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ON THE SPECTRAL RADIUS OF BICYCLIC GRAPHS

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A b s t r a c t. *Let K_3 and K_3' be two complete graphs of order 3 with disjoint vertex sets. Let $B_n^*(0)$ be the 5-vertex graph, obtained by identifying a vertex of K_3 with a vertex of K_3'. Let $B_n^{**}(0)$ be the 4-vertex graph, obtained by identifying two vertices of K_3 each with a vertex of K_3'. Let $B_n^*(k)$ be graph of order n, obtained by attaching k paths of almost equal length to the vertex of degree 4 of $B_n^*(0)$. Let $B_n^{**}(k)$ be the graph of order n, obtained by attaching k paths of almost equal length to a vertex of degree 3 of $B_n^{**}(0)$. Let $\mathcal{B}_n(k)$ be the set of all connected bicyclic graphs of order n, possessing k pendent vertices. One of the authors recently proved that among the elements of $\mathcal{B}_n(k)$, either $B_n^*(k)$ or $B_n^{**}(k)$ have the greatest spectral radius. We now show that for $k \geq 1$ and $n \geq k+5$, among the elements of $\mathcal{B}_n(k)$, the graph $B_n^*(k)$ has the greatest spectral radius.*

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1. *Introduction*

The spectral radius (the greatest graph eigenvalue, also called "index") is an important and much studied spectral property of graphs [1–3]. In a recent work [4] one of the present authors examined the spectral radius of connected bicyclic graphs of order n, possessing k pendent vertices (vertices of degree 1), and arrived at the following result.

Let K_3 and K_3' be two complete graphs of order 3 with disjoint vertex sets. Let $B_n^*(0)$ be the 5-vertex graph, obtained by identifying a vertex of K_3 with a vertex of K_3'. Let $B_n^{**}(0)$ be the 4-vertex graph, obtained by identifying two vertices of K_3 each with a vertex of K_3'. In other words, $B_n^{**}(0)$ is the graph obtained by deleting an edge from K_4.

By P_ℓ is denoted the path of order ℓ. Two paths P_ℓ and $P_{\ell'}$ are said to be of almost equal length, if $|\ell - \ell'| \leq 1$.

The set of all connected bicyclic graphs of order n, possessing k pendent vertices will be denoted by $\mathcal{B}_n(k)$.

The graph $B_n^*(k) \in \mathcal{B}_n(k)$ is obtained by attaching k paths of almost equal length to the vertex of degree 4 of $B_n^*(0)$. The graph $B_n^{**}(k) \in \mathcal{B}_n(k)$ is obtained by attaching k paths of almost equal length to the vertex of degree 3 of $B_n^{**}(0)$.

Note that both $B_n^*(k)$ and $B_n^{**}(k)$ exist if and only if $k \geq 1$ and $n \geq k+5$.

Theorem 1 [4]. *Provided that both $B_n^*(k)$ and $B_n^{**}(k)$ exist, among the elements of $\mathcal{B}_n(k)$, either $B_n^*(k)$ or $B_n^{**}(k)$ have the greatest spectral radius.*

The obvious question that emerges from Theorem 1 is which of the two graphs $B_n^*(k)$, $B_n^{**}(k)$ has greater spectral radius. Solving this seemingly simple problem it turned out to be not quite easy. In this paper we offer its solution:

Theorem 2. *Provided that both $B_n^*(k)$ and $B_n^{**}(k)$ exist, the spectral radius of $B_n^*(k)$ is greater than the spctral radius of $B_n^{**}(k)$.*

In order to prove Theorem 2 we need some preparations.

2. *Some Auxiliary Results*

Let G be a simple graph (i.e., a graph without loops, multiple and/or directed and/or weighted edges). Its vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The graph G has n vertices, i.e., $|V(G)| = n$.

The eigenvalues of G will be denoted by $\lambda_i = \lambda_i(G)$ and, as usual [1], it is assumed that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. If the graph G is connected, then $\lambda_1 > \lambda_2$.

The characteristic polynomial of the graph G is denoted by $\phi(G,\lambda)$.

We need the following well known Lemmas [1].

Lemma 1. *Let v be a vertex of G and let $\mathcal{C}(v)$ be the set of all cycles of G that contain v. Then*

$$\phi(G,\lambda) = \lambda\,\phi(G-v,\lambda) - \sum_{(u,v)\in E(G)} \phi(G-u-v,\lambda) - 2\sum_{Z\in\mathcal{C}(v)} \phi(G-V(Z),\lambda),$$

where $G-V(Z)$ is the graph obtained by removing from G the vertices belonging to Z.

Lemma 2. *Let v be a vertex of G, let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the graph G, and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of $G-v$. Then the inequalities*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$$

hold. If G is connected, then $\lambda_1 > \mu_1$.

Lemma 3. *The characteristic polynomial of the n-vertex path P_n satisfies the expression*

$$\phi(P_n,\lambda) = \frac{1}{\sqrt{\lambda^2-4}}\left(x_1^{n+1} - x_2^{n+1}\right),$$

where

$$x_1 = \frac{1}{2}\left(\lambda + \sqrt{\lambda^2-4}\right) \quad \textit{and} \quad x_2 = \frac{1}{2}\left(\lambda - \sqrt{\lambda^2-4}\right)$$

are the roots of the equation $x^2 - \lambda x + 1 = 0$.

Lemma 4. *If the graphs G and H have exactly one eigenvalue greater than some constant a, and if $\phi(G,\lambda_1(H)) > 0$, then $\lambda_1(G) < \lambda_1(H)$.*

In the proof that follows the special case of Lemma 4, for $a = 2$ will be used.

3. *Proof of Theorem 2*

The graphs $B_n^*(k)$ and $B_n^{**}(k)$ are defined above. Evidently, in the case of $B_n^*(k)$ it must be $k \leq n-5$ whereas in the case of $B_n^{**}(k)$ it must be $k \leq n-4$. If $k = n-4$ then $B_n^*(k)$ does not exist, and then among the elements of $\mathcal{B}_n(k)$ the graph $B_n^{**}(k)$ has the greatest spectral radius. Therefore in the following we assume that $k < n-4$. If so, then at least one path attached to $B_n^{**}(k)$ possesses at least two vertices ($\ell \geq 2$).

The vertex of $B_n^{*}(k)$ that has degree $k+4$ is denoted by v. Also the vertex of $B_n^{**}(k)$ that has degree $k+3$ is denoted by v. Denote by ℓ the maximal number of vertices of a path attached to the vertex v of $B_n^{**}(k)$. As already explained, $\ell \geq 2$.

Let B^{**} be the graph analogous to $B_n^{**}(k)$ in which all paths attached to vertex v have ℓ vertices.

Let B^{*} be the graph analogous to $B_n^{*}(k)$ in which all paths attached to vertex v have $\ell-1$ vertices.

Evidently, B^{*} is an induced subgraph of $B_n^{*}(k)$ whereas $B_n^{**}(k)$ is an induced subgraph of B^{**}. Therefore, by Lemma 2,

$$\lambda_1(B^{*}) \leq \lambda_1(B_n^{*}(k))$$

with equality if and only if $n = (\ell-1)k+5$. Also,

$$\lambda_1(B^{**}) \geq \lambda_1(B_n^{**}(k))$$

with equality if and only if $n = \ell\, k + 4$.

Thus for the proof of the Theorem it is sufficient to show that $\lambda_1(B^{**}) < \lambda_1(B^{*})$. We do this in the following.

Because of Lemma 2, the graphs B^{**} and B^{*} have exactly one eigenvalue greater than 2. (This is because all components of the subgraphs $B^{**}-v$ and $B^{*}-v$ are paths, and the spectral radii of paths are less than 2. Therefore $\lambda_2(B^{**}) < 2$ and $\lambda_2(B^{*}) < 2$. By direct calculation we check that in the case $n=6$, $k=1$, the greatest eigenvalues of B^{**} and B^{*} are greater than 2. Therefore the greatest eigenvalues of B^{**} and B^{*} are greater than 2 for all values of n and k.)

Consequently, Lemma 4 is applicable to B^{**} and B^{*} and it is sufficient to show that $\phi(B^{**}, \lambda_1(B^{*})) > 0$.

By applying Lemma 1 to the vertex v of B^{**} we obtain

$$\phi(B^{**},\lambda) = \lambda\, \phi(P_\ell,\lambda)^{k-1} \left[(\lambda^3 - 5\,\lambda - 4)\, \phi(P_\ell,\lambda) - k\,(\lambda^2-2)\, \phi(P_{\ell-1},\lambda)\right] .$$

In an analogous manner we obtain

$$\phi(B^{*},\lambda) = (\lambda^2-1)\, \phi(P_{\ell-1},\lambda)^{k-1} \left[(\lambda^3-5\lambda-4)\, \phi(P_{\ell-1},\lambda) - k\,(\lambda^2-1)\, \phi(P_{\ell-2},\lambda)\right] .$$

Denote the greatest eigenvalue of B^* by r. For $n = 6$ and $k = 1$ the greatest eigenvalue of B^* is $2.709\ldots$. Therefore, for any n and k,

$$r = \lambda_1(B^*) \geq 2.709 \quad .$$

From the above expression for $\phi(B^*, \lambda)$ it is seen that r satisfies the equation

$$(r^3 - 5\,r - 4)\,\phi(P_{\ell-1}, r) - k\,(r^2 - 1)\,\phi(P_{\ell-2}, r) = 0$$

from which

$$k = \frac{(r^3 - 5\,r - 4)\,\phi(P_{\ell-1}, r)}{(r^2 - 1)\,\phi(P_{\ell-2}, r)} \ .$$

Now, the inequality $\phi(B^{**}, r) > 0$ holds if and only if

$$r\,\phi(P_\ell, r)^{k-1}\,\Big[(r^3 - 5\,r - 4)\,\phi(P_\ell, r) - k\,(r^2 - 2)\,\phi(P_{\ell-1}, r)\Big] > 0$$

if and only if

$$(r^3 - 5\,r - 4)\,\phi(P_\ell, r) - k\,(r^2 - 2)\,\phi(P_{\ell-1}, r) > 0$$

if and only if

$$(r^3 - 5\,r - 4)\,\phi(P_\ell, r) - \frac{(r^3 - 5\,r - 4)\,\phi(P_{\ell-1}, r)}{(r^2 - 1)\,\phi(P_{\ell-2}, r)}\,(r^2 - 2)\,\phi(P_{\ell-1}, r) > 0 \ .$$

Now, the expression $r^3 - 5\,r - 4$ is positive–valued for $r \geq 2.709$. Therefore the above inequality holds if and only if

$$(r^2 - 2)\,\phi(P_{\ell-1}, r)^2 < (r^2 - 1)\,\phi(P_\ell, r)\,\phi(P_{\ell-2}, r) \ .$$

From Lemma 3 we get

$$\phi(P_n, r) = \frac{1}{\sqrt{r^2 - 4}}\,\left(r_1^{n+1} - r_2^{n+1}\right),$$

where

$$r_1 = \frac{1}{2}\,\left(r + \sqrt{r^2 - 4}\right) \qquad \text{and} \qquad r_2 = \frac{1}{2}\,\left(r - \sqrt{r^2 - 4}\right)$$

are the roots of the equation $x^2 - r\,x + 1 = 0$. From the Vieta formulas,

$$r_1 + r_2 = r \quad ; \quad r_1\,r_2 = 1$$

and therefore

$$r_1^2 + r_2^2 = (r_1 + r_2)^2 - 2\, r_1\, r_2 = r^2 - 2$$

$$r_1^4 + r_2^4 = (r_1^2 + r_2^2)^2 - 2\, r_1^2\, r_2^2 = (r^2 - 2)^2 - 2 \ .$$

In view of the above, $\phi(B^{**}, r) > 0$ holds if and only if

$$\frac{1}{r^2 - 4}\,(r^2 - 2)\,(r_1^\ell - r_2^\ell)^2 < \frac{1}{r^2 - 4}\,(r^2 - 1)\,(r_1^{\ell+1} - r_2^{\ell+1})(r_1^{\ell-1} - r_2^{\ell-1})$$

if and only if

$$(r^2 - 2)\,(r_1^{2\ell} + r_2^{2\ell} - 2) < (r^2 - 1)\,[r_1^{2\ell} + r_2^{2\ell} - (r^2 - 2)]$$

if and only if

$$r_1^{2\ell} + r_2^{2\ell} > (r^2 - 2)(r^2 - 3) \ .$$

We now demonstrate that for $\ell \geq 2$ the series $a_\ell = r_1^{2\ell} + r_2^{2\ell}$ strictly increases.

Because

$$r_1^{2\ell} + r_2^{2\ell} = \frac{r_1^{4\ell} + 1}{r_1^{2\ell}}$$

we get that

$$\frac{a_{\ell+1}}{a_\ell} = \frac{r_1^{4\ell+4} + 1}{r_1^{4\ell+2} + r_1^2}$$

will be greater than unity (in which case a_ℓ increases) if and only if

$$r_1^{4\ell+4} + 1 > r_1^{4\ell+2} + r_1^2$$

i.e., if

$$\left(r_1^{4\ell+2} - 1\right)(r_1^2 - 1) > 0$$

which is evidently obeyed since $r_1 > 1$.

We have previously shown that $\phi(B^{**}, r) > 0$ holds if and only if

$$r_1^{2\ell} + r_2^{2\ell} > (r^2 - 2)(r^2 - 3) \ .$$

Now, if this inequality is satisfied for $\ell = 2$ it will be satisfied for all $\ell \geq 2$.

For $\ell = 2$ we get

$$r_1^4 + r_2^4 > (r^2 - 2)(r^2 - 3)$$

if and only if

$$(r^2 - 2)^2 - 2 > (r^2 - 2)(r^2 - 3)$$

if and only if $r^2 > 4$, which is evidently satisfied.

Thus we have demonstrated that

$$\phi(B^{**}, \lambda_1(B^*)) > 0$$

which, by Lemma 4, implies

$$\lambda_1(B^{**}) < \lambda_1(B^*)$$

which, in turn, is sufficient for the validity of Theorem 2. It is interesting to note that we managed to verify the above inequalities without knowing the actual value of $r = \lambda_1(B^*)$.

By this the proof of the Theorem 2 is completed.

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