma} L(t^{\rho}\xi(t))}{(\sigma+\rho\gamma+1)(\sigma+\rho\gamma+2)}, \qquad t \to \infty$$

or

$$x(t) \sim \frac{t^{\sigma+2} l(t) x(t)^{\gamma} L(x(t))}{(\sigma+\rho\gamma+1)(\sigma+\rho\gamma+2)} = \frac{t^2 q(t) \phi(x(t))}{(\sigma+\rho\gamma+1)(\sigma+\rho\gamma+2)}, \qquad t \to \infty \tag{3.6}$$

Since $(\sigma+\rho\gamma+1)(\sigma+\rho\gamma+2) = \rho(\rho-1)$, (3.6) is equivalent to (3.3), leading to (3.4). Using the fact that the inverse ψ of the function $\phi(x)/x$ is regularly varying of index $1/(\gamma-1)$ (cf. Bigham et al. [3, Theorem 1.5.12]), we see from (3.4) that $x(t)$ is a regularly varying function of index $\rho = (\sigma+2)/(1-\gamma)$, which, combined with the assumption $\rho > 1$, implies that $\sigma < -\gamma - 1$ if $\gamma > 1$ and $\sigma > -\gamma - 1$ if $0 < \gamma < 1$.

(The "if" part): **(A)** We begin by considering the case where $\gamma > 1$. Suppose that $\sigma < -\gamma - 1$. Let $\rho > 1$ be the constant given by (3.2). Our task is to prove the existence of a positive solution $x(t) \in \mathrm{RV}(\rho)$ of (A) such that $x(t) \sim X_0(t)$, $t \to \infty$, where one defines $X_0(t) \in \mathrm{RV}(\rho)$ by

$$X_0(t) = \psi\Big[\rho(\rho-1)\Big(t^2 q(t)\Big)^{-1}\Big], \tag{3.7}$$

or

$$\frac{\phi(X_0(t))}{X_0(t)} = \rho(\rho-1)\Big(t^2 q(t)\Big)^{-1}. \tag{3.8}$$

For this purpose we proceed as follows. Clearly $X_0(t)$ exists on $[b, \infty)$ for some sufiiciently large $b > 0$ and so we first construct a solution $x_0(t)$ of (A) on some interval $[t_0, \infty)$, $t_0 > b$, such that

$$x_0(t) \sim ct, \quad t \to \infty, \tag{3.9}$$

for any constant $c > 0$. Then by means of the substitution $x(t) = x_0(t)y(t)$ we transform (A) into

$$\Big(x_0(t)^2 y'(t)\Big)' = x_0(t) q(t)\Big[\phi\Big(x_0(t)y(t)\Big) - \phi(x_0(t))y(t)\Big], \tag{3.10}$$

and find a positive solution $y_0(t)$ of equation (3.10) existing on some interval $[T, \infty)$, $T > t_0$, and satisfying

$$y_0(t) \sim c^{-1} Y_0(t) := c^{-1} t^{-1} X_0(t) \in \mathrm{RV}(\rho - 1), \quad t \to \infty. \tag{3.11}$$

Finally we form the product $x(t) = x_0(t)y_0(t)$, which gives an RV(ρ)-solution of (A) on $[T, \infty)$ with the desired asymptotic property $x(t) \sim X_0(t)$, $t \to \infty$.

(Step 1): We first note that

$$\int_a^\infty q(t)\phi(t)dt = \int_a^\infty t^{\sigma+\gamma} l(t)L(t)dt < \infty$$

because $\sigma + \gamma < -1$ and $l(t)L(t) \in \mathrm{SV}$. Since $\phi(t) \in \mathrm{RV}(\gamma)$ satisfies

$$\lim_{t\to\infty} \frac{\phi(ct)}{\phi(t)} = c^\gamma \quad \text{for all } \ c > 0, \tag{3.12}$$

and the convergence is uniform with respect to $c \in (0, 1]$ (cf. Bingham et al. [3, Theorem 1.5.2]), we see that there exists $t_0 > b$ such that

$$\phi(ct) \le 2c^\gamma \phi(t), \quad \text{for } \ t \ge t_0 \ \text{ and all } c \in (0, 1], . \tag{3.13}$$

Let $c \in (0, 1)$ be such that

$$\int_{t_0}^\infty q(t)\phi(t)\, dt \le \frac{c^{1-\gamma}}{4}. \tag{3.14}$$

From (3.13) and (3.14) we find that

$$\int_{t_0}^\infty q(t)\phi(ct)\, dt \le \frac{c}{2}.$$

Let us now define the integral operator

$$\mathcal{F}x(t) = ct - \int_{t_0}^t \int_s^\infty q(r)\phi(x(r))drds, \quad t \ge t_0,$$

and the set

$$\mathcal{X} = \Big\{ x(t) \in C[t_0, \infty) : \frac{1}{2}ct \le x(t) \le ct, \quad t \ge t_0 \Big\}.$$

It is a matter of routine computation to show that $\mathcal{F}$ is a continuous self-map on the closed convex set $\mathcal{X}$ and sends it into a relatively compact subset of $C[t_0, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem $\mathcal{F}$ has a fixed point $x_0(t) \in \mathcal{X}$, which gives birth to a solution of equation (A) such that $x_0(t) \sim ct$ as $t \to \infty$.

(Step 2): We choose a constant $c \in (0, 1)$ so that it satisfies in addition to (3.14) the following inequality

$$2^{\gamma+1} c^{\gamma + \frac{1}{\gamma} - 2} \le 1, \tag{3.15}$$

which is possible because $\gamma > 1$. Let $x_0(t)$ be the solution of equation (A) constructed in Step 1. Using it we want to obtain a solution $y_0(t)$ of equation (3.10) satisfying (3.11) as a solution of the integral equation

$$y(t) = 1+\int_T^t x_0(s)^{-2} \int_T^s x_0(r)q(r)\Big[\phi(x_0(r)y(r))-\phi(x_0(r))y(r)\Big]drds, \quad t \geq T, \tag{3.16}$$

for some $T > t_0$. To this end a crucial role will be played by the fact that

$$\int_{t_0}^t s^{-2} \int_{t_0}^s rq(r)\phi(rY_0(r))drds \sim Y_0(t) \quad \text{as} \quad t \to \infty, \tag{3.17}$$

which is a consequence of the following computation. Letting $X_0(t) = t^\rho \xi(t)$ and using (3.8) and (3.11) one finds:

$$\begin{aligned}
\int_{t_0}^t s^{-2} \int_{t_0}^s rq(r)\phi(rY_0(r))drds &= \int_{t_0}^t s^{-2} \int_{t_0}^s rq(r)\phi(X_0(r))drds \\
&= \int_{t_0}^t s^{-2} \int_{t_0}^s rq(r)\frac{\phi(X_0(r))}{X_0(r)}\, X_0(r)drds \\
&= \int_{t_0}^t s^{-2} \int_{t_0}^s rq(r) \left[\rho(\rho-1)\Big(r^2 q(r)\Big)^{-1}\right] X_0(r)drds \\
&= \rho(\rho-1) \int_{t_0}^t s^{-2} \int_{t_0}^s r^{\rho-1}\xi(r)drds \\
&\sim (\rho-1) \int_{t_0}^t s^{\rho-2}\xi(s)ds \sim t^{\rho-1}\xi(t) = t^{-1}X_0(t) = Y_0(t), \quad t \to \infty.
\end{aligned}$$

Consider the integral

$$\int_{t_0}^t x_0(s)^{-2} \int_{t_0}^s x_0(r)q(r)\phi(kx_0(r)Y_0(r))drds,$$

where $k > 0$ is a constant. Using (3.9) and (3.12) we have that

$$\phi(kx_0(t)Y_0(t)) \sim \phi(kctY_0(t)) \sim (kc)^\gamma \phi(t\,Y_0(t)), \qquad t \to \infty\,. \tag{3.18}$$

By combining (3.17) and (3.18) we see that

$$\int_{t_0}^t x_0(s)^{-2} \int_{t_0}^s x_0(r)q(r)\phi(kx_0(r)Y_0(r))drds \sim c^{\gamma-1}k^\gamma Y_0(t), \quad t \to \infty,$$

from which it follows that there exists $T > t_0$ depending on k such that

$$\int_T^t x_0(s)^{-2} \int_T^s x_0(r)q(r)\phi(kx_0(r)Y_0(r))drds \leq 2c^{\gamma-1}k^\gamma Y_0(t)\,, \quad t \geq T\,. \tag{3.19}$$

In what follows we make a special choice of $k = 2c^{-\frac{1}{\gamma}}$ and require additionally that T is such that

$$Y_0(t) \geq c^{\frac{1}{\gamma}} \quad \text{for} \;\; t \geq T. \tag{3.20}$$

Define the integral operator

$$\mathcal{G}y(t) = 1 + \int_T^t x_0(s)^{-2} \int_T^s x_0(r)q(r)\Big[\phi(x_0(r)y(r)) - \phi(x_0(r))y(r)\Big]drds, \tag{3.21}$$

for $t \geq T$ and let it act on the set $\mathcal{Y}$ consisting of continuous functions $y(t)$ on $[T, \infty)$ satisfying

$$1 \leq y(t) \leq 2c^{-\frac{1}{\gamma}} Y_0(t), \quad t \geq T,$$

and

$$y(t) \sim c^{-1} Y_0(t), \quad t \to \infty\,. \tag{3.22}$$

It is clear that $\mathcal{Y}$ is a closed convex subset of the locally convex space $C[T, \infty)$. It can be verified that $\mathcal{G}$ is a continuous self-map on $\mathcal{Y}$ with the porperty that $\mathcal{G}(\mathcal{Y})$ is relatively compact in $C[T, \infty)$.

(i) $\mathcal{G}(\mathcal{Y}) \subset \mathcal{Y}$. Let $y(t) \in \mathcal{Y}$. Then, since $y(t) \geq 1$, $t \geq T$, increasing nature of $\phi(x)/x$ implies

$$\phi(x_0(t)y(t)) \geq \phi(x_0(t))y(t) \quad \text{for} \;\; t \geq T\,, \tag{3.23}$$

and so we have $\mathcal{G}y(t) \geq 1$, $t \geq T$. Moreover, using (3.15), (3.19) and (3.20), we obtain

$$\begin{aligned} \mathcal{G}y(t) \;\; &\leq \;\; 1 + \int_T^t x_0(s)^{-2} \int_T^s x_0(r)q(r)\phi\Big(2c^{-\frac{1}{\gamma}} x_0(r) Y_0(r)\Big)drds \\ &\leq \;\; 1 + 2c^{\gamma-1}\Big(2c^{-\frac{1}{\gamma}}\Big)^{\gamma} Y_0(t) \leq c^{-\frac{1}{\gamma}} Y_0(t) + c^{-\frac{1}{\gamma}} Y_0(t) = 2c^{-\frac{1}{\gamma}} Y_0(t), \quad t \geq T\,. \end{aligned}$$

As for the asymptotic behavior of $\mathcal{G}y(t)$, using (1.1), (3.8) and (3.22), we have

$$\begin{aligned} \frac{\phi(t)Y_0(t)}{\phi(tY_0(t))} = \frac{\phi(t)}{t} \cdot \frac{X_0(t)}{\phi(X_0(t))} = \\ = t^{\gamma-1} L(t)\, (\varrho(\varrho-1))^{-1} t^2 q(t) = (\varrho(\varrho-1))^{-1} t^{\sigma+\gamma+1} L(t)\, l(t)\,. \end{aligned}$$

Since $\sigma + \gamma + 1 < 0$ and $M(t) = (\varrho(\varrho-1))^{-1} L(t)\, l(t) \in \mathrm{SV}$, it follows that $t^{\sigma+\gamma+1} M(t) \to 0$ as $t \to \infty$, so that

$$\frac{\phi(t)Y_0(t)}{\phi(tY_0(t))} \to 0 \quad \text{as} \quad t \to \infty\,.$$

Therefore,

$$\phi(x_0(t)y(t)) - \phi(x_0(t))y(t) \sim \phi(ct \cdot c^{-1}Y_0(t)) - c^{\gamma-1}\phi(t)Y_0(t) \sim \phi(tY_0(t))\,, \quad t \to \infty\,,$$

and accordingly by (3.17)

$$\mathcal{G}y(t) \sim c^{-1} \int_T^t s^{-2} \int_T^s rq(r)\phi(rY_0(r))drds \sim c^{-1}Y_0(t), \quad t \to \infty.$$

This shows that $\mathcal{G}y(t) \in \mathcal{Y}$.

(ii) $\mathcal{G}(\mathcal{Y})$ *is relatively compact.* The inclusion $\mathcal{G}(\mathcal{Y}) \subset \mathcal{Y}$ implies that the set $\mathcal{G}(\mathcal{Y})$ is locally uniformly bounded on $[T, \infty)$. Since for any $y(t) \in \mathcal{Y}$

$$0 \le \Big(\mathcal{G}y\Big)'(t) \le x_0(t)^{-2} \int_T^t x_0(s)q(s)\phi\Big(2c^{-\frac{1}{\gamma}} x_0(s)Y_0(s)\Big)ds, \quad t \ge T,$$

the upper bound for $\Big(\mathcal{G}y\Big)'(t)$ is independent of $y(t)$. It follows that $\mathcal{G}(\mathcal{Y})$ is locally equicontinuous on $[T, \infty)$. The Arzela-Ascoli lemma then ensures the relative compactness of $\mathcal{G}(\mathcal{Y})$.

(iii) $\mathcal{G}$ *is continuous.* Let $\{y_n(t)\}$ be a sequence in $\mathcal{Y}$ converging to $y(t) \in \mathcal{Y}$ uniformly on compact subintervals of $[T, \infty)$. We have to prove that $\mathcal{G}y_n(t) \to \mathcal{G}y(t)$ as $n \to \infty$ uniformly on any compact subinterval of $[T, \infty)$. But this follows from the Lebesgue dominated convergence theorem applied to the inequality

$$|\mathcal{G}y_n(t) - \mathcal{G}y(t)| \le \int_T^t x_0(s)^{-2} \int_T^s x_0(r)q(r)F_n(r)drds, \quad t \ge T,$$

where

$$F_n(t) = |\phi(x_0(t)y_n(t)) - \phi(x_0(t)y(t))| + \phi(x_0(t))|y_n(t) - y(t)|.$$

In fact $F_n(t) \to 0$, $n \to \infty$, at every point $t \in [T, \infty)$ and since $F_n(t)$ satisfies

$$F_n(t) \le 4\phi(2X_0(t)), \quad t \ge T,$$

its upper bound is independent of n.

Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled, and so there exists a fixed point $y_0(t) \in \mathcal{Y}$ of $\mathcal{G}$, which satisfies the integral equation (3.16):

$$y_0(t)=1+\int_T^t x_0(s)^{-2}\int_T^s x_0(r)q(r)\Big[\phi(x_0(r)y_0(r))-\phi(x_0(r))y_0(r)\Big]drds, \ \ t \geq T.$$

It is clear that $y_0(t)$ has the asymptotic behavior $y_0(t) \sim c^{-1}Y_0(t)$ as $t \to \infty$. Since $x(t) = x_0(t)y_0(t)$ is a solution of equation (A) on $[T, \infty)$ such that

$$x(t) \sim ct{\cdot}c^{-1}Y_0(t) = X_0(t), \quad t \to \infty,$$

this completes the proof for the superlinear case of (A).

(B) We now turn to the case where $0 < \gamma < 1$. Suppose that $\sigma > -\gamma - 1$ and define ρ by (3.2). Let $X_0(t)$ be an RV(ρ)-function on $[b, \infty)$ defined by (3.8). Note that $X_0(t)$ has the asymptotic property:

$$\int_b^t \int_b^s q(r)\phi(X_0(r))drds \sim X_0(t), \quad t \to \infty. \tag{3.24}$$

In this case we are able to construct a desired solution $x(t)$ of (A) satisfying $x(t) \sim X_0(t)$, $t \to \infty$, directly as a solution of the integral equation

$$x(t) = 1 + \int_T^t \int_T^s q(r)\phi(x(r))drds, \quad t \geq T, \tag{3.25}$$

for some $T > b$. In fact, choose $T > b$ so that $X_0(t) \geq 1$ and

$$\int_T^t \int_T^s q(r)\phi(X_0(r))drds \leq 2X_0(t), \quad \text{for } t \geq T\,, \tag{3.26}$$

which is possible because of (3.24), and define the integral operator

$$\mathcal{H}x(t) = 1 + \int_T^t \int_T^s q(r)\phi(x(r))drds, \quad t \geq T\,. \tag{3.27}$$

Let $K > 1$ be a constant such that $K^{1-\gamma} \geq 2^{\gamma+2}$ and consider the set $\mathcal{Z}$ consisting of continuous functions $x(t)$ on $[T, \infty)$ satisfying

$$1 \leq x(t) \leq 2K\, X_0(t), \quad t \geq T \qquad \text{and} \qquad x(t) \sim X_0(t), \quad t \to \infty.$$

Clearly, $\mathcal{Z}$ is closed and convex in $C[T, \infty)$.

Since $\phi(x) \in \mathrm{RV}(\gamma)$ satisfies

$$\lim_{t\to\infty} \frac{\phi(2KX_0(t))}{\phi(X_0(t))} = (2K)^\gamma,$$

we may assume that $T > b$ is large enough so that, in addition to (3.26), the following inequality holds:

$$\phi(2KX_0(t)) \le 2^{\gamma+1}K^\gamma \phi(X_0(t)), \quad t \ge T. \tag{3.28}$$

Let $x(t) \in \mathcal{Z}$. Then, using (3.24), (3.26) and (3.28), we have $\mathcal{H}x(t) \sim X_0(t)$, $t \to \infty$, and

$$\begin{aligned} 1 &\le \mathcal{H}x(t) \\ &\le 1 + \int_T^t \int_T^s q(r)\phi(2KX_0(r))drds \le 1 + 2^{\gamma+1}K^\gamma \int_T^t \int_T^s q(r)\phi(X_0(r))drds \\ &\le 1 + 2^{\gamma+2}K^\gamma X_0(t) \le 2KX_0(t), \quad t \ge T. \end{aligned}$$

This shows that $\mathcal{H}x(t) \in \mathcal{Z}$, that is, $\mathcal{H}$ maps $\mathcal{Z}$ into itself. Furthermore it can be verified without difficulty that $\mathcal{H}$ is a continuous map and that $\mathcal{H}(\mathcal{Z})$ is relatively compact in $C[T,\infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a fixed point $x(t)$ of $\mathcal{H}$ which satisfies the integral equation (3.25), and hence the differential equation (A) on $[T,\infty)$. This completes the proof of Theorem. ⊠

We emphasize that Theorem 3.1 states that increasing solutions $x(t)$ of (A) are of the form

$$x(t) = t^{(\sigma+2)/(1-\gamma)}\xi(t)$$

where slowly varying function $\xi(t)$ satisfies the asymptotic relation

$$\xi(t)^{\gamma-1} L\Big(t^{(\sigma+2)/(1-\gamma)}\xi(t)\Big) \sim K\, l(t)^{-1}, \quad t \to \infty\,, \tag{3.29}$$

with constant $K > 0$ which could be easily computed. To find $\xi(t)$ from (3.29) is possible only for some restricted classes of L.

4. *Increasing regularly varying solutions of index 1*

Let us turn to the problem of constructing RV(1)-solutions for equation (A), that is, members of $\mathcal{R}(1)$ given in Theorem 2.1. A solution $x(t) \in$

$\mathcal{R}(1)$ of (A) is called a *trivial* RV(1)-solution or a *nontrivial* RV(1)-solution according as

$$\lim_{t\to\infty} \frac{x(t)}{t} = \text{const} > 0,$$

or

$$\lim_{t\to\infty} \frac{x(t)}{t} = \infty.$$

The totality of trivial RV(1)-solutions (respectively nontrivial RV(1)-solutions) of (A) is denoted by tr-$\mathcal{R}(1)$ (respectively ntr-$\mathcal{R}(1)$).

The existence of a trivial RV(1)-solution of both superlinear and sublinear equation (A) has been completely characterized with the following proposition

Proposition 4.1.*Equation* (A) *possesses a trivial* RV(1)−*solution if and only if*

$$\int_a^\infty q(t)\phi(t)dt < \infty.$$

Notice that the "if" part of this Proposition is proved in the Step 1 in the proof of the "if" part of Theorem 3.1., while for the proof of the "only if" part see book Agarwal et al. [1].

So, we need only to focus our attention on its nontrivial RV(1)-solutions. Since the problem of finding such solutions of (A) seems to be difficult for general $\phi(x) \in \mathrm{RV}(\gamma)$, we will be content to restrict ourselves to a smaller class of $\phi(x) \in \mathrm{RV}(\gamma)$ by imposing the following additional requirement

$$u(t) \in \mathrm{SV}\cap C^1 \quad \Longrightarrow \quad \phi(tu(t)) \sim \phi(t)u(t)^\gamma, \;\; t \to \infty, \tag{4.1}$$

which amounts to requiring that the slowly varying part $L(x)$ of $\phi(x)$ satisfies

$$u(t) \in \mathrm{SV}\cap C^1 \quad \Longrightarrow \quad L(tu(t)) \sim L(t), \;\; t \to \infty\,. \tag{4.2}$$

It is easy to check that (4.2) is satisfied by

$$L(t) = \prod_{k=1}^{N} (\log_k t)^{\alpha_k}, \quad \alpha_k \in \mathbf{R},$$

but not by

$$L(t) = \exp\left(\prod_{k=1}^{N} (\log_k t)^{\beta_k}\right), \quad \beta_k \in (0,1),$$

where $\log_k t = \log\log_{k-1} t$.

Theorem 4.1 *Suppose that $\phi(x)$ satisfies (4.1). Equation* (A) *possesses a nontrivial* RV(1)*−solution if and only if $\sigma = -\gamma - 1$, and*

$$\int_{\alpha}^{\infty} q(t)\phi(t)dt < \infty \qquad \text{for } \ \gamma > 1 \tag{4.3}$$

$$\int_{\alpha}^{\infty} q(t)\phi(t)dt = \infty \qquad \text{for } \ 0 < \gamma < 1 \tag{4.4}$$

in which case any such solution $x(t)$ has the precise asymptotic behavior

$$x(t) \sim t\Big[(\gamma - 1)\int_{t}^{\infty} q(s)\phi(s)ds\Big]^{\frac{1}{1-\gamma}}, \quad t \to \infty \qquad \text{if} \quad \gamma > 1\,, \tag{4.5}$$

$$x(t) \sim t\Big[(1 - \gamma)\int_{a}^{t} q(s)\phi(s)ds\Big]^{\frac{1}{1-\gamma}}, \quad t \to \infty \qquad \text{if} \quad 0 < \gamma < 1. \tag{4.6}$$

Proof. (The "only if" part): Suppose that (A) has a nontrivial RV(1)-solution $x(t)$ on $[t_0, \infty)$. Let $x(t) = t\xi(t)$, $\xi(t) \in \text{SV}$. Then, since

$$x'(t) \sim \int_{t_0}^{t} q(s)\phi(x(s))ds = \int_{t_0}^{t} s^{\sigma+\gamma} l(s)\xi(s)^{\gamma} L(s\xi(s))ds, \quad t \to \infty, \tag{4.7}$$

and $x'(t) \to \infty$ as $t \to \infty$, σ must satisfy $\sigma + \gamma \geq -1$. It is impossible, however, that $\sigma + \gamma > -1$. In fact, if this would be the case, then integrating (4.7) from t_0 to t and applying Karamata's integration theorem, we would obtain

$$x(t) \sim \frac{t^{\sigma+\gamma+2} l(t)\xi(t)^{\gamma} L(t\xi(t))}{(\sigma + \gamma + 1)(\sigma + \gamma + 2)} \in \text{RV}(\sigma + \gamma + 2), \quad t \to \infty\,,$$

which is impossible because $\sigma + \gamma + 2 > 1$. Thus, we must have $\sigma + \gamma = -1$, i.e., $\sigma = -\gamma - 1$, and using the condition (4.2), (4.7) becomes

$$x'(t) \sim \int_{t_0}^{t} s^{-1} l(s)\xi(s)^{\gamma} L(s\xi(s))ds \sim \int_{t_0}^{t} q(s)\phi(s)\xi(s)^{\gamma} ds \in \text{SV}, \quad t \to \infty\,.$$

Integrating the above from t_0 to t then gives

$$x(t) \sim t\int_{t_0}^{t} q(s)\phi(s)\xi(s)^{\gamma} ds, \quad t \to \infty,$$

or

$$\xi(t) \sim \int_{t_0}^{t} q(s)\phi(s)\xi(s)^{\gamma} ds, \quad t \to \infty. \tag{4.8}$$

Let the integral in (4.8) be denoted by $Y(t)$. It is easy to see that $Y(t)$ satisfies

$$Y(t)^{-\gamma}Y'(t) \sim q(t)\phi(t), \quad t \to \infty\,, \tag{4.9}$$

and $Y(t) \to \infty$, $t \to \infty$. If $\gamma > 1$, then (4.9) can be integrated over $[t, \infty)$, implying the validity of (4.3) and yielding the asymptotic formula

$$Y(t) \sim \left[(\gamma - 1)\int_t^\infty q(s)\phi(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

from which the formula (4.5) for $x(t)$ follows, whereas if $0 < \gamma < 1$, then integration of (4.9) over $[a, t]$ yields

$$Y(t) \sim \left[(1 - \gamma)\int_a^t q(s)\phi(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t \to \infty,$$

from which (4.4) and (4.6) follows immediately.

(The "if" part): **(A)** Let $\gamma > 1$. Suppose that (4.3) holds. As in the proof of the "if" part of Theorem 3.1 one can choose a (sufficiently small) constant $c \in (0, 1)$ such that (3.14) holds, so that (A) has a solution $x_0(t)$ on $[t_0, \infty)$ such that $x_0(t) \sim ct$ as $t \to \infty$. We may assume that c satisfies (3.15). The substitution $x(t) = x_0(t)y(t)$ transforms (A) into the differential equation (3.10). Our task is reduced to constructing a solution $y_1(t)$ of (3.10) which is a nontrivial slowly varying function having the property $y_1(t) \sim c^{-1}Y_1(t)$ as $t \to \infty$, where $Y_1(t)$ is a nontrivial SV$-$function given by

$$Y_1(t) = \left[(\gamma - 1)\int_t^\infty q(s)\phi(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t \geq t_0.$$

Our aim is to obtain $y_1(t)$ as a solution of the integral equation (3.16) for some suitably chosen $T > t_0$.

Crucial to the later discussions is the following asymptotic formula for $Y_1(t)$:

$$\int_{t_0}^t s^{-2}\int_{t_0}^s rq(r)\phi(krY_1(r))drds \sim k^\gamma Y_1(t), \quad t \to \infty, \tag{4.10}$$

where $k > 0$ is a constant. To verify this, since $\phi(ktY_1(t)) \sim (kY_1(t))^\gamma\phi(t)$ as $t \to \infty$ and $tq(t)(kY_1(t))^\gamma\phi(t) \in$ SV, using Karamata's integration theorem, we see that

$$\int_{t_0}^t sq(s)\phi(ksY_1(s))ds \sim k^\gamma t^2 q(t)\phi(t)Y_1(t)^\gamma, \quad t \to \infty,$$

and hence

$$\int_{t_0}^{t} s^{-2}\int_{t_0}^{s} rq(r)\phi(krY_1(r))drds \sim k^{\gamma}\int_{t_0}^{t} q(s)\phi(s)Y_1(s)^{\gamma}ds$$
$$= k^{\gamma}\int_{t_0}^{t} q(s)\phi(s)\Big[(\gamma-1)\int_{s}^{\infty} q(r)\phi(r)dr\Big]^{\frac{\gamma}{1-\gamma}}ds \sim k^{\gamma}Y_1(t), \quad t\to\infty.$$

In view of the relation

$$\int_{t_0}^{t} x_0(s)^{-2}\int_{t_0}^{s} x_0(r)q(r)\phi(ckrY_1(r))drds \sim c^{\gamma-1}k^{\gamma}Y_1(t)\,, \quad t\to\infty\,,$$

which is an immediate consequence of (4.10), there exists $T > t_0$ depending on k such that

$$\int_{t_0}^{t} x_0(s)^{-2}\int_{t_0}^{s} x_0(r)q(r)\phi(ckrY_1(r))drds \le 2c^{\gamma-1}k^{\gamma}Y_1(t) \quad \text{for} \quad t\ge T. \tag{4.11}$$

We may assume that T is so large that

$$c^{-\frac{1}{\gamma}}Y_1(t)\ge 1 \quad \text{for} \quad t\ge T. \tag{4.12}$$

We define $\mathcal{Y}_1$ to be the set of continuous functions $y(t)$ satisfying

$$1\le y(t)\le 2c^{-\frac{1}{\gamma}}Y_1(t), \quad t\ge T,$$

and

$$y(t)\sim c^{-1}Y_1(t), \quad t\to\infty.$$

Clearly, $\mathcal{Y}_1$ is a closed convex subset of $C[T,\infty)$. It can be verified that the integral operator $\mathcal{G}$ defined by (3.21) is continuous and maps $\mathcal{Y}_1$ into a relatively compact subset of $\mathcal{Y}_1$. In fact, for $y(t)\in\mathcal{Y}_1$, since $\phi(x_0(t)y(t))\ge \phi(x_0(t))y(t)$, it is clear that $\mathcal{G}y(t)\ge 1$, $t\ge T$. On the other hand, using (3.15) and (4.11) with $k=2c^{-\frac{1}{\gamma}}$, we obtain

$$\int_{T}^{t} x_0(s)^{-2}\int_{T}^{s} x_0(r)q(r)\phi(x_0(r)y(r))drds \le 2^{\gamma+1}c^{\gamma-1}Y_1(t)\le c^{-\frac{1}{\gamma}}Y_1(t),$$

and so

$$\mathcal{G}y(t)\le 1+c^{-\frac{1}{\gamma}}Y_1(t)\le 2c^{-\frac{1}{\gamma}}Y_1(t), \quad t\ge T.$$

Furthermore, from (4.10) for $k = 1$, we have

$$\begin{aligned}\mathcal{G}y(t) &\sim \int_T^t x_0(s)^{-2}\int_T^s x_0(r)q(r)\phi(x_0(r)y(r))drds \\ &\sim \int_T^t (cs)^{-2}\int_T^s crq(r)\phi(rY_1(r))drds \sim c^{-1}Y_1(t), \quad t\to\infty.\end{aligned}$$

Therefore, it follows that $\mathcal{G}y(t) \in \mathcal{Y}_1$. The proof of the continuity of $\mathcal{G}$ and the relative compactness of $\mathcal{G}(\mathcal{Y}_1)$ is essentially the same as that of Theorem 3.1. Therefore, there exists a fixed point $y_1(t) \in \mathcal{Y}_1$ of $\mathcal{G}$, which solves the integral equation (3.16) and hence the differential equation (3.10). Then, the function $x(t) = x_0(t)y_1(t)$ provides a solution of (A) on $[T,\infty)$ having the desired asymptotic behavior $x(t) \sim tY_1(t)$ as $t\to\infty$. This completes the proof for the case $\gamma > 1$.

(B) Next we let $0 < \gamma < 1$. Let $X_1(t)$ denote the function

$$X_1(t) = t\left[(1-\gamma)\int_a^t q(s)\phi(s)ds\right]^{\frac{1}{1-\gamma}}, \quad t \ge a.$$

A straightforward computation with the aid of (4.1) shows that

$$\int_a^t\int_a^s q(r)\phi(X_1(r))drds \sim X_1(t) \quad t\to\infty,$$

which implies that

$$\int_a^t\int_a^s q(r)\phi(kX_1(r))drds \sim k^{\gamma}X_1(t) \quad t\to\infty,$$

for any constant $k > 0$. Let $K > 1$ be a constant such that $K^{1-\gamma} \ge 2^{\gamma+1}$ and define $\mathcal{Z}_1$ to be the set of continuous functions $x(t)$ on $[T,\infty)$ satisfying

$$1 \le x(t) \le 2KX_1(t), \quad t\ge T, \qquad \text{and} \qquad x(t) \sim X_1(t), \quad t\to\infty,$$

where $T > a$ is chosen so that

$$X_1(t) \ge 1 \quad \text{and} \quad \int_T^t\int_T^s q(r)\phi(2KX_1(r))drds \le 2(2K)^{\gamma}X_1(t), \quad t \ge T.$$

Clearly, $\mathcal{Z}_1$ is closed and convex in $C[T,\infty)$.

We now let the integral operator $\mathcal{H}$ defined by (3.27) act on the set $\mathcal{Z}_1$. As in the last part of the proof of Theorem 3.1 it can be shown that $\mathcal{H}$ is a self-map on $\mathcal{Z}_1$ and sends $\mathcal{Z}_1$ continuously into a relatively compact subset of $C[T,\infty)$, so that $\mathcal{H}$ has a fixed point $x(t) \in \mathcal{Z}_1$ which generates a solution of equation (A) with the asymptotic property $x(t) \sim X_1(t)$, $t\to\infty$. This completes the proof of Theorem 4.1. ⊠

5. *Concluding remarks and examples*

On the basis of Theorems 2.1, 3.1 and 4.1 one can determine the structure of increasing regularly varying solutions of (A) in terms of γ, σ and the integrability of $q(t)\phi(t)$ on $[a,\infty)$, i.e. either (4.3) or (4.4).

(I) Concluding remarks for superlinear case $\gamma > 1$:

(i) If $\sigma < -\gamma - 1$, then by Theorem 3.1 equation (A) has increasing regularly varying solutions of index $(\sigma+2)/(1-\gamma)$. Since (4.3) holds in this case, (A) has trivial RV(1)-solutions as well. Therefore it follows from Theorem 2.1 that

$$\mathcal{R}_+ = \mathcal{R}(1) \cup \mathcal{R}\left(\frac{\sigma+2}{1-\gamma}\right), \quad \mathcal{R}(1) = tr - \mathcal{R}(1).$$

(ii) If $\sigma = -\gamma - 1$ and (4.3) holds, then Theorem 4.1 shows that (A) has both trivial and nontrivial RV(1)-solutions, but no RV(ρ)-solutions for any $\rho > 1$. Therefore, Theorem 2.1 implies that

$$\mathcal{R}_+ = \mathcal{R}(1) = tr - \mathcal{R}(1) \cup tr - \mathcal{R}(1).$$

(iii) If $\sigma = -\gamma - 1$ and (4.4) holds, or if $\sigma > -\gamma - 1$, then there is no increasing regularly varying solution of (A), that is, $\mathcal{R}_+ = \emptyset$.

(II) Concluding remarks for sublinear case $0 < \gamma < 1$:

(i) If $\sigma > -\gamma - 1$, then by Theorem 3.1 equation (A) has increasing regularly varying solutions of index $(\sigma+2)/(1-\gamma)$, and Theorem 2.1 implies that

$$\mathcal{R}_+ = \mathcal{R}\left(\frac{\sigma+2}{1-\gamma}\right).$$

(ii) If $\sigma = -\gamma - 1$ and (4.4) holds, then Theorem 4.1 ensures that

$$\mathcal{R}_+ = \mathcal{R}(1), \quad \mathcal{R}(1) = ntr - \mathcal{R}(1).$$

(iii) If $\sigma = -\gamma - 1$ and (4.3) holds, or if $\sigma < -\gamma - 1$, then

$$\mathcal{R}_+ = \mathcal{R}(1), \quad \mathcal{R}(1) = tr - \mathcal{R}(1).$$

Example 5.1. Let γ be a positive constant different from 1 and consider the differential equation

$$x''(t) = q_1(t)\phi(x(t)), \tag{5.1}$$

with

$$\phi(x) = x^\gamma \log(x+1) \quad \text{and} \quad q_1(t) = \frac{\rho(\rho-1)\log t + 2\rho - 1}{t^{\rho(\gamma-1)+2}(\log t)^\gamma \log(t^\rho \log t + 1)}, \tag{5.2}$$

where $\rho > 1$ is a constant. The function $q_1(t)$ is a regularly varying function of index $\sigma = -\rho(\gamma-1) - 2$, which satisfies

$$\sigma < -\gamma - 1 \quad \text{if} \quad \gamma > 1, \quad \text{and} \quad \sigma > -\gamma - 1 \quad \text{if} \quad 0 < \gamma < 1.$$

It is easy to check that

$$\begin{aligned} \rho(\rho-1)\Big(t^2 q_1(t)\Big)^{-1} &= \frac{t^{\rho(\gamma-1)}(\log t)^\gamma \log(t^\rho \log t + 1)}{\log t + (2\rho-1)/\rho(\rho-1)} \\ &\sim (t^\rho \log t)^{\gamma-1} \log(t^\rho \log t + 1), \quad t \to \infty. \end{aligned}$$

Therefore, from Theorem 3.1 it follows that equation (5.1)-(5.2) possesses increasing regularly varying solutions $x(t)$ of index ρ, all of which are governed by the asymptotic formula

$$x(t)^{\gamma-1} \log(x(t)+1) \sim (t^\rho \log t)^{\gamma-1} \log(t^\rho \log t + 1), \quad t \to \infty,$$

which implies that

$$x(t) \sim t^\rho \log t, \quad t \to \infty.$$

One easily check that $x(t) = t^\rho \log t$ is an exact solution of equation (5.1)–(5.2).

Example 5.2. Let γ be a positive constant different from 1 and consider the differential equation

$$x''(t) = q_2(t)\phi(x(t)), \tag{5.3}$$

with

$$\phi(x) = x^\gamma \log(x+1) \quad \text{and} \quad q_2(t) = \frac{1}{t^{\gamma+1}(\log t)^\gamma \log(t \log t + 1)}. \tag{5.4}$$

Note that $\phi(x)$ fulfills the condition (4.1). Clearly, $q_2(t)$ is a regularly varying function of index $\sigma = -\gamma - 1$ and satisfies

$$q_2(t)\phi(t) = \frac{t^\gamma \log(t+1)}{t^{\gamma+1}(\log t)^\gamma \log(t \log t + 1)} \sim \frac{1}{t(\log t)^\gamma}, \quad t \to \infty.$$

This shows that

$$\int_t^\infty q_2(s)\phi(s)ds \sim \frac{(\log t)^{1-\gamma}}{(\gamma-1)}, \quad t\to\infty \qquad \text{if} \quad \gamma>1$$

$$\int_e^t q_2(s)\phi(s)ds \sim \frac{(\log t)^{1-\gamma}}{1-\gamma}, \quad t\to\infty \qquad \text{if} \quad 0<\gamma<1$$

so that from Theorem 4.1 we conclude that equation (5.3)–(5.4) possesses nontrivial regularly varying solutions $x(t)$ of index 1, all of which have one and the same asymptotic behavior

$$x(t) \sim t\log t, \quad t\to\infty.$$

In fact equation (5.3)–(5.4) possesses an exact solution $x(t) = t\log t$.

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