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ON THE COMPRESSED ELASTIC ROD WITH ROTARY INERTIA ON A VISCOELASTIC FOUNDATION

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A b s t r a c t. We study transversal vibrations of an elastic axially compressed rod on a fractional derivative type of viscoelastic foundation. We assume that the axial force has a constant and a time dependent part given by Dirac distributions. The dynamics of the system is described by a system of two partial differential equations, having integer and fractional derivatives. The solution of this system is obtained in the space of distributions and its asymptotic behavior is investigated. It is shown that the foundation increases the stability bound.

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1. *Formulation of the Problem*

Consider an elastic rod simply supported at both ends. Let L be the length of the rod. We assume that the rod is loaded by an axial force $\overline{P}\,(t)$ that is a known function of time and has a fixed direction coinciding with the rod axis in the initial (undeformed) state. In this work we shall generalize our previous analysis [1] in two directions. First, we assume here that the axial force is given by generalized functions and second we assume that the rod has a (not negligible) rotary inertia. Thus we allow for impulsive (described by a Dirac distribution) axial loading of the rod. The rod is positioned on a viscoelastic foundation described by a fractionl derivative type constitutive equation (see Figure 1).

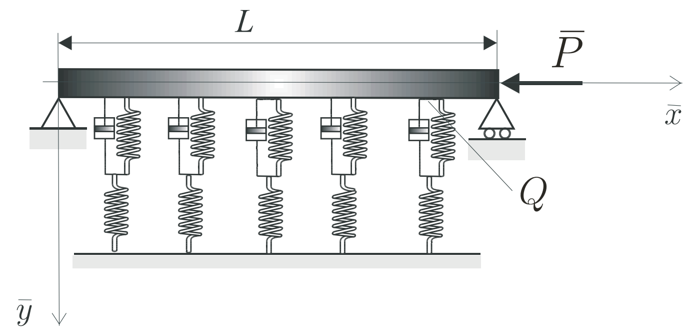

Figure 1. Coordinate system and load configuration

Such foundations are important for vibration damping and have been recently analyzed in the context of the railpad in the railway track model (see [2] and [3] for the physical explanation of the model). The equilibrium equations of active and inertial forces are (see [4] p. 338)

$$\begin{aligned}
&\frac{\partial H}{\partial S} = \rho_0 \frac{\partial^2 x}{\partial t^2} - q_x, \quad \frac{\partial V}{\partial S} = \rho_0 \frac{\partial^2 y}{\partial t^2} - q_y, \\
&\frac{\partial M}{\partial S} = -V \cos\vartheta + H \sin\vartheta - J \frac{\partial^2 \vartheta}{\partial t^2}, \\
&\frac{\partial x}{\partial S} = \cos\vartheta, \quad \frac{\partial y}{\partial S} = \sin\vartheta, \\
&\frac{\partial \vartheta}{\partial S} = \frac{M}{EI}, \quad S \in (0, L)\,, \quad t \geq 0,
\end{aligned} \tag{1}$$

where x and y are coordinates of an arbitrary point on the rod axis, S is the arc length of the rod axis in the undeformed state so that $S \in (0, L)$, t is the time, H and V are components of the force in an arbitrary cross–section of the rod along the $\bar{x}$ and $\bar{y}$ axes of a rectangular Cartesian coordinate system $\bar{x} - B - \bar{y}$, respectively, M is the bending moment, and q_x, q_y are the intensities of the distributed forces per unit length of the rod axis in the undeformed state, J is moment of inertia of a part of the rod of unit length, ϑ is the angle between the tangent to the rod axis and the $\bar{x}$ axis, ρ_0 is the line density of the rod, EI is the bending rigidity of the rod. For the rod shown in Figure 1 the boundary conditions are

$$\begin{aligned} &y(0,t)=0, \quad x(0,t)=0; \quad y(L,t)=0, \\ &M(0,t)=0, \quad M(L,t)=0; \quad H(L,t)=-\overline{P}, \quad t\geq 0. \end{aligned} \tag{2}$$

Suppose that the rod is positioned on a viscoelastic foundation. We assume that the foundation is of the fractional derivative type. If the foundation is made of a fractional type viscoelastic material, then the force in the foundation Q and deformation Δ of the foundation (in our case $\Delta = y$) are connected as

$$Q+\tau_Q Q^{(\beta)} = E_p\left(y+\tau_y y^{(\beta)}\right), \tag{3}$$

with $0<\beta<1$. In (3) we used $(\cdot)^{(\beta)}$ to denote the β-th derivative of a function $(\cdot)$ taken in Riemann-Liouville form as (see [5], and [6])

$$g^{(\beta)} \equiv \frac{d^\beta}{dt^\beta} g(t) \equiv \frac{d}{dt}\frac{1}{\Gamma(1-\beta)}\int_0^t \frac{g(\xi)\,d\xi}{(t-\xi)^\beta}. \tag{4}$$

The dimension of the constants τ_y and τ_Q is $[\text{time}]^\alpha$. The constants E_p, τ_Q and τ_y in (3) are called the instantaneous modulus of the pad and the relaxation times, respectively. In Figure 1 the rheological model of the foundation is presented, as given in [7], for example. We assume that the following inequality, as a consequence of the second law of thermodynamics, is satisfied (see [9] and [8])

$$E>0, \qquad \tau_Q>0, \qquad \tau_y>\tau_Q. \tag{5}$$

We assume that the pads are positioned under the rod so that

$$q_x = 0, \quad q_y = -bQ, \tag{6}$$

where b is a constant depending on the part of the rod's width that is supported by pads. Note that in the case $\beta = 1$ the foundation becomes a standard viscoelastic solid.

The trivial solution to the system (1),(2),(3) and (5) in which the rod axis is straight reads

$$\begin{aligned} &H^0(S,t)=-\overline{P}, \quad V^0(S,t)=0, \quad M^0(S,t)=0, \;\; x^0(S,t)=S, \\ &y^0(S,t)=0, \;\; \vartheta^0(S,t)=0, \quad Q^0(S,t)=0. \end{aligned} \tag{7}$$

Let the solution to (1),(2),(3) and (5) be written in the form $H = H^0 + \Delta H, ...Q = Q^0 + \Delta Q$, where $\Delta H, ..., \Delta Q$ are perturbations, assumed to be small. By substituting this in (7) and neglecting the higher order terms in the perturbations $\Delta H, ..., \Delta Q$, we obtain

$$
\begin{aligned}
&\frac{\partial \Delta H}{\partial S} = \rho_0 \frac{\partial^2 \Delta x}{\partial t^2}, \quad \frac{\partial \Delta V}{\partial S} = \rho_0 \frac{\partial^2 \Delta y}{\partial t^2} + b\Delta Q, \\
&\frac{\partial \Delta M}{\partial S} = -\Delta V - \overline{P}\Delta\vartheta - J\frac{\partial^2 \Delta\theta}{\partial t^2}, \quad \frac{\partial \Delta x}{\partial S} = 0, \quad \frac{\partial \Delta y}{\partial S} = \Delta\vartheta, \\
&\frac{\partial \Delta\vartheta}{\partial S} = \frac{\Delta M}{EI}, \quad \Delta Q + \tau_Q \Delta Q^{(\alpha)} = E_p\left(\Delta y + \tau_y \Delta y^{(\alpha)}\right),
\end{aligned}
\tag{8}
$$

subject to

$$
\begin{aligned}
&\Delta H\,(L,t) = 0, \quad \Delta M(0,t) = 0, \qquad \Delta M(L,t) = 0, \\
&\Delta x\,(0,t) = 0, \quad \Delta y(0,t) = 0, \qquad \Delta y(L,t) = 0.
\end{aligned}
\tag{9}
$$

Introducing the dimensionless quantities

$$
\begin{aligned}
&\lambda = \frac{\overline{P}L^2}{EI}, \quad \tau = \frac{t}{\sqrt{\frac{\rho_0 L^4}{EI}}}, \quad \mu = \sqrt{\frac{EAL^2}{EI}}, \quad u = \frac{\Delta y}{l}, \\
&\xi = \frac{S}{L}, \quad \gamma = b\frac{E_p L^4}{EI}, \quad F = \frac{\Delta Q}{LE_p}, \quad \alpha = \frac{J}{\rho_0 L^2}, \\
&\tau_q = \tau_Q\left(\frac{EI}{\rho_0 L^4}\right)^{\alpha/2}, \quad \tau_u = \tau_y\left(\frac{EI}{\rho_0 L^4}\right)^{\alpha/2}.
\end{aligned}
\tag{10}
$$

From (8),(9) we obtain

$$
\frac{\partial^4 u}{\partial \xi^4} - \alpha\frac{\partial^4 u}{\partial \xi^2 \partial \tau^2} + \lambda\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \tau^2} + \gamma F = 0,
$$

$$
F + \tau_q F^{(\beta)} = u + \tau_u u^{(\beta)}. \tag{11}
$$

where $F^{(\beta)} = \frac{d}{d\tau}\frac{1}{\Gamma(1-\beta)}\int_0^\tau \frac{F(\xi)d\xi}{(\tau-\xi)^\beta}$ and

$$
u\,(0,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}\,(0,\tau) = 0, \quad u\,(1,\tau) = 0, \quad \frac{\partial^2 u}{\partial \xi^2}\,(1,\tau) = 0. \tag{12}
$$

Note that $(5)_{2,3}$ imply

$$
\tau_u > \tau_q > 0. \tag{13}
$$

Suppose that the solution to (11),(12) is assumed to be of the form

$$u(\xi,\tau)=T_k(\tau)\sin k\pi\xi, \quad F(\xi,\tau)=S_k(\tau)\sin k\pi\xi, \quad k\in\mathbb{N}. \tag{14}$$

where T_k and S_k satisfy the system

$$\begin{aligned}(\alpha(\pi k)^2+1)T_k^{(2)}(\tau)+(\pi k)^2((\pi k)^2-\lambda)T_k(\tau)+\gamma S_k(\tau)=0,\\ aS_k^{(\beta)}(\tau)+S_k(\tau)=bT_k^{(\beta)}(\tau)+T_k(\tau),\ \ \tau>0,\end{aligned} \tag{15}$$

for every $k\in\mathbb{N}$, where $a=\tau_q, b=\tau_u; 0<a<b$ is a consequence of the Second law of the thermodynamics. In this paper we look for solutions, classical and generalized, to system (11), (12) in the form (14). Thus we shall study the system (15). The problem of existence or nonexistence of other solutions to (15) we do not treat here and is reported elsewhere [10].

2. *Mathematical preliminaries*

We repeat some definitions and facts that we need in our method of solving system (11), which are related to the space of distributions and to the Laplace transform of distributions (cf. [18],[19]).

Let Ω denote an open subset of $\mathbb{R}^n$ (Ω can be $\mathbb{R}^n$ on the whole).

The *support of a function* φ defined on Ω (suppφ) is the closure in Ω of the set $\{x\in\Omega;\ \varphi(x)\neq 0\}$.

The space $\mathcal{D}(\Omega)$ is the space $\{\varphi\in\mathcal{C}^\infty(\mathbb{R}^n);\ \mathrm{supp}\varphi\subset\Omega\}$. A sequence $\varphi_j\in\mathcal{D}(\Omega)$ converges in $\mathcal{D}(\Omega)$ to zero if and only if there exists the compact set $K\subset\Omega:$

1. $\mathrm{supp}\varphi_\mathrm{j}\subset \mathrm{K},\ \ \mathrm{j}\in\mathbb{N};$
2. for every $\alpha=(\alpha_1,...,\alpha_n)\in(\mathbb{N}\cup\{0\})^n\equiv\mathbb{N}_0^n$, $\varphi_j^{(\alpha)}\to 0$ uniformly on K;

$$\varphi_j^{(\alpha)}=\left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\right)\varphi_j.$$

$\mathcal{D}'(\Omega)$ is the space of all continuous linear functionals on $\mathcal{D}(\Omega)$. It is called *the space of distributions* on Ω. The value of a distribution f at a function $\varphi\in\mathcal{D}'(\Omega)$ will be denoted by $\langle f,\varphi\rangle$. The support of a distribution f is the least closed set D such that $\langle f,\varphi\rangle=0$ for all $\varphi\in\mathcal{D}(\mathbb{R}^n\setminus D)$. Let $\mathcal{D}'_+$ denote: $\mathcal{D}'_+=\{f\in\mathcal{D}'(\mathbb{R});\mathrm{suppf}\subset[0,\infty)\}$.

Every locally integrable function f on Ω defines the *regular distribution* $[f]$, $\langle[f],\varphi\rangle=\int\limits_\Omega f(x)\varphi(x)dx,\ \varphi\in\mathcal{D}(\Omega)$. Two functions $f,g\in\mathcal{L}^1_{loc}(\Omega)$ define the same distribution $[f]=[g]$ on Ω if and only if $f=g$ a.e. on Ω.

We list some properties of the derivatives of distributions:

1. Every distribution has all derivatives D^{α_i} and $D^{\alpha_i}D^{\alpha_j} = D^{\alpha_j}D^{\alpha_i}$, $i, j = 1, ..., n$.

2. The differentiation of distributions is a linear and continuous mapping $\mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$.

3. In particular, every regular distribution has derivatives of any order. In this sense every locally integrable function has distributional derivatives. The derivative of a regular distribution is not necessarily a regular distribution.

4. If $F \in \mathcal{C}^{\alpha}(\Omega),\ \alpha = (\alpha_1, ..., \alpha_n)$, then $D^{\alpha}[F] = [F^{(\alpha)}]$.

5. Let $f \in \mathcal{C}^{(p)}((-\infty, b)),\ p \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and θ_a be a function such that $\theta_a(x) = 0,\ -\infty < x < a < b;\ \theta_a(x) = 1,\ 0 \le a \le x < b$. Denote by $[\theta_a f]$ the regular distribution defined by $\theta_a f$. Hence, $[\theta_a f] \in \mathcal{D}'((-\infty, b)),\ \mathrm{supp}[\theta_\mathrm{a}\mathrm{f}] \subset [\mathrm{a}, \mathrm{b})$ or $[\theta_a f] \in \mathcal{D}'([a, b))$, as well. By $[f_*^{(p)}],\ p \in \mathbb{N}$, we denote the distribution defined by the function $f_*^{(p)}$ equal to $f^{(p)}(x),\ x \in (a, b)$ and equal to zero for $x \in (-\infty, a)$ and is not defined for $x = a$.

Since the function $(\theta_a f)^{(k)}$ has in general a discontinuity of the first kind for $x = a,\ k = 0, 1, ..., p$, by the well-known formula (cf. [18])

$$D^p[\theta_a f] = [f_*^{(p)}] + f^{(p-1)}(a)\delta(x - a) + ... + f(a)\delta^{(p-1)}(x - a).$$

An important subspace of $\mathcal{D}'(\mathbb{R}^n)$ is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. Let us define it.

By $\mathcal{S}(\mathbb{R}^n)$ we denote the *space of rapidly decreasing functions* φ with the property that for every pair of multi–indices $\alpha, \beta \in \mathbb{N}_0^n$, $\sup\limits_{x \in \mathbb{R}^n} |x^{\alpha}\varphi^{(\beta)}(x)| < \infty$.

The space of linear continuous functionals on $\mathcal{S}(\mathbb{R}^n)$ is called the *space of tempered distributions* and is denoted by $\mathcal{S}'(\mathbb{R}^n)$.

We will use the Laplace transform of a subspace of distributions, namely of the space $e^{\sigma t}\mathcal{S}'(\overline{\mathbb{R}}_+),\ \sigma \ge 0$. For an $f \in e^{\sigma t}S'(\overline{\mathbb{R}}_+),\ f(t) = e^{\sigma t}g(t)$ the Laplace transform, denoted by $\mathcal{L}$ is $\mathcal{L}(f)(s) \stackrel{\mathrm{def}}{=} \langle g(t), e^{-(s-\sigma)t}\rangle,\ \mathrm{Re}\, s > \sigma$ (cf. [19]). The formalism we use in applications of the so defined Laplace transform to differential equations is just the same as for the classical Laplace transform. Moreover, if a function $\widehat{F}(s)$ is the classical Laplace transform of $F(t) = e^{\sigma t}G(t)$, such that $[G] \in \mathcal{S}'(\overline{\mathbb{R}}_+)$, then $\widehat{F}(s) = \mathcal{L}([F])(s)$, as well.

3. *Solutions for the case* $\lambda = A + B\delta(\tau - \tau_0),\ \tau_0 > 0$

3.1. *Existence and character of the solution*

In order for system (15) to have a meaning, $T_k(\tau),\ k \in \mathbb{N}$, has to be defined by a continuous function on $(0,\infty)$. Then δ can be regarded as a measure and we have that the second term on the left hand side of (15) becomes

$$(\pi k)^2((\pi k)^2-\lambda)T_k(\tau) = (\pi k)^2((\pi k)^2-A)T_k(\tau)-(\pi k)^2 BT_k(\tau_0)\delta(\tau-\tau_0),\ k \in \mathbb{N}.$$

If the function $T_k(\tau)$ does not have classical derivatives equations have to be satisfied with the derivatives in the sense of distributions. Therefore we construct the system in $\mathcal{D}'([0,\infty))$ which corresponds to system (15). First we use the relation

$$D^2\left[T_k\right] = \left[\left(T_k^{(2)}\right)_*\right] + T_k^{(1)}(0)\,\delta(\tau) + T_k(0)\,\delta^{(1)}(\tau)$$

If we introduce the notation $T_k^{(1)}(0) \equiv T_{k0}^1$ and $T_k(0) \equiv T_{k0}$ into (15) it corresponds in $\mathcal{D}'([0,\infty))$ to the following equation

$$\begin{aligned}(\alpha(\pi k)^2+1)D^2[T_k] + (\pi k)^2((\pi k)^2 - A)[T_k] + \gamma[S_k]&\\ = (\pi k)^2 BT_k(\tau_0)\delta(\tau-\tau_0) + T_{k0}^1\delta(\tau) + T_{k0}\delta^{(1)}(\tau)&\\ aD^\beta[S_k] + [S_k] = bD^\beta[T_k] + [T_k].&\end{aligned} \tag{16}$$

Applying the LT to (16) and supposing that for every $k \in \mathbb{N}$, $T_k(t)$ and $S_k(\tau)$ are bounded on $[0,\eta]$ for an $\eta > 0$, we have

$$\begin{aligned}\left[(\alpha(\pi k)^2+1)s^2 + (\pi k)^2((\pi k)^2 - A)\right]\widehat{T}_k(s) + \gamma\widehat{S}_k(s)&\\ = (k\pi)^2 BT_k(\tau_0)e^{-s\tau_0} + (\alpha(\pi k)^2+1)(T_{k0}s + T_{k0}^1)&\\ (bs^\beta+1)\widehat{T}_k(s) - (as^\beta+1)\widehat{S}_k(s) = 0, \quad k \in \mathbb{N}\,.&\end{aligned} \tag{17}$$

To solve linear algebraic system (17) we have to find the determinants Δ_{0k}, Δ_{1k} and Δ_{2k}. To shorten the form of $\Delta_{ik},\ i = 0,1,2$, we introduce the following notations

$$\begin{aligned}&M = (\alpha(\pi k)^2+1), \qquad N = a(\pi k)^2((\pi k)^2 - A) + \gamma b,\\ &P = (\pi k)^2((\pi k)^2 - A) + \gamma.\end{aligned} \tag{18}$$

It is easily seen that $M > 0$ and $N = aP + (b-a)\gamma$. Consequently if $P \geq 0$, then $N > 0$.

Now, we have

$$\Delta_{0k} = -(aMs^{2+\beta} + Ms^2 + Ns^\beta + P) \equiv -\Delta^*_{0k}, \tag{19}$$

and

$$\Delta_{1k} = -(s^\beta + \frac{1}{a})(aMT_{k0}s + aMT^1_{k0} + a(k\pi)^2 BT_k(t_0)e^{-s\tau_0}), \tag{20}$$

$$\Delta_{2k} = -(s^\beta + \frac{1}{b})(bMT_{k0}s + bMT^1_{k0} + b(k\pi)^2 BT_k(t_0)e^{-s\tau_0}). \tag{21}$$

The solutions to (17) for $k \in \mathbb{N}$ are:

$$\widehat{T}_k(s) = \frac{s^\beta + \frac{1}{a}}{\Delta^*_{0k}(s)}\left(aMT_{k0}s + aMT^1_{k0} + a(k\pi)^2 BT_k(t_0)e^{-s\tau_0}\right), \tag{22}$$

and

$$\widehat{S}_k(s) = \frac{s^\beta + \frac{1}{b}}{\Delta^*_{0k}(s)}\left(bMT_{k0}s + bMT^1_{k0} + b(k\pi)^2 BT_k(t_0)e^{-s\tau_0}\right), \tag{23}$$

where Δ^*_{0k} is given by (19).

Let us remark that the domain of the analycity of $\widehat{T}_k(s)$ and $\widehat{S}_k(s)$ depends on the numbers of s such that $\Delta^*_{0k}(s) = 0$. For such numbers cf. Section 5.

To analyze the existence of $T_k(t)$ and $S_k(t)$ such that $\mathcal{L}(T_k)(s) = \widehat{T}_k(s)$ and $\mathcal{L}(S_k)(s) = \widehat{S}_k(s)$, where $\widehat{T}_k(s)$ and $\widehat{S}_k(s)$ are given by (22) and (23), respectively, we give another form to $\widehat{T}_k(s)$ and $\widehat{S}_k(s)$using elementary algebraic operations

$$\begin{aligned}\widehat{T}_k(s) \quad &= T_{k0}\left(\frac{N}{aM}\frac{Ns^\beta + P}{s^3\Delta^*_{k0}(s)} + \frac{(b-a)\gamma}{a}\frac{1}{s\Delta^*_{k0}(s)}\right. \\ &\left. -\frac{N}{aMs^3} + \frac{1}{s}\right) + T^1_{k0}\left(\frac{-Ns^\beta - P}{s^2\Delta^*_{k0}(s)} + \frac{1}{s^2}\right) \\ &+\frac{(k\pi)^2}{M}BT_k(t_0)e^{-st_0}\left(\frac{-Ns^\beta - P}{s^2\Delta^*_{k0}(s)} + \frac{1}{s^2}\right),\end{aligned} \tag{24}$$

and

$$
\begin{aligned}
\widehat{S}_k(s) &= \frac{b}{a}\widehat{T}_k(s) - b\left(\frac{1}{a}-\frac{1}{b}\right)\frac{MT_{k0}s + MT_{k0}^1 + (k\pi)^2 BT_k(t_0)e^{-st_0}}{\Delta_{k0}^*(s)} \\
&= \frac{b}{a}\widehat{T}_k(s) - b\left(\frac{1}{a}-\frac{1}{b}\right)\left[MT_{k0}\left(\frac{1}{aMs^{1+\beta}} - \frac{Ms^2+Ns^\beta+P}{aMs^{1+\beta}\Delta_{k0}^*(s)}\right)\right. \\
&\quad \left. + \frac{MT_{k0}^1 + (k\pi)^2 BT_k(t_0)\, e^{-st_0}}{\Delta_{k0}^*(s)}\right].
\end{aligned} \tag{25}
$$

Let us consider first $\widehat{T}_k(s)$. We have only to prove that there exist $\phi_i(\tau),\ i=2,3$ and $\phi_1(\tau)$ such that $\mathcal{L}(\phi_i)(s) = \dfrac{Ns^\beta+P}{s^i\Delta_{0k}^*}$, $i=2,3$ and $\mathcal{L}(\phi_1)(s) = \dfrac{1}{s\Delta_{0k}^*(s)}$. It is easily seen by Theorem 3 and Theorem 5 in [11], part I, ch. 7, § 2, that such continuous functions exist for $\tau \geq 0$. Consequently, for $\tau \geq 0$ and $k \in \mathbb{N}$,

$$
\begin{aligned}
T_k(\tau) &= T_{k0}\frac{N}{aM}\phi_3(\tau) + \frac{b-a}{a}\gamma\,\phi_1(\tau) - \frac{N}{2aM}\tau^2 + 1 \\
&\quad - T_{k0}^1\left(\phi_2(\tau)-\tau\right) - \frac{(k\pi)^2 B}{M}T_k(\tau_0)H(\tau-\tau_0)\left(\phi_2(\tau-\tau_0)-(\tau-\tau_0)\right),
\end{aligned} \tag{26}
$$

where H is Heaviside's function.

With regard to $\widehat{S}_k(s)$, given by (25), we need to prove the existence of a new continuous function ϕ_4, beside $\phi_i,\ i=1,2,3$, such that

$$
\mathcal{L}(\phi_4)(s) = \frac{Ms^2+Ns^\beta+P}{aMs^{1+\beta}\Delta_{0k}^*(s)}.
$$

But this follows just from the same two theorems we cited from [11]. Consequently, we have for $\tau \geq 0$ and $k \in \mathbb{N}$:

$$
\begin{aligned}
S_k(\tau) &= \frac{b}{a}T_k(\tau) - b\left(\frac{1}{a}-\frac{1}{b}\right)\left[MT_{k0}\left(\frac{\tau^\beta}{aM\Gamma(1+\beta)} - \phi_4(\tau)\right)\right. \\
&\quad \left. + MT_{k0}^1\phi_1(\tau) + (k\pi)^2 BT_k(\tau_0)H(\tau-\tau_0)\phi_1(\tau-\tau_0)\right].
\end{aligned} \tag{27}
$$

The next step is to analyze the character of the found solution given by (26), (27), with respect to system (15).

1) *If* $B = 0$ *the solution is a classical one and it can be obtained by the classical LT.*

To prove this we use the relation between the classical LT and integration (cf. [11], II, p. 18]): If there exist $F_i(\tau),\ i = 1, 2, 3$, such that

$$\mathcal{L}(F_i)(s) = \frac{s(Ns^\beta + P)}{s^i \Delta_{0k}^*(s)}, \quad i = 2, 3, \quad \mathcal{L}(F_1)(s) = \frac{1}{\Delta_{0k}^*(s)}$$

converge for an $s_0, \operatorname{Re} s_0 > 0$, then there exist

$$\mathcal{L}(\int_0^t F_1(\tau)d\tau)(s) = \frac{1}{s\Delta_{0k}^*}, \qquad \mathcal{L}(\int_0^t F_i(\tau)d\tau)(s) = \frac{Ns^\beta + P}{s^i \Delta_{0k}^*(s)}, \ \ i = 2, 3$$

By mentioned Theorem 3 in [11] I, ch. 7, § 2 $F_i,\ i = 1, 2, 3$, exist. By the uniqueness of the inverse LT, we have that $\phi_i(\tau) = \int_0^\tau F_i(t)dt$. Consequently, $\phi_i(\tau) \to 0, \tau \to 0$. If we repeat the same operation, then we obtain that $F_i(\tau) = \int_0^\tau \psi_i(t)dt$ and $\phi_i(\tau) = \int_0^\tau \int_0^t \psi_i(u)du\, dt,\ i = 1, 2, 3.$

In such a way we prove that every $\phi_i(\tau),\ i = 1, 2, 3$, belongs to $\mathcal{C}^{(2)}((0, \infty))$ and that $\phi_i(\tau) \to 0, \tau \to 0$. Now it is easily seen that $T_k \in \mathcal{C}([0, \infty)) \cap \mathcal{C}^{(2)}((0, \infty)),\ k \in \mathbb{N}$, as well and that $S_k(\tau) \in \mathcal{C}([0, \infty)),\ k \in \mathbb{N}$. The proved properties of T_k and S_k guarantee that they are a classical solution to (15) which can be obtained by the classical LT.

Remark. Let us notice that in the case $B = 0$ system (16) represents the case when λ is only a constant, $\lambda = A$. Consequently, if we take in (26), (27) $B = 0$, then these T_k and S_k constitute a solution to system (11), with conditions (12) and with $\lambda = A$ (cf. (14)).

2) *If* $B \neq 0$, *then we have a generalized solution to (11), (12), defined by continuous function (regular distributions).*

The problem is concentrated in the last term in the expression for $T_k(\tau)$ (cf. (26)) and at the point $\tau = \tau_0$. The function $H(\tau - \tau_0)\,(\phi_2(\tau - \tau_0) - (\tau - \tau_0))$ is continuous because $\phi_2(\tau) \to 0, \tau \to 0^+$, but the derivative of $H(\tau-\tau_0)[\phi_2(\tau-\tau_0)-(\tau-\tau_0)]$ does not tend to zero when $\tau \to \tau_0^+$ (it tends to 1) and $H(\tau - \tau_0)\,(\phi_2(\tau - \tau_0) - (\tau - \tau_0))$ does not belong to $\mathcal{C}^1((0, \infty))$. We have only that $T_k \in \mathcal{C}([0, \infty))$. The same conclusion is valued for $S_k, k \in \mathbb{N}$.

We can be more precise leaving no doubt about properties of the function

$$F(\tau) \equiv H(\tau - \tau_0)\,(\phi_2(\tau - \tau_0) - (\tau - \tau_0)) \,.$$

It defines the regular distribution $[F]$. Let $[F_*^{(i)}(\tau)]$, $i = 1, 2$, denote the regular distribution defined by $F_*^{(i)}(\tau) = F^{(i)}(\tau), \tau \neq \tau_0, \tau_0 > 0$. Let F_{τ_0} denote the jump of the function F at $\tau = \tau_0$. Then we have the following relation between the distributional derivative $D^i[F]$ and $[F_*^{(i)}(\tau)]$ (cf. Section 2):

$$D^2[F] = [F_*^{(2)}(\tau)] + F_{\tau_0}\delta^{(1)}(\tau - \tau_0) + F_{\tau_0}^{(1)}\delta(\tau - \tau_0)$$

$$D^1[F] = [F_*^{(1)}(\tau)] + F_{\tau_0}\delta(\tau - \tau_0).$$

In our case it gives

$$D^2[F] = [F_*^{(2)}(\tau)] + \delta(\tau - \tau_0) \text{ and } D^1[F] = [F_*^{(1)}]. \tag{28}$$

Let us also remark that a function f has a derivative in the sense of distributions if the regular distribution $[f]$ defined by this function has the demanded derivative.

Now we can give meaning to the sentence: "We have a generalized solution to (15)": Solutions T_k and S_k to (15) are given by the continuous functions on $[0, \infty)$ and $T_k(\tau)$ has two derivatives in the sense of distributions, which have been defined by (28). In this case the restrictions of T_k on two intervals $(0, \tau_0)$ and (τ_0, ∞) are classical solutions to $(15)_1$ on these intervals. What occurs at the point $\tau = \tau_0$ one can explain by (28).

As matters stand now, u_k and F_k, defined by (14), are distribution valued functions in $\xi \in [0, 1)$ (cf. [12], p. 99). $u_k(\xi, \tau)$ has four derivatives with respect to ξ.

3.2. *Analytical form of the solution when $\lambda = A + B\delta(\tau - \tau_0), \tau > 0$*

As it is easily seen, the main role in describing $\widehat{T}_k(s)$ and $\widehat{S}_k(s)$ is played by the function

$$\widehat{f}(s) = \frac{1}{\Delta_{0k}^*(s)}.$$

There exist many possibilities to find the analytical form of $f(t)$, such that $\mathcal{L}(f)(s) = \widehat{f}(s)$. We choose to express it by a functional series.

First we give a different form of $\Delta_{0k}^*(s)$, given by (19). Introducing the notation $e = \dfrac{b-a}{a}\gamma > 0$ we have

$$\begin{aligned}\frac{1}{\Delta^*_{0k}(s)} &= \frac{1}{aM}\,\frac{1}{(s^2+\frac{N}{Ma})(s^\beta+1/a)-e}\\ &= \frac{1}{aM}\,\frac{1}{s^2+\frac{N}{aM}}\,\frac{1}{s^\beta+\frac{1}{a}}\left(1+\sum_{i=1}^{\infty}e^i\left(\frac{1}{s^2+\frac{N}{Ma}}\right)^i\left(\frac{1}{s^\beta+1/a}\right)^i\right)\end{aligned}$$

Let $w(\tau) = \beta\tau^{\beta-1}E^{(1)}_\beta(z)$, where $z = -\left(\frac{1}{a}\tau^\beta\right), \tau \geq 0$, and $E_\beta(z)$ is Mittag–Leffler's function (cf. [13]). It is an entire function given by

$$E_\beta(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}.$$

We know that $\mathcal{L}(w)(s) = \left(s^\beta + \dfrac{1}{a}\right)^{-1}$ (cf. [13]). To find the inverse function to $\dfrac{1}{s^2+\frac{N}{Ma}}$ we have to consider three different cases which depend on the sign of N. Thus we have

$$\frac{1}{s^2+\frac{N}{Ma}} = \begin{cases}\sqrt{\frac{Ma}{N}}\mathcal{L}\left(\sin\sqrt{\frac{N}{Ma}}\tau\right), & N>0\\[2ex] \mathcal{L}(\tau), & N=0\\[2ex] \sqrt{-\frac{Ma}{N}}\mathcal{L}\left(\sin h\sqrt{-\frac{N}{Ma}}\tau\right), & N<0.\end{cases}$$

Let us prove the existence of a function $\psi_N(\tau)$ such that $\mathcal{L}(\psi_N)(s) = \widehat{\psi}_N(s)$ and

$$\widehat{\psi}_N(s) = \sum_{i=1}^{\infty}e^i\left(\frac{1}{s^2+\frac{N}{Ma}}\right)^i\left(\frac{1}{s^\beta+1/a}\right)^i.$$

For this proof we need some properties of the Mittag-Leffler function By [14], p.36: $E^{(1)}_\beta(z) = \frac{1}{\Gamma(1-\alpha)}\frac{1}{z^2} + O\left(\frac{1}{z^3}\right),\ z\to\infty, |arg(-z)| < (1-\frac{3}{4}\beta)\pi$. With this property of $E^{(1)}_\beta(z)$ it follows that

$$w(\tau) \sim \frac{1}{\Gamma(1+\beta)}\tau^{\beta-1},\ \ \tau\to 0,$$

$$w(\tau) \sim \frac{a^2\beta}{\Gamma(1-\beta)}\tau^{-(1+\beta)},\ \ \tau\to\infty.$$

Consequently, there exists a constant C_1 such that

$$|w(\tau)| \le C_1 \tau^{\beta-1}, \quad 0<\tau<\infty.$$

In case $N>0$, we have for $\tau \ge 0$

$$\left|\sqrt{\frac{Ma}{N}}\sin\sqrt{\frac{N}{Ma}}\tau * w(\tau)\right| \le \sqrt{\frac{Ma}{N}} C_1 \int_0^\tau u^{\beta-1}du \le \sqrt{\frac{Ma}{N}} C_2 \frac{\tau^\beta}{\Gamma(\beta+1)},$$

and

$$\left|\left(\sqrt{\frac{Ma}{N}}\sin\sqrt{\frac{N}{Ma}}\tau * w(\tau)\right)^{*i}\right| \le C_2^i \left(\frac{Ma}{N}\right)^{\frac{i}{2}} \frac{\tau^{(\beta+1)i-1}}{\Gamma(i(\beta+1))}, \tag{29}$$

where g^{*i} means $i-$ fold convolution of g (C_2 does not depend on i). The existence of the function $\psi_N(\tau)$,

$$\psi_N(\tau) = \sum_{i=1}^{\infty} e^i \left(\sqrt{\frac{Ma}{N}}\sin\sqrt{\frac{N}{Ma}}\tau * w(\tau)\right)^{*i} \tag{30}$$

follows from Theorem 2 in [11], I, p. 305.

Therefore, in case $N>0$, the function $f(\tau)$ such that $\mathcal{L}(f)(s) = \dfrac{1}{\Delta_{0k}^*(s)}$ is

$$\begin{aligned} f(\tau) &= \frac{1}{aM}\sqrt{\frac{Ma}{N}}\sin\sqrt{\frac{N}{Ma}}\tau * w(\tau) \\ &+\Big(\frac{1}{aM}\sqrt{\frac{Ma}{N}}\sin\sqrt{\frac{N}{Ma}}\tau * w(\tau)\Big) * \psi_N(\tau). \end{aligned} \tag{31}$$

In the other two cases, $N=0$ and $N<0$ the procedure is just the same.

Now we can give the analytic form to $\mathcal{L}^{-1}\left(\frac{s^\beta+1/a}{\Delta_{0k}^*(s)}\right)$ as

$$\begin{aligned} \mathcal{L}^{-1}\Big(\frac{s^\beta+1/a}{\Delta_{0k}^*(s)}\Big) &= \frac{1}{aM}\sqrt{\frac{Ma}{N}}\sin\sqrt{\frac{N}{Ma}}\tau \\ &+ \frac{1}{aM}\sqrt{\frac{Ma}{N}}\sin\sqrt{\frac{N}{Ma}}\tau * \psi_N(\tau) \equiv \varphi(\tau). \end{aligned} \tag{32}$$

Thus

$$\mathcal{L}^{-1}\Big(\frac{s^\beta+1/a}{\Delta_{0k}^*}s\Big) = \varphi^{(1)}(t),$$

because $\varphi(0)=0$, (cf. [11] I, p. 99).

The functions $T_k(\tau)$ and $S_k(\tau)$ given by (26) and (27) can be also represented by

$$T_k(\tau) = aMT_{k0}\varphi^{(1)}(\tau) + aMT_{k0}^1\varphi(\tau) + a(k\pi)^2BT_k(\tau_0)H(\tau-\tau_0)\varphi(\tau-\tau_0), \tag{33}$$

and

$$\begin{aligned} S_k(\tau) \quad &= \frac{b}{a}T_k(\tau) - \bigg(\frac{b-a}{a}(MT_{k0}f^{(1)}(\tau) + MT_{k0}^1 f(\tau) \\ &\quad + (k\pi)^2BT_k(\tau_0)H(\tau-\tau_0)f(\tau-\tau_0)\bigg), \end{aligned} \tag{34}$$

where $\varphi(\tau)$ and $f(\tau)$ have been given by (32) and (31) respectively.

These forms of T_k and S_k have the property that both functions have been expressed by the same function $\psi_N(\tau)$ (via f and φ). However this function can be approximated by a finite sum

$$\sum_{i=1}^{N} e^i \left(\sqrt{\frac{Ma}{N}}\sin\sqrt{\frac{N}{Ma}}\tau * w(\tau)\right)^{*i},$$

and the elements of this sum may be estimated by $\left(\frac{Ma}{N}\right)^{i/2} C_2^i \frac{\tau^{(\beta+1)i-1}}{\Gamma(i(\beta+1))}$ (cf. (29)). Moreover, with the same relation (29) one can estimate the error that is introduced by taking only a finite number of terms in $\psi_N(\tau)$.

4. *Solution in case* $\lambda = A + BH(\tau-\tau_0)$

We consider the system (11), (12) not on $(0,1)\times(0,\infty)$, but on two domains: $(0,1)\times(0,\tau_0)$ and $(0,1)\times(\tau_0,\infty)$. In the case when the domain is $(0,1)\times(0,\tau_0)$, the first equation of system (11) has the form

$$\left(\frac{\partial^4}{\partial\xi^4} - \alpha\frac{\partial^4}{\partial\xi^2\partial t^2} + A\frac{\partial^2}{\partial\tau^2} + \frac{\partial^2}{\partial\tau^2}\right)u(\xi,\tau) + \gamma F(\xi,\tau) = 0. \tag{35}$$

In this case a solution to (35) is constituted by the restriction on $(0,\tau_0)$, $T_k^0(t)$ and $S_k^0(\tau)$, of functions $T_k(\tau)$ and $S_k(\tau)$, respectively in which $B=0$ (cf. Remark in Section 3.1). Thus

$$u_k^0(\xi,\tau) = C_k\sin k\pi\xi\cdot T_k^0(\tau),\ F_k^0(\xi,\tau) = C_k\sin k\pi\xi\cdot S_k^0(\tau),$$

is a classical solution to (11) with $\lambda = A$, for $(\xi,\tau)\in(0,1)\times(0,\tau_0)$ and the boundary condition (12).

If the domain is $(0,1)\times(\tau_0,\infty)$, the first equation in system (11) has the form (35) but in which A is replaced by $A+B$. A solution for this domain has been constituted by the restrictions on (τ_0,∞), $T_k^{\tau_0}$ and $S_k^{\tau_0}$, of $T_k(\tau)$ and $S_k(\tau)$ given by (26) and (27), respectively but in which instead of A stands $A+B$ and instead of B stands zero. For the analytical form of T_k and S_k cf. Section 3.2.

A question arises: Is it possible to extend the function $(T_k^0, T_k^{\tau_0})$ and $(S_k^0, S_k^{\tau_0})$, which are continuous on $(0,\tau_0)\cup(\tau_0,\infty)$ to continuous functions on the whole interval $(0,\infty)$? It is easily seen that it is not possible in the general case. But we can extend the distribution $[(T_k^0, T_k^{\tau_0})]$ and $[(S_k^0, S_k^{\tau_0})]$ to distributions defined on $(0,\infty)$ (cf. [12], p. 44). Let us denote them by $\tilde{T}_k$ and $\tilde{S}_k$ respectively.

A solution to system (11) with $\lambda = A + BH\left(\tau-\tau_0\right), \tau_0>0$ is now

$$u_k(\xi,\tau) = C_k \sin k\pi\xi \cdot \tilde{T}_k \quad \text{and} \quad F_k(\xi,\tau) = C_k \sin k\pi\xi \cdot \tilde{S}_k.$$

The explanation of the character of this solution is just the same as we give in case $\lambda = A + B\delta(\tau-\tau_0),\ B\neq 0$ (cf. Section 3.1).

5. *Asymptotic behavior of the solution to (15)*

Since the domain of the analycity of $\widehat{T}_k(s)$ and $\widehat{S}_k(s)$ (cf. (22), (23)) and also the asymptotic behavior of the solution found for (15) depend on the real parts of the values of s for which $\Delta_{0k}(s)=0$ (cf. [11], I chapter 13), we consider such complex numbers s. In Section 3.1. we have found $\Delta_{0k}(s)$ and have introduced Δ_{0k}^*

$$\Delta_{0k}^*(s) = aMs^{2+\beta} + Ms^2 + Ns^\beta + P, \tag{36}$$

where

$$M = (\alpha(\pi k)^2+1)>0,\ N=a(\pi k)^2((\pi k)^2-A)+b\gamma,\ P=(\pi k)^2((k\pi)^2-A)+\gamma.$$

It is easily seen that $N = aP + (b-a)\gamma$, consequently if $P\geq 0$ then $N>0$. This connection we will use in the following.

Our analysis of values of s such that $\Delta_{0k}(s)=0$ we divide in three parts: $P<0,\ P=0,\ P>0$, taking into account that s^β means the principal branch.

Case 1. *If $P<0$ then there exists at least one $\rho>0$ such that $\Delta_{0k}\left(s\right) = 0, s=\rho$.*

To show this we only have to write

$$aM\rho^{2+\beta} + M\rho^2 = (-N)\,\rho^\beta + (-P)\,. \tag{37}$$

Then, the existence of a $\rho > 0$ follows from the graph of the two functions $y_1 = aM\rho^{2+\beta} + M\rho^2$ and $y_2 = (-N)\rho^\beta + (-P)$ (see Figure 2).

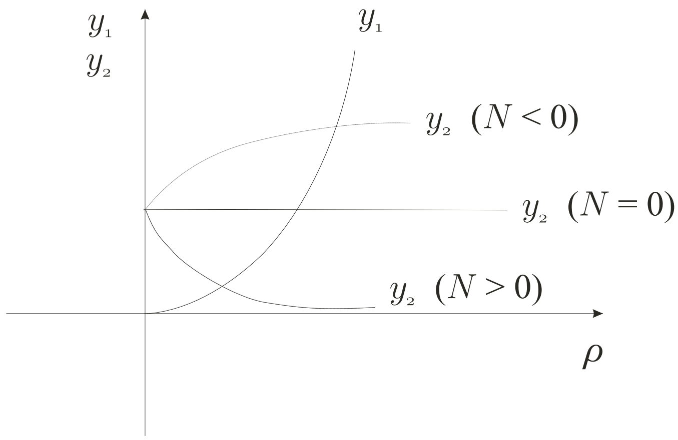

Figure 2. Graph of the functions $y_1\,(\rho)$ and $y_2\,(\rho)$

Case 2. If $P = 0$. *Then there are three values* s_1, s_2 *and* s_3 *such that* $\Delta_{0k}(s_i) = 0,\ i = 1,2,3.$ *Namely,* $s_1 = 0$, and for s_2 and s_3 we have $\operatorname{Re} s_2 < 0, \operatorname{Re} s_3 < 0$.

To show this, note that by the relation between P and N it follows $N > 0$. If we apply the well–known result that $\sum\limits_{j=1}^{p} w_j$ cannot vanish if $\gamma \le \arg w_j < \gamma + \pi,\ j = 1, ..., p$, where γ is a real constant (cf. [16]), then $s^\beta(aMs^2 + Ms^{2-\beta} + N)$ cannot vanish for $s, \operatorname{Re} s > 0$. Neither can it vanish for $s = \rho e^{\pm i\pi/2},\ \rho \ne 0$ because

$$\operatorname{Im}\,(aM\rho^2 e^{\pm\pi} + M\rho^{2-\beta}e^{\pm(2-\beta)\pi/2} + N) = M\rho\sin(2-\beta)\pi/2.$$

The only complex number s for which $\Delta_{0k}(s) = 0,\ \operatorname{Re} s \ge 0$ (if $P = 0$) is $s = 0$ and this is a branch point. To prove the existence of the points s_1, s_2 we use:

The Argument Principle [17] which says: *Let* $f(z)$ *be a single valued function on the domain* G *and suppose that in* G *it has no singular points. Let* Γ *be a closed Jordan rectifying curve which with its interior belongs to* G *and which does not contain points* s *such that* $f(s) = 0$. *The number of* *zeros of the function f in the interior of Γ equals to the number of entire turns of the vector $f(s)$ around the point $s = 0$ when the point s makes a round on Γ in the positive sense.*

To apply the Argument Principle, note first that $s = \rho e^{\pm i\pi}$, $\rho > 0$, cannot be a solution of $\Delta_{0k}(s) = 0$. To prove this, suppose there exists ρ, such that

$$f(s) = aM\rho^2 + M\rho^{2-\beta}(\cos\beta\pi \pm i\sin\beta\pi) + N = 0.$$

so that $M\rho^{2-\beta}\sin\beta\pi = 0$. This contradicts with $\rho > 0$.

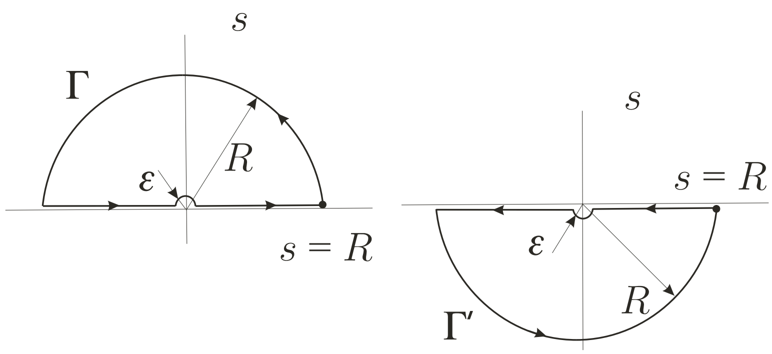

Figure 3. The contour Γ and corresponding curve Γ'

Now we apply the Argument Principle to the curve Γ shown in the left part of Figure 3 with R large enough and $\varepsilon > 0$ small enough. Let the point s run along Γ in the positive sense from the point $s = R$. Then $f(s)$ describes the curve such that when $R \to \infty$, the vector $f(s)$ turns once around the point $s = 0$. Consequently in the interior of Γ we have single s_1 such that $f(s_1) = 0$. Since we proved that $\mathrm{Re}\, s_1$ can not be nonnegative it follows that $\mathrm{Re}\, s_1 < 0, \mathrm{Im}\, s_1 > 0$.

If we take a curve Γ' symmetric to Γ with respect to the real axis we can prove the existence of s_2, $\mathrm{Re}\ s_2 < 0$, $\mathrm{Im}\ s_2 < 0$.

Case 3. *Suppose that $P > 0$. Then there exists no complex number $s = \rho e^{i\varphi}, \rho \geq 0, -\pi/2 \leq \varphi \leq \pi/2$, such that $\Delta_{0k}(s) = 0$. But there exists two numbers s_i,* $\mathrm{Re}\ s_i < 0$, *such that $\Delta_{0k}(s_i) = 0,\ i = 1, 2$.*

Let us analyze first the case $\varphi = \pm\pi/2$. Then

$$\begin{aligned} \mathrm{Im}\,(-\Delta_{0k}(s)) &= \rho^\beta(N - aM\rho^2)\sin(\pm\beta\frac{\pi}{2}), \\ \mathrm{Re}\,(-\Delta_{0k}(s)) &= \rho^\beta(N - aM\rho^2)\cos(\pm\beta\frac{\pi}{2}) - M\rho^2 + P. \end{aligned} \tag{38}$$

If there exists $\rho > 0$ such that $\Delta_{0k}(\rho e^{\pm\pi/2}) = 0$, we would have Im $(-\Delta_{0k}(s)) = 0$ and Re $(-\Delta_{0k}(s)) = 0$ or $N - aM\rho^2 = 0$ and $M\rho^2 - P = 0$. This is possible only if $N = aP$. Consequently, we would have $a = b$ or $\gamma = 0$ which is contrary to our suppositions.

In order to prove that there is no complex number $s = \rho e^{i\varphi}$, $\rho > 0$, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, such that $\Delta_{0k}(s)$, we start from another form of the equation $\Delta_{0k}(s) = 0$, namely

$$\left(Ms^2 + \frac{N}{Ma}\right)\left(\frac{1}{a} + s^\beta\right) = \frac{1}{a}\frac{b-a}{\gamma}. \tag{39}$$

From (37) it follows directly that s cannot be a real positive number. Consequently we can consider two cases $0 < \varphi < \pi/2$ and $-\pi/2 < \varphi < 0$.

Let there existed a φ, $0 < \varphi < \pi/2$ such that $s = \rho e^{i\varphi}$ satisfies (39). Then we would have

$$\rho^2 e^{i2\varphi} + \frac{N}{Ma} = r_1 e^{i\theta} \text{ and } \rho^\beta e^{i\beta\varphi} + \frac{1}{a} = r_2 e^{-i\theta},$$

where $r_1 r_2 = \frac{1}{a}\frac{b-a}{\gamma}$ and $\theta \in \mathbb{R}$.

Since $0 < \varphi < \pi/2$, then

$$0 < arg(\rho^2 e^{i2\varphi} + \frac{N}{Ma}) < \pi; \text{ and } 0 < arg(\rho^\beta e^{i\beta\varphi} + \frac{1}{a}) < \pi/2.$$

Consequently, θ would satisfy two inequalities: $0 < \theta < \pi$ and $0 < -\theta < \pi/2$ which is impossible.

The same conclusion holds if $-\frac{\pi}{2} < \varphi < 0$. Thus we proved the first part of the assertion in case $P > 0$.

From (39) it follows that there exists no $\rho > 0$ such that $s = \rho e^{i\pi}$ and vanishing $\Delta_{0k}(s)$. Because in this case for such $s = \rho e^{i\pi}$ in (39) we would have that the product of a real and a complex number equals to a real number.

The existence of s_1 and s_2, with Re $s_1 < 0$ and Re $s_2 < 0$ we can prove just in the same manner as we did for the case $P = 0$. When $R \to \infty$, the vector $f(s)$ turns once around the point $s = 0$. Consequently, we have a point s_1 in the interior of Γ and $\operatorname{Re} s_1 < 0, \operatorname{Im} s_1 > 0$. The existence of s_2 one can prove taking the curve Γ' symmetric to Γ with respect to the real axis (cf. Figure 3).

Remark 1 *Note that in all cases of system (11) treated here, that is,* $\lambda = A$, $\lambda = A + B\delta(t - t_0)$, $\lambda = A + BH(t - t_{\tau_0})$, $B \neq 0$. $\tau_0 > 0$, *we have the* *same value of* Δ^*_{0k}. *Only in the case* $\lambda = A + BH(t-\tau_0),\ B \neq 0$, *we have to take in* Δ^*_{0k}, $A+B$ *instead of* A. *Also, since* P *does not depend on* α, *the discussion with different* P *gives the same result for any* $\alpha > 0$. *Thus, the rotary inertia of the rod does not influence the asymptotic behavior of the rod.*

6. *Conclusions*

In this work we studied transversal vibrations of an elastic axially compressed rod on a fractional derivative type of viscoelastic foundation. We assumed that the axial force has a constant and a time dependent part. Thus, we generalized the results of our work [1].

Our main result concerns the influence of the fractional type viscoelastic foundation on the stability of the rod.

We compare the asymptotic behavior of solutions to system (11) and the equation which is obtained from system (11) when $\gamma = 0$, to judge the contribution of a viscoelastic foundation to the stability of the rod.

Let in system (11) λ be $\lambda = A + B\delta(\tau - \tau_0), \tau_0 > 0$ with A and B constants (see Section 3). Then, we distinguish two cases:

Case 1: $B = 0$ and $\lambda = A$:

If $\lambda < (\pi k)^2 + \dfrac{\gamma}{(\pi k)^2}$, then $P > 0$ and we have two complex numbers $s_i,\ Re\, s_i < 0$ such that $\Delta_{0k}(s_i) = 0,\ i = 1, 2$.

If $\lambda = (\pi k)^2 + \dfrac{\gamma}{(k\pi)^2}$, then $P = 0$ and there are three numbers $s_1 = 0,\ s_2, s_3,\ Re\, s_i < 0,\ i = 2, 3$, such that $\Delta_{0k}(s_i) = 0,\ i = 1, 2, 3$.

If $\lambda > (\pi k)^2 + \dfrac{\gamma}{(k\pi)^2}$, then $P < 0$ and there is at least one $s,\ Re\, s > 0$ such that $\Delta_{0k}(s) = 0$.

Suppose now that in system (11) $\gamma = 0$. Then for the function $T_k(t)$ (see (15)) we have the equation

$$T_k^{(2)}(t) - q_k T_k(t) = 0, \quad t > 0,$$

where

$$q_k = \frac{(k\pi)^2}{\alpha(k\pi)^2 + 1}(\lambda - (k\pi)^2).$$

If $\lambda < (k\pi)^2$, then $q_k < 0$ and

$$T_k(t) = C_1 \cos\sqrt{-q_k}t + C_2 \sin\sqrt{-q_k}t. \tag{40}$$

If $\lambda = (k\pi)^2$, then $q_k = 0$ and

$$T_k(t) = C_1 t + C_2. \tag{41}$$

If $\lambda > (k\pi)^2$, then $q_k > 0$ and

$$T_k(t) = C_1 \cos h\sqrt{q_k}t + C_2 \sin h\sqrt{q_k}t. \tag{42}$$

We note that the initial condition $\frac{\partial u}{\partial t}(x,0) = 0$ leads to $C_1 = 0$ in (41). In this case both (40) and (41) are bounded functions of time and the solution $u(\xi,\tau) = T_k(\tau)\sin k\pi\xi$ (see (14)) is also bounded while for case (42) the solution $u(\xi,\tau) = T_k(\tau)\sin k\pi\xi$ is unbounded.

In application we are interested in lowest mode $k = 1$ the stability is guaranteed if (note that $\lambda = A$):

In the case $\gamma \neq 0$ and $k = 1$

$$P = \pi^2(\pi^2 - A) + \gamma, \tag{43}$$

or

$$A \leq \pi^2 + \frac{\gamma}{\pi^2}. \tag{44}$$

In the case $\gamma = 0$ we conclude that in the cases (40), (41) we have stability. Thus with $k = 1, \lambda = A$ we obtain

$$A = \lambda \leq \pi^2. \tag{45}$$

The stability bound (45) is in agreement with the results of static (cf. [4]) and dynamic methods (cf. [20] for the case without rotary inertia). By comparing (43) and (45) we conclude that the foundation increases the stability boundary of the rod.

If $B = 0$ in $\lambda = A + B\delta(\tau - \tau_0)$ we have a classical solution $T_k(\tau)$ and $S_k(\tau)$.

Case 2: $B \neq 0$ in $\lambda = A + B\delta(\tau - \tau_0)$:

If $B \neq 0$ we have a generalized solution defined by a continuous function (see remark in Section 3.1). The solution to (11), $u_k(x,\tau), F_k(x,\tau)$ may be viewed as a distribution valued function in $x \in [0,1)$. As far as stability is concerned we have similar conclusions. The only difference being that in (44) we have to take $A + B$ instead of A. Thus the stability bound in this case reads

$$A + B \leq \pi^2 + \frac{\gamma}{\pi^2}. \tag{46}$$

Again, the foundation increases the stability bound. The rotary inertia does not increase the stability bound in both cases.

REFERENCES

[1] T. M. A t a n a c k o v i c, B. S t a n k o v i c, *Stability of an Elastic rod on a Fractional derivative type of Foundation,* Journal of Sound and Vibration 227 (2004) 149–161.

[2] Å. F e n a n d e r, *A fractional derivative railpad model included in a railway track model,* Journal of. Sound and Vibration 212 (1998) 889–903.

[3] A. C h a t t e r j e e, *Statistical origin of fractional derivatives in viscoelasticity,* Journal of Sound and Vibration (in press).

[4] T. M. A t a n a c k o v i c, *Stability theory of elastic rods*, World Scientific, River Edge, 1997.

[5] K. B. O l d h a m, J. S p a n i e r, *The fractional calculus*, Academic Press, New York, 1974.

[6] S. G. S a m k o, A. A. K i l b a s, O. I. M a r i c h e v, *Fractional integrals and derivatives*, Gordon and Breach, Amsterdam, 1993.

[7] A. S c h m i d t, L. G a u l, *Implementation von Stoffgesetzen mit fraktionalen Ableitungen in der Finite elemente Methode,* Zietschrift fur Angewandte Mathematik und Mechanik (ZAMM) 83 (2003) 26–37.

[8] T. M. A t a n a c k o v i c, *A modified Zener model of a viscoelastic body,* Continuum Mechanics and Thermodynamics 14 (2002) 137–148.

[9] R. L. B a g l e y, P. J. T o r v i k,*On the fractional calculus model of viscoelastic behavior,* Journal of Rheology 30 (1986) 133–155.

[10] B. S t a n k o v i c, T. M. A t a n a c k o v i c, *Distribution valued functions and Their Applications,* Integral Transforms and Special Functions 2005 (in press).

[11] G. D o e t s c h, *Handbuch der Laplace Transformation*, I, II. Birkhäuser, Basel, 1950, 1955.

[12] Z. S z m y d t, *Fourier transformation and linear differential equations*, D. Reidel Publishing Company, Dordrecht 1977.

[13] A. E r d é l y i (Editor), *Higher Transcendental Functions*, McGraw–Hill, New York, 1955.

[14] R. G o r e n f l o, F. M a i n a r d i, *Fractional calculus: Integral and differential equations of fractional order.* In: Fractional Calculus in Continuum Mechanics (Editors, A. Carpinteri, E. Mainardi), 223–276, Springer, Wien 1997.

[15] L. B e r g, *Asymptotishe Darstellung und Entwicklungen*, VEB, Deutscher Verlag der Wissenschaften, Berlin 1968.

[16] M. M a r d e n, *The Geometry of the Zeros of a polynomial in a complex variable*, American Mathematical Society, New York, 1949.

[17] M. J. A b l o w i t z, A. S. F o k a s, *Complex Analysis* (second edition). Cambridge University Press, Cambridge 2003.

[18] L. S c h w a r t z, *Théorie des Distributions*, Hermann, Paris 1955.

[19] V. S. V l a d i m i r o v, *Equations of the mathematical physics.* Nauka, Moscow, (In Russian) 1988.

[20] R. J. K n o p s, E. W. W i l k e s, *Theory of Elastic Stability*. In Handbuch der Physik, VIa/3, (Editor C. Truesdell), 125–302, Springer, Berlin, 1973.

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