Bulletin T.CXXXVII de l'Académie serbe des sciences et des arts − 2008
Classe des Sciences mathématiques et naturelles
Sciences mathématiques, № 33

AN EQUATION IN THE LEFT AND RIGHT FRACTIONAL DERIVATIVES OF THE SAME ORDER

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(Presented at the 2nd Meeting, held on March 28, 2008)

A b s t r a c t. The linear equation in left and right fractional derivatives is considered in a subspace $\mathcal{D}'_b$ of the space $\mathcal{D}'$ of distributions.

AMS Mathematics Subject Classification (2000): 34G10
Key Words: Left and right fractional derivatives, tempered distributions.

1. *Introduction*

Equations with the left and right fractional derivatives have many applications and have been elaborated in many papers and books (cf. for example the books: [4], [5], [2]). In the book [3] articles from different part of physics have been collected in which fractional derivatives have an important role.

The equation with left and right fractional derivatives, we consider on a bounded interval, has many applications. It is in direct connection with the generalized Abel equation (cf. [6], §30.3). In [6]. p. 689 one can find a list of papers treating problems from physics which can be connected with equation (3.1). But there are only a few papers with equations containing the both kinds of fractional derivatives, left and right.

We consider equation

$$D^{\beta}_{0^+}f + AD^{\beta}_{b^-}f = C, \quad \beta = k+\alpha,\ k \in \mathbf{N}_0,\ \alpha \in (0,1)$$

in the subspace $\mathcal{D}'_b$, of the space of distributions $\mathcal{D}'(-\infty, b)$ which is large enough to contain "singular solutions" which can appear in mathematical models of mechanics. Such "singular" solutions have been given many times by distributions which are locally regular except in some points of $[0, b]$, i.e., which are locally classic.

Let us remark that: a) Solutions of the quoted equation give the possibility to compare the two fractional derivatives, left and right ones (cf. Remark after Theorem 1). b) Lemma 2 asserts that the left (and the right) fractional derivative on $\mathcal{D}'_b$ is a generalization of the classical one.

2. *Preliminaries*

We recall some definitions and results: $\mathcal{S}' \equiv \mathcal{S}'(\mathbf{R})$ is the space of tempered distributions, $\mathcal{S}'_+ = \{T \in \mathcal{S}', suppT \subset [0, \infty)\}$. $\mathcal{S}'_+$ is a convolution algebra which is commutative and associative. (cf. for example [10] and [7]), $\mathcal{D}'([0, b)) = \{T \in \mathcal{D}'(-\infty, b), suppT \subset [0, b)\}$.

$\mathcal{O}_M$ denotes the space of multipliers of $\mathcal{S}$. Then, if $F \in \mathcal{O}_M$ and $g \in \mathcal{S}', Fg \in \mathcal{S}'$, as well. (cf. [10], p.14).

$\{f_\beta; \beta \in \mathbf{R}\}$ is a class of distributions

$$f_\beta(t) = \begin{cases} H(t)t^{\beta-1}/\Gamma(\beta),\ \beta > 0, \\ f^{(m)}_{\beta+m}(t),\ \beta \le 0,\ \beta+m > 0, m \in \mathbf{N}, \end{cases} \tag{2.1}$$

which belong to $\mathcal{S}'_+$ and has an important role in definition of the fractional derivatives of distributions; $f^{(m)} \equiv D^m,\ m \in \mathbf{N}_0$, denotes the $m-$th derivative in the distributional sense and H is Heaviside's function.

By $f^{(-\beta)}$ for $f \in \mathcal{S}'_+$ we denote $f_\beta * f$, where $*$ is the sign for the convolution and $\beta \in \mathbf{R}$. If $\beta > 0,\ f^{(-\beta)}$ is termed the operator of fractional integral of order β, but if $\beta < 0,\ f^{(-\beta)}$ is the operator of fractional derivative of order $-\beta$ (cf. [11], p. 36) and [10], p. 89).

The class $\{f_\beta; \beta \in \mathbf{R}\}$ with the operation convolution forms an Abelian group: $f_{\beta_1} * f_{\beta_2} = f_{\beta_1+\beta_2},\ f_0 = \delta$.

If $T \in \mathcal{S}'_+$ is the *regular distribution* defined by the function f, then we write $T = [f]$.

2.2. *The space* $\mathcal{D}'_b$

Let f be defined as: $f(x)$ is integrable in the sense of Lebesgue on $(-\infty, b)$ and $f(x) = 0,\ x \in \mathbf{R}_-$. This function defines the regular distribution $[f] \in \mathcal{D}'(-\infty, b),\ \mathrm{supp}[f] \subset [0, b)$. There always exists the distribution $[\overline{f}] \in \mathcal{S}'_+$, defined by $\overline{f} \in L^1((-\infty, \infty))$ with the properties: 1. $f(x) = \overline{f}(x),\ x \in (-\infty, b)$; 2. $\overline{f}(x) = 0,\ x < 0$.

We denote by $\mathbf{R}\mathcal{S}'_+$ the associative and commutative ring (with operation convolution, denoted by $*$), without divisors of zeros (Titchmarsh's theorem) consisting of regular distributions f defined by functions belonging to $L^1(-\infty, \infty)$, $\mathrm{supp} f \subset [0, \infty)$. Since all functions f defining $[f] \in \mathbf{R}\mathcal{S}'$ equal zero on $(-\infty, 0)$, we do not separately repeat this fact. Let $\mathcal{A}$ be the ideal of $\mathbf{R}\mathcal{S}'_+$, $\mathcal{A} = \{T \in \mathbf{R}\mathcal{S}'_+,\ \mathrm{supp} T \subset [b, \infty)\}$. In $\mathbf{R}\mathcal{S}'_+$ we can define the following equivalence relation $f \sim g \Longleftrightarrow f - g \in \mathcal{A}$. An element $\dot{[\overline{f}]}$ of the quotient space $\mathbf{R}\mathcal{S}'_+/\mathcal{A}$ is the class defined by $T = [\overline{f}] \in \mathbf{R}\mathcal{S}'_+$.

Taking care of the property of the δ distribution: $\delta^{(k)} * f = D^k f \equiv f^{(k)},\ f \in \mathcal{S}'_+$, we introduce two families of spaces.

Definition 2.1. *Let* $\mathcal{B}_m = \{T = \delta^{(m)} * \dot{[\overline{U}]};\ \dot{[\overline{U}]} \in \mathbf{R}\mathcal{S}'_+/\mathcal{A}\},\ m \in \mathbf{N}_0\}$ $(\mathbf{N}_0 = \mathbf{N} \cup \{0\})$ *and* $\mathcal{D}'_m = \{v = \delta^{(m)} * [U];\ U = \overline{U}|_{(-\infty, b)},\ [\overline{U}] \in \mathbf{R}\mathcal{S}'_+\},\ m \in \mathbf{N}_0$. *Then* $\mathcal{D}'_b = \bigcup\limits_{m \in \mathbf{N}_0} \mathcal{D}'_m$ *and* $\mathcal{B} = \bigcup\limits_{m \in \mathbf{N}_0} \mathcal{B}_m$.

It is easily seen that

Lemma 2.1. $\mathcal{D}'_b$ *is algebraically isomorphic to* $\mathcal{B}$ *by the mapping:* $v = \delta^{(m)} * [U] \in \mathcal{D}'_m \to \delta^{(m)} * \dot{[\overline{U}]} \in \mathcal{B}_m$.

The set $\{\delta^{(m)} * \mathbf{R}\mathcal{S}'_+,\ m \in \mathbf{N}_0\}$ is a large subset of $\mathcal{S}'_+$. This follows from the structure theorem of the space $\mathcal{S}'$, which says that if $f \in \mathcal{S}'$, then there exists a continuous function g of slow growth and $m \in \mathbf{N}_0$ such that $f = D^m[g] \equiv \delta^{(m)} * [g]$.

In $\mathcal{D}'_b$ we define the convolution: Let $\delta^{(m)} * [f]$ and $\delta^{(k)} * [g]$ belong to $\mathcal{D}'_b$ and let $\delta^{(m)} * [\overline{f}]$ and $\delta^{(k)} * [\overline{g}]$ be from $\delta^{(m)} * \mathbf{R}\mathcal{S}'_+$ and $\delta^{(k)} * \mathbf{R}\mathcal{S}'_+$, respectively, the representatives of corresponding elements from $\mathcal{B}$, then $(\delta^{(m)} * [f]) * (\delta^{(k)} * [g]) = \delta^{(m+k)} * [(\overline{f} * \overline{g})|_{(-\infty, b)}] \in \mathcal{D}'_b$.

It is easy to prove that this definition does not depend on the representatives we choose.

We will denote by Q an operator defined as:

Definition 2.2. *Let* $T = \delta^{(m)} * [f] \in \mathcal{D}'_b$, *then* $QT = (-1)^m \delta^{(m)} * [Qf]$, *where* $Qf(x) = \overline{f}(b-x),\ 0 \le x < b.\ (Qf(x) = 0,\ x < 0).$

The properties of the operator Q, defined on $\mathcal{D}'_b$, we use, are: 1) $QQ = I$; 2) If A, B are constants, then $Q(Af_b+Bg_b) = AQf_b+BQg_b$; 3)$(Q(D^k g_b))(x) = (-1)^k D^k((Qg_b))(x)$, where $f_b,\ g_b \in \mathcal{D}'_b$.

Now we can extend the operators D^{β}_{0+} and D^{β}_{b-}, $\beta > 0$, onto $\mathcal{D}'_b$. The classial definitions of these operators for $\beta = k+\gamma,\ k \in \mathbf{N}_0,\ \gamma \in (0,1)$, is

$$D^{\beta}_{0+}\eta = \Big(\frac{d}{dx}\Big)^{k+1} I^{1-\gamma}_{0+}\eta\,,\ \ D^{\beta}_{b-}\eta = \Big(-\frac{d}{dx}\Big)^{k+1} I^{1-\gamma}_{b-}\eta\,,$$

where I^{β}_{0+} and I^{β}_{b-} denote the fractional integrals. The conditions on f that $D^{\beta}_{0+}f$ and $D^{\beta}_{b-}f$ exist one can find for example in [6], Lemma 2.2 and Theorem 14.9.

Definition 2.3. *Let* $T = \delta^{(m)} * [\eta] \in \mathcal{D}'_m \subset \mathcal{D}'_b$ *and* $\beta = k+\gamma,\ k \in \mathbf{N}_0,\ \gamma \in (0,1)$. *Then by definition:*

$$D^{\beta}_{0+}T = \delta^{(m+k+1)} * \Big[(I^{1-\gamma}_{0+}\eta)\Big|_{[0,b)}\Big] = \delta^{(m+k+1)} * [(f_{1-\gamma} * \eta)|_{[0,b)}] \tag{2.2}$$

$$D^{\beta}_{b-}T = QD^{\beta}_{0+}QT = (-1)^{k+1}\delta^{(k+1)} * [(I^{1-\gamma}_{b-} * \eta)|_{[0,b)}]\,. \tag{2.3}$$

Consequently, D^{β}_{0+} and D^{β}_{b-} are defined for every element of $\mathcal{D}'_b$ and map $\mathcal{D}'_b$ into $\mathcal{D}'_b$.

The next Lemma gives the connection between the operator D^{β}_{0+} on the space of functions for $\beta = k+\gamma,\ k \in \mathbf{N}_0,\ \gamma \in (0,1)$.

Lemma 2.2. *Let* $\eta \in L((-\infty,b))$, supp$\eta \subset [0,b)$, $\overline{\eta}(x) = \eta(x), x \in (0,b)$ *and such that:*

1) there exists $(f_{1-\gamma}*\overline{\eta})^{(k+1+m)}(x)$ *for* $x \in (0,b)$, *and belongs to* $L^1_{loc}(-\infty,b)$;

2) $\lim\limits_{x\to 0^+}(f_{1-\gamma} * \eta)^{(i)} = c_i,\ i = 0,1,...,k+m$. *Then there exists* $D^{\beta}_{0+}T = [(D^{\beta}_{0+}T] + \sum\limits_{o=0}^{k+m} c_i\delta^{(k+m-i)},\ T = \delta^{(m)} * [\eta]$.

P r o o f. By Definition 2.3. we have

$$D^{\beta}_{0+}T = \delta^{(m+k+1)} * \Big[(I^{1-\gamma}_{0+}\eta)|_{[0,b)}\Big].$$

We can use the connection between the classical derivative and the derivative in the sense of distributions (cf. [12], p. 51) to obtain

$$\begin{aligned} D^{\beta}_{0+}T &= \Big[(I^{1-\gamma}_{0+}\eta)^{(m+k+1)}|_{(0,b)}\Big] + \sum_{i=0}^{m+k} c_i\delta^{(k+m-i)} \\ &= \Big[(\delta^{(k+1)} * \delta^{(m)} * f_{1-\gamma} * \eta)|_{(0,b)}\Big] + \sum_{i=0}^{m+k} c_i\delta^{(k+m-i)} \\ &= [D^{\beta}_{0+}T\Big|_{(0,b)}] + \sum_{i=0}^{m+k} c_i\delta^{(k+m-i)}\,. \end{aligned}$$

2.3. *Some spaces of numerical functions ([6], p.246)*

$$\begin{aligned} \mathcal{H}^{\lambda}([0,b]) &= \{f;\ |f(x_1)-f(x_2)| \le A|x_1-x_2|^{\lambda}, \\ &\quad x_1,x_2\in[0,b],\ \ 0<\lambda\le 1\}; \end{aligned}$$

$$\mathcal{H} = \bigcup_{0<\lambda\le 1}\mathcal{H}^{\lambda};$$

$$\mathcal{H}^* \equiv \mathcal{H}^*(a,b) = \Big\{f;\ f(x) = \frac{f^*(x)}{x^{1-\varepsilon_1}(b-x)^{1-\varepsilon_2}},\ \ 0<x<b,\ \varepsilon_1>0;$$

$$\varepsilon_2>0,\ \ 0<\lambda\le 1,\ \ f^*\in\mathcal{H}^{\lambda}([0,b])\Big\};$$

$$\mathcal{H}^{\lambda}_0(\varepsilon_1,\varepsilon_2) = \{f\in\mathcal{H}^*,\ \ f^*(0)=f^*(b)=0\};$$

$$\mathcal{H}^*_{\alpha} = \bigcup_{\substack{\alpha<\lambda\le 1\\ \varepsilon_1.\varepsilon_2>0}} \mathcal{H}^{\lambda}_0(\varepsilon_1,\varepsilon_2)\,.$$

3. *Solutions to equation*

$$D^{\beta}_{0+}f + AD^{\beta}_{b-}f = C, \tag{3.1}$$

in $\mathcal{D}'_m$, where $\beta = k+\alpha,\ k\in\mathbf{N}_0,\ \alpha\in(0,1),\ m\in\mathbf{N}_0$.

Theorem 3.1. *A necessary and sufficient condition that equation (3.1) has a solution* $f=\delta^{(m)}*[\eta],\ \eta\in\mathcal{H}^*$ *is that* $C=\delta^{m+k+1}*[\xi],\ \xi\in\mathcal{H}^*_{1-\alpha}$. *If*

$(-1)^{k+1}A > 0$, *the solution is unique in the space* $\{f = \delta^{(m)} * [\eta];\ \eta \in \mathcal{H}^*\} \subset \mathcal{D}'_m$; *if* $(-1)^{k+1}A < 0$ *it contains an arbitrary constant. The analytical form of* η *is:*

$$\eta(x) = \frac{c}{x^{1-\alpha+\theta/2\pi}(b-x)^{1-\theta/2\pi}} + \frac{\sin(1-\alpha)\pi}{N\pi}\frac{d}{dx}\int_0^x \frac{\xi(t)}{(x-t)^{1-\alpha}}dt$$

$$-(-1)^{k+1}\frac{A}{N}\Big(\frac{\sin(1-\alpha)\pi}{\pi}\Big)^2 \frac{d}{dx}\int_0^x \frac{Z(t)dt}{(x-t)^{1-\alpha}}\frac{d}{dt}\int_0^t \frac{d\tau}{(t-\tau)^\alpha}\int_\tau^b \frac{\xi(s)ds}{Z(s)(s-\tau)^{1-\alpha}}, \tag{3.2}$$

where $c = 0$, *if* $(-1)^{k+1}A > 0$ *and* c *is arbitrary, if* $(-1)^{k+1}A < 0$;

$$N = 1+(-1)^{k+1}A\cos(1-\alpha)\pi+A^2;\ \theta = \arg\frac{1+e^{(\alpha-1)\pi i}(-1)^{k+1}A}{1+e^{(1-\alpha)\pi i}(-1)^{k+1}A},\ 0<\theta<2\pi;$$

$$Z(t) = t^{2-(1-\alpha)-\theta/2\pi}(b-t)^{-\alpha+\theta/2\pi} \ \textit{if}\ (-1)^{k+1}A > 0 \ \textit{and}$$

$$Z(t) = (t/(b-t))^{\alpha-\theta/2\pi},\quad \textit{if}\ \ (-1)^{k+1}A < 0\ .$$

P r o o f. By definition of $D^\beta_{0^+}$ and $D^\beta_{b^-}$ in $\mathcal{D}'_{m+k+1}$ (cf. (2.2) and (2.3)) and by the analytical form of $C = \delta^{(m+k+1)} * [\xi]$, equation (3.1) can be written as:

$$\delta^{(k+m+1)} * \bigg(\Big[(I^{1-\alpha}_{0^+}\eta)|_{[0,b)}\Big] + (-1)^{k+1}A\Big[(I^{1-\alpha}_{b^-}\eta)|_{[0,b)}\Big] - \Big[\xi|_{[0,b)}\Big]\bigg) = 0.$$

To this equation there corresponds in $\mathcal{S}'$

$$\delta^{(k+m+1)} * \bigg([I^{1-\alpha}_{0^+}\overline{\eta}] + (-1)^{k+1}A[I^{1-\alpha}_{b^-}\overline{\eta}] - [\overline{\xi}] - [M]\bigg) = 0,$$

where $[M]$ is any element of $\mathcal{A}$.

By the properties of δ distribution and definition of the primitive of a distribution (cf. [7], Chapter I, §4), we have

$$\Big[I^{1-\alpha}_{0^+}\overline{\eta}\Big] + (-1)^{k+1}A\Big[I^{1-\alpha}_{b^-}\overline{\eta}\Big] - \Big[\overline{\xi}\Big] = \Big[\sum_{i=0}^{m+k} a_i x^i\Big] + [M]$$

in $\mathcal{S}'$, where $a_i,\ i = 0, 1, ..., m + k$ are undefined constants. Consequently,

$$I_{0+}^{1-\alpha}\overline{\eta} + (-1)^{k+1} A I_{b-}^{1-\alpha}\overline{\eta} - \overline{\xi} - M = \sum_{i=0}^{m+k} a_i x^i, \tag{3.3}$$

for almost every $x \in \mathbf{R}$. But this is possible only if $a_i = 0,\ i = 0, 1, ..., m+k$, because of the support of the function on the left-hand side of equation (3.3).

In such a way we reduced equation (3.1) to the form

$$(I_{0+}^{1-\alpha}\eta)(x) + (-1)^{k+1} A (I_{b-}^{1-\alpha}\eta)(x) = \xi(x), \ \ 0 < x < b. \tag{3.4}$$

We used of the property of M, $\mathrm{supp} M \subset [b, \infty)$.

By Theorem 30.7 in [6] equation (3.4) is solvable in the space $\mathcal{H}^*$ whatever $\xi(x) \in \mathcal{H}^*_{1-\alpha}$ was. Its solution is given by (3.2).

A solution $f = \delta^{(m)} * [\eta],\ \eta \in \mathcal{H}^*$ can exist if and only if C can be given as: $C = \delta^{(k+m+1)} * [\xi],\ \xi \in \mathcal{H}^*_{1-\alpha}$. This follows from Theorem 13.14 in [6], p. 248, which says that the fractional integration operators I_{0+}^{α} and I_{b-}^{α}, $0 < \alpha < 1$, map $\mathcal{H}^*$ one-to-one onto $\mathcal{H}^*_\alpha : I_{0+}^{\alpha}(\mathcal{H}^*) = I_{b-}^{\alpha}(\mathcal{H}^*) = \mathcal{H}^*_\alpha$.

This completes the proof of the theorem. □

Remarks. Let us analyse the result of Theorem 1.

1. The parameter θ can be zero if and only if $A = 0$.

2. If $f = \delta^{(m)} * [\eta],\ m \in \mathbf{N}_0$ and $\eta \in \mathcal{H}^*$, then $D_{0+}^{\beta} f \neq (-1)^k A D_{b-}^{\beta} f$ for every $A,\ (-1)^k A > 0,\ f \neq 0$.

3. There exists $f = \delta^{(m)} * [\eta],\ m \in \mathbf{N}_0$ and $\eta \in \mathcal{H}^*$, such that $D_{0+}^{\beta} f = (-1)^k A D_{b-}^{\beta} f$, for every $A,\ (-1)^k A < 0$ and

$$\eta = \frac{c}{x^{1-\alpha+\theta/2\pi}(b-x)^{1-\theta/2\pi}}\ , 0 < \theta < \alpha 2\pi,$$

where c is an arbitrary constant.

REFERENCES

[1] T. M. Atanacković, B. Stanković, *On a class of differential equations with left and right fractional derivatives,* Z. Angew. Math. Mech., **87,** No 7, (2007), 537–546.

[2] R. Gorenflo, F. Mainardi, *Fractional Calculus, Integral and Differential Equations of Fractional Order.* In fractional Calculus in Continuum Mechanics (Ed-s: A. Carpinteri, F. Mainardi), (Berlin, Springer), 223–276.

[3] R. H i l f e r, *Fractional Calculus and Regular Variation in Thermodinamics,* in Applications of Fractional Calculus in Physics, Editor R. Hilfer, (Singapore, World Scientific), (2000), 429–463.

[4] A. A. K i l b a s, H. M. S r i v a s t a v a, J. I. T r u j i l l o, *Theory and Applications of Fractional Differential Equations,* (Amsterdam, Elsevier), (2006).

[5] I. P o d l u b n y, *Fractional defferential equations,* (San Diego, Academic Press), (1999).

[6] S. G. S a m k o, A. A. K i l b a s, and O. I. M a r i c h e v, *Fractional Integrals and Derivatives,* (Amsterdam, Gordon and Breach), (1993).

[7] L. S c h w a r t z, *Théorie des distributions,* T. **I,** (Paris, Hermann), (1957).

[8] B. S t a n k o v i ć, Equations with left and right fractional derivatives, *Publ. Inst. Math.,* Beograd, T. **80(94)**, (2006), 257–272.

[9] B. S t a n k o v i ć, A method to solve an equation with left and right fractional derivatives, *Bull. Acad. Serbe Sci. Math.* **32,** (2007), 59–67.

[10] V. S. V l a d i m i r o v, *Generalized functions in Mathematical Physics,* (Moscow, Mir Publishers), (1979).

[11] V. S. V l a d i m i r o v, Yu. N. D r o z h i n o v, B. I. Z a v i a l o v, *Tauberian Theorems for Generalized Functions,* (Dordreht, Kluwer Acad. Publishers), (1988).

[12] Z. S z m y d t, *Fourier Transformation and Linear Differential Equations,* (Dordreht, Reidel Publishing Company), (1977).

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