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NORDHAUS-GADDUM-TYPE RELATIONS FOR THE ENERGY AND LAPLACIAN ENERGY OF GRAPHS

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A b s t r a c t. Let $\overline{G}$ denote the complement of the graph G. If $I(G)$ is some invariant of G, then relations (identities, bounds, and similar) pertaining to $I(G) + I(\overline{G})$ are said to be of Nordhaus-Gaddum type. A number of lower and upper bounds of Nordhaus-Gaddum type are obtained for the energy and Laplacian energy of graphs. Also some new relations for the Laplacian graph energy are established.

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1. *Introduction*

In this paper we are concerned with simple graphs. Let G be such a graph, and let n and m denote, respectively, the number of its vertices and edges. Then G is said to be an (n, m)-graph.

The (ordinary) spectrum of G is the spectrum of its adjacency matrix [6], and consists of the numbers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The Laplacian spectrum of G is the spectrum of its Laplacian matrix [10, 11, 21, 22], and consists of the numbers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$.

The *energy* of a graph G, denoted by $E(G)$, is defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \ .$$

This graph-spectrum-based invariant has its origin in theoretical chemistry (for details see [13, 14]), but has recently attracted the interest of mathematicians. The basic mathematical properties of graph energy can be found in the review [12], whereas some most recent mathematical studies in the papers [3, 4, 25–30, 32, 33, 35].

The *Laplacian energy* of a graph G, denoted by $LE(G)$, has been recently defined as [15]

$$LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right|$$

and was aimed at being the Laplacian-spectral analog of graph energy. Until now, only two papers [15, 37] are devoted to the study of Laplacian graph energy.

As usual, $\overline{G}$ will symbolize the complement of the graph G. The number of vertices and edges of the complement of an (n,m)-graph will be denoted by $\overline{n}$ and $\overline{m}$, respectively.

Nordhaus and Gaddum [23] reported bounds for the sum of the chromatic numbers of a graph and its complement. Eventually, Norhhaus-Gaddum-type relations were established for many other graph invariants [1, 2, 5, 8, 9, 16, 17, 20, 31, 34, 36]. In this paper we obtain bounds of this kind for the graph energy and Laplacian graph energy.

2. *Nordhaus-Gaddum-Type Bounds for Graph Energy*

Let $\overline{\lambda_1}$ be the largest eigenvalue of $\overline{G}$. Nosal [24] demonstrated that for a graph G with n vertices,

$$n - 1 \leq \lambda_1 + \overline{\lambda_1} < \sqrt{2}\, n \tag{1}$$

which itself is a Nordhaus-Gaddum-type relation. In connection with the right-hand side inequality in (1), it was shown in [17] that

$$\lambda_1 + \overline{\lambda_1} \leq \sqrt{\left(2 - \frac{1}{\omega} - \frac{1}{\overline{\omega}}\right) n(n-1)}\ , \tag{2}$$

where ω and $\overline{\omega}$ denote the clique numbers of G and $\overline{G}$, respectively.

Theorem 2.1. *Let G be a graph with n vertices. Then*

$$E(G) + E(\overline{G}) \geq 2(n-1) \tag{3}$$

with equality if and only if G is the complete graph K_n or its complement, the empty graph (the n-vertex graph without edges).

P r o o f. We first observe that $E(G) \geq 2\,\lambda_1$ with equality if and only if G has at most one positive eigenvalue, i.e., if G is the empty graph or a complete multipartite graph [6]. Therefore,

$$E(G) + E(\overline{G}) \geq 2(\lambda_1 + \overline{\lambda_1}) \geq 2(n-1)\ .$$

If equality holds in (3), then both G and $\overline{G}$ are empty or complete multipartite graphs, and so G must be the complete graph or the empty graph. Conversely, knowing the spectrum of K_n and $\overline{K_n}$, see [6], it is easily shown that (3) is an equality if $G \cong K_n$ or $G \cong \overline{K_n}$. □

In [19] it was shown that for an (n,m)-graph G,

$$E(G) \leq \lambda_1 + \sqrt{(n-1)\,(2m - \lambda_1^2)}\ . \tag{4}$$

From this upper bound it could be deduced that [18]

$$E(G) \leq \frac{n}{2}\ (\sqrt{n}+1)$$

which immediately implies

$$E(G) + E(\overline{G}) \leq n\ (\sqrt{n}+1)\ .$$

In what follows we improve the latter upper bound.

Theorem 2.2. *Let G be a graph with n vertices. Then*

$$E(G) + E(\overline{G}) < \sqrt{2}\,n + (n-1)\sqrt{n-1}\ . \tag{5}$$

P r o o f. Let m and $\overline{m}$ denote, respectively, the number of edges of G and $\overline{G}$. By (4) and (1), we have

$$E(G) + E(\overline{G}) \ \leq\ \lambda_1 + \overline{\lambda_1} + \sqrt{(n-1)\ (2m - \lambda_1^2)} + \sqrt{(n-1)\ \left(2\overline{m} - \overline{\lambda_1}^2\right)}$$

$$\begin{aligned}
&\le \lambda_1 + \overline{\lambda_1} + \sqrt{2(n-1)\left[2m + 2\overline{m} - \left(\lambda_1^2 + \overline{\lambda_1}^2\right)\right]} \\
&\le \lambda_1 + \overline{\lambda_1} + \sqrt{2(n-1)\left[n(n-1) - \frac{1}{2}\left(\lambda_1 + \overline{\lambda_1}\right)^2\right]} \\
&< \sqrt{2}\,n + \sqrt{2(n-1)\left[n(n-1) - \frac{1}{2}(n-1)^2\right]} \\
&= \sqrt{2}\,n + (n-1)\sqrt{n-1}\ .
\end{aligned}$$

This completes the proof. □

Remark 2.3. Let G be an n-vertex regular graph of degree r. Then (4) becomes $E(G) \le r + \sqrt{(n-1)r(n-r)}$ and we have

$$\begin{aligned}
E(G) + E(\overline{G}) &\le n - 1 + \sqrt{(n-1)}\left[\sqrt{r(n-r)} + \sqrt{(r+1)(n-r-1)}\right] \\
&\le (n-1)\left(\sqrt{n+1} + 1\right)
\end{aligned}$$

which for $n \ge 6$ is better than (5).

Remark 2.4. A strongly regular graph G with parameters (n, r, ρ, σ) is an r-regular graph on n vertices, in which each pair of adjacent vertices has ρ common neighbors and each pair of non-adjacent vertices has σ common neighbors. If $\sigma \ge 1$ and G is non-complete, then the eigenvalues of G are [6] r, s, and t, with multiplicities 1, m_s, and m_t, where s and t are the solutions of the equation $x^2 + (\sigma - \rho)x + (\sigma - r) = 0$, and m_s and m_t are determined by $m_s + m_t = n - 1$ and $r + m_s\, s + m_t\, t = 0$. If G is a strongly regular graph with parameters $(n, (n + \sqrt{n})/2, (n + 2\sqrt{n})/4, (n + 2\sqrt{n})/4)$ (for some conveniently chosen value of n), then

$$E(G) + E(\overline{G}) = \frac{n}{2}\,(\sqrt{n} + 1) + \frac{n}{2}\,(\sqrt{n} + 1) - \sqrt{n} - 2 = (n-1)\,(\sqrt{n} + 1) - 1\ .$$

If we consider a Paley graph H, which is a strongly regular graph with parameters $(n, (n-1)/2, (n-5)/4, (n-1)/4)$, then

$$E(H) + E(\overline{H}) = (n-1)(\sqrt{n} + 1)\ .$$

The results stated in Remark 2.4 show that the bound (5) is asymptotically tight.

Remark 2.5. Using (2), from the proof of Theorem 2.2, we have

$$E(G)+E(\overline{G}) \leq \sqrt{\left(2-\frac{1}{\omega}-\frac{1}{\overline{\omega}}\right) n(n-1)}+(n-1)\sqrt{n-1}\ .$$

3. *Some Properties of the Laplacian Graph Energy*

Details of the theory of Laplacian graph spectra are found in the reviews [10, 11, 21, 22]. For the following consideration we need the properties: $\mu_n = 0$ for all graphs, and $\mu_{n-1} > 0$ if and only if G is connected.

Let $G_1 * G_2$ denote the join of the graphs G_1 and G_2, i.e., the graph obtained from the disjoint union of G_1 and G_2 by adding all possible edges between vertices of G_1 and vertices of G_2.

Theorem 3.1. *Let G_1 and G_2 be (n,m)-graphs. Then*

$$LE(G_1 * G_2) = LE(G_1) + LE(G_2) + 2n - \frac{4m}{n}\ .$$

P r o o f. Let $\mu'_1, \mu'_2, \ldots, \mu'_n$ be the Laplacian eigenvalues of G_1 and $\mu''_1, \mu''_2, \ldots, \mu''_n$ the Laplacian eigenvalues of G_2. Then the Laplacian eigenvalues of $G_1 * G_2$ are [22]

$$2n\ ,\ n+\mu'_1\ ,\ n+\mu''_1\ ,\ n+\mu'_2\ ,\ n+\mu''_2\ ,\ldots,\ n+\mu'_{n-1}\ ,\ n+\mu''_{n-1}\ ,\ 0\ .$$

Note that $G_1 * G_2$ is a $(2n, 2m+n^2)$-graph with average vertex degree $(2m+n^2)/n$. Therefore,

$$\begin{aligned} LE(G_1 * G_2) &= 2n + \sum_{i=1}^{n-1}\left|n+\mu'_i - \frac{2m+n^2}{n}\right| + \sum_{i=1}^{n-1}\left|n+\mu''_i - \frac{2m+n^2}{n}\right| \\ &= 2n + \sum_{i=1}^{n-1}\left|\mu'_i - \frac{2m}{n}\right| + \sum_{i=1}^{n-1}\left|\mu''_i - \frac{2m}{n}\right| \\ &= 2n + LE(G_1) - \frac{2m}{n} + LE(G_2) - \frac{2m}{n}\ . \end{aligned}$$

The result follows. □

Remark 3.2. Let G_1 and G_2 be regular graphs of degrees r' and r'', respectively, with n' and n'' vertices, respectively. Then

$$E(G_1 * G_2) = E(G_1) + E(G_2) + \sqrt{(r' - r'')^2 + 4\,n'\,n''} - r' - r'' \ .$$

Let $G_1 \times G_2$ denote the Cartesian product of graphs G_1 and G_2. Then $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and (u_1, u_2) is adjacent to (v_1, v_2) if and only if $u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$, or $u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$.

Theorem 3.3. *Let G_1 and G_2 be, respectively, (n, m_1)– and (n, m_2)-graphs. Then*

$$LE(G_1 \times G_2) \leq n\,LE(G_1) + n\,LE(G_2) \ .$$

P r o o f. Let the notation be the same as in the proof of Theorem 3.1. Then the Laplacian eigenvalues of $G_1 \times G_2$ are [22]

$$\mu_i' + \mu_j'' \ , \ i, j = 1, 2, \ldots, n \ .$$

Note that $G_1 \times G_2$ is an $(n^2, n(m_1 + m_2))$-graph with average vertex degree $(2m_1 + 2m_2)/n$. Therefore,

$$\begin{aligned} LE(G_1 \times G_2) &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \mu_i' + \mu_j'' - \frac{2\,m_1 + 2\,m_2}{n} \right| \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left| \mu_i' - \frac{2\,m_1}{n} \right| + \left| \mu_j'' - \frac{2\,m_2}{n} \right| \right) \\ &= n\,LE(G_1) + n\,LE(G_2) \ . \end{aligned}$$

The result follows. □

Let G be an (n, m)-graph. Note that $\mu_1 \geq 2m/n$. Then

$$LE(G) = \mu_1 + \sum_{i=2}^{n-1} \left| \mu_i - \frac{2m}{n} \right| \ .$$

If G is not a complete graph, then $\mu_{n-1} \leq 2m/n$ [7], and therefore

$$LE(G) = \mu_1 - \mu_{n-1} + \frac{2m}{n} + \sum_{i=2}^{n-2} \left| \mu_i - \frac{2m}{n} \right| \ .$$

Theorem 3.4. *Let G be an (n,m)-graph with $n \geq 2$ and $m \geq 1$. Then $LE(G) \geq \mu_1$, with equality if and only if $G \cong K_{n/2,n/2}$, in which case, of course, n must be even.*

P r o o f. It is easy to see that $LE(G) \geq \mu_1$, with equality if and only if $n = 2$ or for $n \geq 3$, if $\mu_2 = \cdots = \mu_{n-1} = \frac{2m}{n}$. Suppose that $n \geq 3$ and $LE(G) = \mu_1$. Then by a result from [37], G is a regular complete k-partite graph with $1 < k \leq n$. Then

$$n - \frac{n}{k} + (k-1)\,\frac{n}{k} = n\ ,$$

implying $k = 2$. Thus, $G \cong K_{n/2,n/2}$. Conversely, if $G \cong K_{n/2,n/2}$, then it is easy to verify that $LE(G) = \mu_1$. □

In a similar manner we arrive at

Theorem 3.5. *Let G be an (n,m)-graph, such that $n \geq 3$ and $m \geq 1$. Then*

$$LE(G) \geq \mu_1 - \mu_{n-1} + \frac{2m}{n}$$

with equality if and only if $n = 3$ or for $n \geq 4$, if $\mu_2 = \cdots = \mu_{n-2} = 2m/n$.

4. *Nordhaus-Gaddum-Type Bounds for Laplacian Graph Energy*

Lemma 4.1. *If G is not the complete graph, and has at least one edge, then $\mu_1 - \mu_{n-1} > 1$.*

P r o o f. Since G has at least one edge, $\mu_1 \geq \Delta + 1$, where Δ is the maximum vertex degree of G [10, 21]. If G is connected, then equality holds if and only if $\Delta = n - 1$.

Suppose that G is connected. Then $\mu_1 - \mu_{n-1} \geq \Delta - 2m/n + 1 \geq 1$. If $\mu_1 - \mu_{n-1} = 1$, then $2m/n = \Delta = n - 1$ and then it would be $G \cong K_n$, a contradiction.

If G is not connected, then $\mu_1 - \mu_{n-1} = \mu_1 \geq \Delta + 1 > 1$. □

Theorem 4.2. *Let G be a graph with n vertices. Then*

$$LE(G) + LE(\overline{G}) \geq 2n - 2$$

with equality if and only if G is isomorphic to K_n or $\overline{K_n}$.

P r o o f. If G is isomorphic to K_n or $\overline{K_n}$, then it is easy to show that $LE(G) + LE(\overline{G}) = 2n - 2$. Suppose that $n \geq 3$ and that G is different from

K_n and $\overline{K_n}$. Then

$$\begin{aligned} LE(G)+LE(\overline{G}) &= \mu_1-\mu_{n-1}+\frac{2m}{n}+\sum_{i=2}^{n-2}\left|\mu_i-\frac{2m}{n}\right| \\ &+\mu_1-\mu_{n-1}+\frac{2\,\overline{m}}{n}+\sum_{i=2}^{n-2}\left|n-\mu_i-\frac{2\,\overline{m}}{n}\right| \\ &\geq 2\,(\mu_1-\mu_{n-1})+n-1+\sum_{i=2}^{n-2}1=2\,(\mu_1-\mu_{n-1})+2n-4\,. \end{aligned}$$

By Lemma 4.1, $LE(G)+LE(\overline{G})>2n-2$. □

Theorem 4.3. *Let G be a graph with n vertices. Then*

$$LE(G)+LE(\overline{G})<n\sqrt{n^2-1}\,.$$

P r o o f. Denote by $d_1,d_2,\ldots,d_n$ the vertex degrees of G. Assume that $n\geq 2$. Let the auxiliary quantity M be defined as [15]

$$M=M(G)=m+\frac{1}{2}\sum_{i=1}^{n}\left(d_i-\frac{2m}{n}\right)^2\,.$$

Then

$$M(\overline{G})=\overline{m}+\frac{1}{2}\sum_{i=1}^{n}\left(d_i-\frac{2m}{n}\right)^2\,.$$

Using the fact

$$\sum_{i=1}^{n}(d_i)^2\leq 2(n-1)m$$

with equality if and only if G is the empty graph or the complete graph, we have

$$\begin{aligned} M(G)+M(\overline{G}) &= \frac{1}{2}\,n(n-1)+\sum_{i=1}^{n}\left(d_i-\frac{2m}{n}\right)^2 \\ &= \frac{1}{2}\,n(n-1)+\sum_{i=1}^{n}(d_i)^2-\frac{4m^2}{n} \\ &\leq \frac{1}{2}\,n(n-1)+2(n-1)m-\frac{4m^2}{n} \\ &\leq \frac{1}{2}\,n(n-1)+\frac{1}{4}\,n(n-1)^2=\frac{1}{4}\,(n-1)n(n+1)\,. \end{aligned}$$

Now, because for $n \geq 2$ the number of edges of K_n and $\overline{K_n}$ differs from $n(n-1)/4$, we have

$$M(G) + M(\overline{G}) < \frac{1}{4}\,(n-1)n(n+1)\ . \tag{6}$$

In [15] it has been shown that $LE(G) \leq \sqrt{2nM}$, which combined with (6) implies

$$LE(G) + LE(\overline{G}) \leq \sqrt{4n\left[M(G) + M(\overline{G})\right]} < n\,\sqrt{n^2-1}\ . \qquad \square$$

Example 4.4. Let $G \cong K_{n/2} \cup \overline{K_{n/2}}$. Then the Laplacian eigenvalues of G are

$$\frac{n}{2}\ \left(\frac{n}{2} - 1 \text{ times}\right) \text{ and } 0\ \left(\frac{n}{2} + 1 \text{ times}\right)$$

and therefore

$$LE(G) = \left(\frac{n}{2} - 1\right)\frac{n+2}{4} + \left(\frac{n}{2} + 1\right)\frac{n-2}{4} = \frac{1}{4}\,(n^2 - 4)\ .$$

The Laplacian eigenvalues of $\overline{G}$ are

$$n\ \left(\frac{n}{2} \text{ times}\right)\ , \frac{n}{2}\ \left(\frac{n}{2} - 1 \text{ times}\right) \text{ and } 0 \ (1 \text{ time })$$

and therefore

$$LE(\overline{G}) = \frac{n}{2}\,\frac{n+2}{4} + \left(\frac{n}{2} - 1\right)\frac{n-2}{4} + \frac{3n-2}{4} = \frac{1}{4}\,(n^2 + 2n)\ .$$

This implies

$$LE(G) + LE(\overline{G}) = \frac{1}{2}\,(n^2 + n - 2)\ .$$

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REFERENCES

[1] N. A c h u t h a n, N. R. A c h u t h a n, L. Caccetta, *On the Nordhaus-Gaddum class problem*, Australas. J. Combin. **2** (1990) 5–27.

[2] Y. A l a v i, M. B e h z a r d, *Complementary graphs and edge chromatic numbers*, SIAM J. Appl. Math. **20** (1971) 161–163.

[3] R. B a l a k r i s h n a n, *The energy of a graph*, Lin. Algebra Appl. **387** (2004) 287–295.

[4] R. B. B a p a t, S. P a t i, *Energy of a graph is never an odd integer*, Bull. Kerala Math. Assoc. **1** (2004) 129–132.

[5] G. C h a r t r a n d, S. S c h u s t e r, *On the independence numbers of complementary graphs*, Trans. New York Acad. Sci. Ser. II **36** (1974) 247–251.

[6] D. C v e t k o v i ć, M. D o o b, H. S a c h s, *Spectra of Graphs – Theory and Application*, Academic Press, New York, 1980; 2nd revised ed.: Barth, Heidelberg, 1995.

[7] M. F i e d l e r, *Algebraic connectivity of graphs*, Czechoslovak Math. J. **25** (1975) 607–618.

[8] W. G o d d a r d, M. A. H e n n i n g, *Nordhaus–Gaddum bounds for independent domination*, Discr. Math. **268** (2003) 299–302.

[9] W. G o d d a r d, M. A. H e n n i n g, H. C. S w a r t, *Some Nordhaus–Gaddum–type results*, J. Graph Theory **16** (1992) 221–231.

[10] R. G r o n e, R. M e r r i s, *The Laplacian spectrum of a graph II*, SIAM J. Discr. Math. **7** (1994) 221–229.

[11] R. G r o n e, R. M e r r i s, V. S. S u n d e r, *The Laplacian spectrum of a graph*, SIAM J. Matrix Anal. Appl. **11** (1990) 218–238.

[12] I. G u t m a n, *The energy of a graph: old and new results*, in: A. Betten, A. Kohnert, R. Laue, A. Wassermann (Eds.), *Algebraic Combinatorics and Applications*, Springer–Verlag, Berlin, 2001, pp. 196–211.

[13] I. G u t m a n, *Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology*, J. Serb. Chem. Soc. **70** (2005) 441–456.

[14] I. G u t m a n, O. E. P o l a n s k y, *Mathematical Concepts in Organic Chemistry*, Springer–Verlag, Berlin, 1986.

[15] I. G u t m a n, B. Z h o u, *Laplacian energy of a graph*, Lin. Algebra Appl. **414** (2006) 29–37.

[16] F. H a r a r y, T. W. H a y n e s, *Nordhaus–Gaddum inequalities for domination in graphs*, Discr. Math. **155** (1996) 99–105.

[17] Y. H o n g, J. S h u, *A sharp upper bound for the spectral radius of the Nordhas–Gaddum type*, Discr. Math. **211** (2000) 229–232.

[18] J. H. K o o l e n, V. M o u l t o n, *Maximal energy graphs*, Adv. Appl. Math. **26** (2001) 47–52.

[19] J. H. K o o l e n, V. M o u l t o n, I. G u t m a n, *Improving the McCelland inequality for total π-electron energy*, Chem. Phys. Lett. **320** (2000) 213–216.

[20] H. L i u, M. L u, F. T i a n, *On the ordering of trees with the general Randić index of the Nordhaus–Gaddum type*, MATCH Commun. Math. Comput. Chem. **55** (2006) 419–426.

[21] R. M e r r i s, *Laplacian matrices of graphs: A survey*, Lin. Algebra Appl. **197–198** (1994) 143–176.

[22] B. M o h a r, *The Laplacian spectrum of graphs*, in: Y. Alavi, G. Chartrand, O. R. Oellermann, A. J. Schwenk (Eds.), *Graph Theory, Combinatorics, and Applications*, Wiley, New York, 1991, pp. 871–898.

[23] E. A. N o r d h a u s, J. W. G a d d u m, *On complementary graphs*, Amer. Math. Monthly **63** (1956) 175–177.

[24] E. N o s a l, *Eigenvalues of Graphs*, M. Sc. thesis, Univ. Calgary, 1970.

[25] J. R a d a, *Energy ordering of catacondensed hexagonal systems*, Discr. Appl. Math. **145** (2005) 437–443.

[26] J. R a d a, A. T i n e o, *Upper and lower bounds for energy of bipartite graphs*, J. Math. Anal. Appl. **289** (2004) 446–455.

[27] H. S. R a m a n e, I. G u t m a n, H. B. W a l i k a r, S. B. H a l k a r n i, *Another class of equienergetic graphs*, Kragujevac J. Math. **26** (2004) 15–18.

[28] H. S. R a m a n e, H. B. W a l i k a r, S. B. R a o, B. D. A c h a r y a, P. R. H a m p i h o l i, S. R. J o g, I. G u t m a n, *Equienergetic graphs*, Kragujevac J. Math. **26** (2004) 5–13.

[29] H. S. R a m a n e, H. B. W a l i k a r, S. B. R a o, B. D. A c h a r y a, P. R. H a m p i h o l i, S. R. J o g, I. G u t m a n, *Spectra and energies of iterated line graphs of regular graphs*, Appl. Math. Lett. **18** (2005) 679–682.

[30] I. S h p a r l i n s k i, *On the energy of some circulant graphs*, Lin. Algebra Appl. **414** (2006) 378–382.

[31] M. S t e n c e l, *Theorems of Nordhaus–Gaddum type for some numbers*, Discuss. Math. **7** (1985) 107–112.

[32] D. S t e v a n o v i ć, *Energy and NEPS of graphs*, Lin. Multilin. Algebra **53** (2005) 67–74.

[33] D. S t e v a n o v i ć, I. S t a n k o v i ć, *Remarks on hyperenergetic circulant graphs*, Lin. Algebra Appl. **400** (2005) 345–348.

[34] M. S t i e b i t z, *On Hadwiger's number – a problem of the Nordhaus–Gaddum type*, Discr. Math. **101** (1992) 307–317.

[35] W. Y a n, L. Y e, *On the minimal energy of trees with a given diameter*, Appl. Math. Lett. **18** (2005) 1046–1052.

[36] L. Z h a n g, B. W u, *The Nordhaus–Gaddum–type inequalities for some chemical indices*, MATCH Commun. Math. Comput. Chem. **54** (2005) 189–194.

[37] B. Z h o u, I. G u t m a n, *On Laplacian energy of graphs*, MATCH Commun. Math. Comput. Chem. **57** (2007) 211–220.

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