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FINE RESOLUTION OF THE SHEAF OF GERMS OF HOLOMORPHIC PROJECTIVE VECTOR FIELDS ON TWO DIMENSIONAL PROJECTIVELY FLAT COMPLEX MANIFOLD

OLIVERA MILENKOVIĆ

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A b s t r a c t. In the article fine resolution of the sheaf of germs of complex projective vector fields on a locally projective complex manifold M^4 of complex dimension $n = 2$ is given. The local coordinate charts on the manifolds are chosen so that they are adapted to the complex projective pseudo-group structure and carry therefore the geometry of the structure.

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1. *Introduction*

In this article fine resolution of the sheaf of germs of complex projective vector fields over the projectively flat manifold of lower $n = 2$ dimensional case is given. General case is handled in article [12]. In order to illustrate how the resolution and operators look like in lower dimesional case in this article more details are provided. Definition of main operators and motivation are taken from [12] and are included for completness of purposes.

Deformations of general Γ- structures and almost Γ- structures were analyzed by Spencer [14], Kodaira [10], Kumpera [11], Griffiths [5] and their followers. Such structures are given by a certain "Maurer-Cartain" form and its perturbation satisfying an integrability condition with a derivation D which acts on a graded Lie algebra consisting of the sheaf of germs of smooth Lie algebra-valued q-forms $\Sigma^* = \bigoplus_{q\geq 0} \Sigma^q$. The sheaf of germs of infinitesimal transformations Θ of Γ enjoys the exact sequence $0 \to \Theta \to \Sigma^0 \xrightarrow{D} \Sigma^1$. Elements of $H^0(X, \Sigma^1)$ satisfying an integrability condition determine deformations.

Construction of a long exact sequence extending the given one, more precisely, the constructions of fine resolutions of the Killing vector fields on various types of manifolds is the subject of cited papers as well as of [2], [3], [4], [5], [6]. The essential problems were related to solving overdetemined systems of linear partial differencial equations established by Spencer, Kodaira, Griffits, Kumpera, Calabi, Goldschmidt, Gasqui and many other successors of this theory (cf. [1]).

Simplified and direct construction of the fine resolution of the sheaf of germs of Killing vector fields in the real (analytic) case was treated in [8]. This and paper [12] were inspired by [8] (see also [9]). The goal was to make a simple construction of the fine resolution of Killing vector fields in the case of locally complex projective manifolds (existence of such resolution can be derived from the general theory). This is done using the graded Lie algebras on the basis of Dolbeault type theorems for cohomology in sheaves of complex projective vector fields (cf. [6], p. 23-25, 44-46 for the notation).

Since the sheaf of holomprphic vector fields is not fine additional level of complexity in the construction of operators had to be introduced comparing to that given in [8].

The resulting fine resolution of the sheaf of germs of holomorphic projective vector fields for the 2 dimensional complex manifold $K_{hproj}(M^4)$ is of the form

$$0 \to K_{hproj}(M^4) \xrightarrow{i} \tilde{\Omega}_L^{(0)} \xrightarrow{D_0} \tilde{\Omega}_L^{(1)} \xrightarrow{D_1} \tilde{\Omega}_L^{(2)} \xrightarrow{D_2} \tilde{\Omega}_L^{(3)} \xrightarrow{D_3} \tilde{\Omega}_L^{(4)} \to 0, \qquad (1)$$

where $\tilde{\Omega}_L^{(k)}$ denotes the sheaf of germs associated to elements of $\Omega_L^{(k)}$ and where the morphisms D_k, defined in the sequel, use operations derived from ones introduced in [8].

One consequence of the result given in this article is that the deformations of the corresponding (one parameter) group of transformations (in the sense of [5]) is made by elements of $H^0(M^4, \tilde{\Omega}_L^{(0,1)} \oplus \tilde{\Omega}_L^{(1,0)})$. Notation is given below.

2. *Fine Resolution*

A locally complex projective manifold M^4 is by definition a complex manifold which carries an atlas $\mathcal{P}$ with coordinate charts $U(z^1, z^2)$, $U'(z'^1, z'^2)$, such that, in overlapping neighborhoods ($U \cap U' \neq \emptyset$), the transition functions are complex projective transformations:

$$z'^i = \frac{a^i_1 z^1 + a^i_2 z^2 + a^i}{a^0_1 z^1 + a^0_2 z^2 + a^0}, \ \det \begin{vmatrix} a^1_1 & a^1_2 & a^1 \\ a^2_1 & a^2_2 & a^2 \\ a^0_1 & a^0_2 & a^0 \end{vmatrix} \neq 0.$$

All work contained herein is expressed with respect to these local coordinate charts.[1]

These local coordinate charts are not arbitrary – they are adapted to the complex projective pseudo-group structure and cary therefore the geometry of the structure. Using them one avoids any type of connection due to the use of the projective pseudogroup structure , calculations are more direct and straightforward and result is given in formulas that are ready to be applied.

The main algebraic tool we need (see [8], [4]) is the bundle of graded Lie-algebras over M^4

$$L = T \oplus EndT \oplus T^* = L_{-1} \oplus L \oplus L_1,$$

where $T = (TM^4)^{(1,0)}$ is the bundle of complex tangent vectors of M^4, $EndT$ is the bundle of endomorphisms of T, and T^* is the bundle dual of T. The Lie algebra law on L is given by

$$\begin{aligned} [(v,h,\omega),(v',h',\omega')] \quad = \quad & \big(h(v') - h'(v) - (\operatorname{tr} h')v + (\operatorname{tr} h)v', \\ & h \circ h' - h' \circ h + v \otimes \omega' - v' \otimes \omega, \\ & {}^t h'(\omega) - {}^t h(\omega') - (\operatorname{tr} h)\omega' + (\operatorname{tr} h')\omega\big) \,. \end{aligned}$$

This article does not contain Lie algebra computation. Lie algebra L is used in order to explain the geometric meaning of the different sheaves which enter into the resolution.

[1]We will assume that all the coordinate functions or coordinates of vector fields and forms which appear in this paper are smooth functions.

We consider *projective differential forms on* M^4 i.e., smooth differential forms on M^4 with values in L satisfying some restrictive conditions as follows. When decomposed in accordance with the grading of L and with the classical type decomposition in the exterior algebra $C^\infty(\Lambda(TM^4)^*)$, a projective differential k-form $\omega^{(k)}$ with smooth coefficients and values in L is represented as a sum

$$\omega^{(k)} = \omega_{-1}^{(0,k)} + \omega_0^{(1,k-1)} + \omega_1^{(2,k-2)}, \quad 0 \le k \le 4. \tag{2}$$

(As usual, if some super-index on the right hand side is less than 0 or grater than 2 (complex dimension of manifold), the corresponding element in the sum is equal to zero). Here, with respect to complex projective local coordinates $U(z^1, z^2)$,

$$\begin{array}{lll} \omega^{(0)} = \omega_{-1}^{(0,0)} & \omega^{(1)} = \omega_{-1}^{(0,1)} + \omega_0^{(1,0)} & \omega^{(2)} = \omega_{-1}^{(0,2)} + \omega_0^{(1,1)} + \omega_1^{(2,0)} \\ \omega^{(3)} = \omega_0^{(1,2)} + \omega_1^{(2,1)} & \omega^{(4)} = \omega_1^{(2,2)} & \end{array}$$

Here, with respect to complex projective local coordinates $U(z^1, z^2)$,

$$\omega_{-1}^{(0,k)}|_U = \sum_{i,J,|J|=k} v_J^i d\bar{z}^J \otimes \frac{\partial}{\partial z^i}, \quad 0 \le k \le 2.$$

The above is a differential k-form of type $(0,k)$ with values in T, e.g.,

$$\omega_{-1}^{(0,0)}|_U = v^1_{(0,0)} \frac{\partial}{\partial z^1} + v^2_{(0,0)} \frac{\partial}{\partial z^2},$$

$$\omega_{-1}^{(0,1)}|_U = v^1_{(1,0)} d\bar{z}^1 \otimes \frac{\partial}{\partial z^1} + v^1_{(0,1)} d\bar{z}^2 \otimes \frac{\partial}{\partial z^1} + v^2_{(1,0)} d\bar{z}^1 \otimes \frac{\partial}{\partial z^2} + v^2_{(0,1)} d\bar{z}^2 \otimes \frac{\partial}{\partial z^2},$$

$$\omega_{-1}^{(0,2)}|_U = v^1_{(1,1)} (d\bar{z}^1 \wedge d\bar{z}^2) \otimes \frac{\partial}{\partial z^1} + v^2_{(1,1)} (d\bar{z}^1 \wedge d\bar{z}^2) \otimes \frac{\partial}{\partial z^2},$$

In formula (2)

$$\omega_0^{(1,k-1)} = \sum_{\substack{i,j,L,M,\\ |L|=1, |M|=k-1}} h^i_{jLM} dz^L \otimes d\bar{z}^M \otimes dz^j \otimes \frac{\partial}{\partial z^i}$$

e.g.

$$\omega_0^{(1,0)}|_U =$$
$$(h^1_{1(1,0)(0,0)} dz^1 + h^1_{1(0,1)(0,0)} dz^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^1} + (h^1_{2(1,0)(0,0)} dz^1 + h^1_{2(0,1)(0,0)} dz^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^1}$$
$$+ (h^2_{1(1,0)(0,0)} dz^1 + h^2_{1(0,1)(0,0)} dz^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^2} + (h^2_{2(1,0)(0,0)} dz^1 + h^2_{2(0,1)(0,0)} dz^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^2}$$

$$\omega_0^{(1,1)}|_U =$$
$$(h^1_{1(1,0)(1,0)}dz^1\otimes d\bar z^1+h^1_{1(0,1)(1,0)}dz^2\otimes d\bar z^1+h^1_{1(1,0)(0,1)}dz^1\otimes d\bar z^2+h^1_{1(0,1)(0,1)}dz^2\otimes d\bar z^2)\otimes dz^1\otimes\tfrac{\partial}{\partial z^1}$$
$$+(h^1_{2(1,0)(1,0)}dz^1\otimes d\bar z^1+h^1_{2(0,1)(1,0)}dz^2\otimes d\bar z^1+h^1_{2(1,0)(0,1)}dz^1\otimes d\bar z^2+h^1_{2(0,1)(0,1)}dz^2\otimes d\bar z^2)\otimes dz^2\otimes\tfrac{\partial}{\partial z^1}$$
$$+(h^2_{1(1,0)(1,0)}dz^1\otimes d\bar z^1+h^2_{1(0,1)(1,0)}dz^2\otimes d\bar z^1+h^2_{1(1,0)(0,1)}dz^1\otimes d\bar z^2+h^2_{1(0,1)(0,1)}dz^2\otimes d\bar z^2)\otimes dz^1\otimes\tfrac{\partial}{\partial z^2}$$
$$+(h^2_{2(1,0)(1,0)}dz^1\otimes d\bar z^1+h^2_{2(0,1)(1,0)}dz^2\otimes d\bar z^1+h^2_{2(1,0)(0,1)}dz^1\otimes d\bar z^2+h^2_{2(0,1)(0,1)}dz^2\otimes d\bar z^2)\otimes dz^2\otimes\tfrac{\partial}{\partial z^2}$$

$$\begin{aligned}
\omega_0^{(1,2)}|_U \quad = \quad & (h^1_{1(1,0)(1,1)}dz^1+h^1_{1(0,1)(1,1)}dz^2)\otimes(d\bar z^1\wedge d\bar z^2)\otimes dz^1\otimes\frac{\partial}{\partial z^1}\\
& +(h^1_{2(1,0)(1,1)}dz^1\otimes +h^1_{2(0,1)(1,1)}dz^2)\otimes(d\bar z^1\wedge d\bar z^2)\otimes dz^2\otimes\frac{\partial}{\partial z^1}\\
& +(h^2_{1(1,0)(1,1)}dz^1+h^2_{1(0,1)(1,1)}dz^2)\otimes(d\bar z^1\wedge d\bar z^2)\otimes dz^1\otimes\frac{\partial}{\partial z^2}\\
& +(h^2_{2(1,0)(1,1)}dz^1+h^2_{2(0,1)(1,1)}dz^2)\otimes(d\bar z^1\wedge d\bar z^2)\otimes dz^2\otimes\frac{\partial}{\partial z^2}
\end{aligned}$$

is a 1-form of type $(1,1)$, with values trace-less endomorphisms of T satisfying[2]

$$\sum_{j,L} h^i_{jLM}dz^L\wedge dz^j=0, \qquad \sum_{i,L}\frac{\partial}{\partial z^i}\rfloor h^i_{jLM}dz^L=0, \tag{3}$$

i.e.,

$$h^i_{2(1,0)M}dz^1\wedge dz^2+h^i_{1(0,1)M}dz^2\wedge dz^1=0, \qquad i=1,2,\quad 0\le |M|\le 2 \tag{4}$$

$$\sum_i\frac{\partial}{\partial z^i}\rfloor(h^i_{j(1,0)M}dz^1+h^i_{j(0,1)M}dz^2)=h^1_{j(1,0)M}+h^2_{j(0,1)M}=0, \qquad j=1,2,\quad 0\le|M|\le 2. \tag{5}$$

So from 3 it follows that

$$h^i_{2(1,0)M}+h^i_{1(0,1)M}=0,\quad h^1_{j(1,0)M}+h^2_{j(0,1)M}=0, \qquad i,j\in\{1,2\},\quad 0\le|M|\le 2, \tag{6}$$

[2] $\frac{\partial}{\partial x^i}\rfloor$ denotes inner product with the vector field $\frac{\partial}{\partial x^i}$

and consequently that

$$
\begin{aligned}
&\omega_0^{(1,0)}|_U = \\
&\quad (h^1_{1(1,0)(0,0)}dz^1 + h^1_{1(0,1)(0,0)}dz^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^1} - (h^1_{1(1,0)(0,0)}dz^1 + h^1_{1(0,1)(0,0)}dz^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^1} \\
&\quad -(h^1_{1(1,0)(0,0)}dz^1 + h^1_{1(0,1)(0,0)}dz^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^2} + (h^1_{1(1,0)(0,0)}dz^1 + h^1_{1(0,1)(0,0)}dz^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^2} \\
&= (h^1_{1(1,0)(0,0)}dz^1 + h^1_{1(0,1)(0,0)}dz^2) \otimes (dz^1 \otimes \tfrac{\partial}{\partial z^1} - dz^2 \otimes \tfrac{\partial}{\partial z^1} - dz^1 \otimes \tfrac{\partial}{\partial z^2} + dz^2 \otimes \tfrac{\partial}{\partial z^2}) \\
&= (h^1_{1(1,0)(0,0)}dz^1 + h^1_{1(0,1)(0,0)}dz^2) \otimes (dz^1 - dz^2) \otimes \tfrac{\partial}{\partial z^1} - (dz^1 - dz^2) \otimes \tfrac{\partial}{\partial z^2} \\
&= (h^1_{1(1,0)(0,0)}dz^1 + h^1_{1(0,1)(0,0)}dz^2) \otimes (dz^1 - dz^2) \otimes (\tfrac{\partial}{\partial z^1} - \tfrac{\partial}{\partial z^2})
\end{aligned}
$$

$$
\begin{aligned}
\omega_0^{(1,1)}|_U = (&h^1_{1(1,0)(1,0)}dz^1 \otimes d\bar{z}^1 + h^1_{1(0,1)(1,0)}dz^2 \otimes d\bar{z}^1 + h^1_{1(1,0)(0,1)}dz^1 \otimes d\bar{z}^2 \\
&+h^1_{1(0,1)(0,1)}dz^2 \otimes d\bar{z}^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^1} \\
&+(h^1_{2(1,0)(1,0)}dz^1 \otimes d\bar{z}^1 + h^1_{2(0,1)(1,0)}dz^2 \otimes d\bar{z}^1 + h^1_{2(1,0)(0,1)}dz^1 \otimes d\bar{z}^2 \\
&+h^1_{2(0,1)(0,1)}dz^2 \otimes d\bar{z}^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^1} \\
&+(h^2_{1(1,0)(1,0)}dz^1 \otimes d\bar{z}^1 + h^2_{1(0,1)(1,0)}dz^2 \otimes d\bar{z}^1 + h^2_{1(1,0)(0,1)}dz^1 \otimes d\bar{z}^2 \\
&+h^2_{1(0,1)(0,1)}dz^2 \otimes d\bar{z}^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^2} \\
&+(h^2_{2(1,0)(1,0)}dz^1 \otimes d\bar{z}^1 + h^2_{2(0,1)(1,0)}dz^2 \otimes d\bar{z}^1 + h^2_{2(1,0)(0,1)}dz^1 \otimes d\bar{z}^2 \\
&+h^2_{2(0,1)(0,1)}dz^2 \otimes d\bar{z}^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^2} \\
= (&h^1_{1(1,0)(1,0)}dz^1 \otimes d\bar{z}^1 + h^1_{1(0,1)(1,0)}dz^2 \otimes d\bar{z}^1 + h^1_{1(1,0)(0,1)}dz^1 \otimes d\bar{z}^2 \\
&+h^1_{1(0,1)(0,1)}dz^2 \otimes d\bar{z}^2) \otimes (dz^1 - dz^2) \otimes (\tfrac{\partial}{\partial z^1} - \tfrac{\partial}{\partial z^2})
\end{aligned}
$$

$$
\begin{aligned}
&\omega_0^{(1,2)}|_U = \\
&\quad (h^1_{1(1,0)(1,1)}dz^1 + h^1_{1(0,1)(1,1)}dz^2) \otimes (d\bar{z}^1 \wedge d\bar{z}^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^1} \\
&\quad -(h^1_{1(1,0)(1,1)}dz^1 + h^1_{1(0,1)(1,1)}dz^2) \otimes (d\bar{z}^1 \wedge d\bar{z}^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^1} \\
&\quad +(h^2_{1(1,0)(1,1)}dz^1 + h^2_{1(0,1)(1,1)}dz^2) \otimes (d\bar{z}^1 \wedge d\bar{z}^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^2} \\
&\quad -(h^2_{1(1,0)(1,1)}dz^1 + h^2_{1(0,1)(1,1)}dz^2) \otimes (d\bar{z}^1 \wedge d\bar{z}^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^2} \\
&= \quad (h^1_{1(1,0)(1,1)}dz^1 + h^1_{1(0,1)(1,1)}dz^2) \otimes (d\bar{z}^1 \wedge d\bar{z}^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^1} \\
&\quad -(h^1_{1(1,0)(1,1)}dz^1 + h^1_{1(0,1)(1,1)}dz^2) \otimes (d\bar{z}^1 \wedge d\bar{z}^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^1} \\
&\quad -(h^1_{1(1,0)(1,1)}dz^1 + h^1_{1(0,1)(1,1)}dz^2) \otimes (d\bar{z}^1 \wedge d\bar{z}^2) \otimes dz^1 \otimes \tfrac{\partial}{\partial z^2} \\
&\quad +(h^1_{1(1,0)(1,1)}dz^1 + h^1_{1(0,1)(1,1)}dz^2) \otimes (d\bar{z}^1 \wedge d\bar{z}^2) \otimes dz^2 \otimes \tfrac{\partial}{\partial z^2} \\
&= \quad (h^1_{1(1,0)(1,1)}dz^1 + h^1_{1(0,1)(1,1)}dz^2) \\
&\quad \otimes(d\bar{z}^1 \wedge d\bar{z}^2) \otimes (dz^1 \otimes \tfrac{\partial}{\partial z^1} - dz^2 \otimes \tfrac{\partial}{\partial z^1} - dz^1 \otimes \tfrac{\partial}{\partial z^2} + dz^2 \otimes \tfrac{\partial}{\partial z^2}) \\
&= \quad (h^1_{1(1,0)(1,1)}dz^1 + h^1_{1(0,1)(1,1)}dz^2) \otimes (d\bar{z}^1 \wedge d\bar{z}^2) \otimes (dz^1 - dz^2) \otimes (\tfrac{\partial}{\partial z^1} - \tfrac{\partial}{\partial z^2})
\end{aligned}
$$

For $2 \leq k \leq 4$,

$$\omega_1^{(2,k-2)}|_U = \sum_{j,M,|M|=k-2} \omega_{jM} dz^j \otimes dz^1 \wedge dz^2 \otimes d\bar{z}^M$$

is a differential k-form of type $(2, k-2)$ with values in $((TM^4)^*)^{(1,0)}$.
The space of projective differential forms on M^4 is denoted Ω_L. When one puts in evidence the grading of L, or the degrees of its differential forms, or their (complex, anti-complex) types, Ω_L admits several splittings:

$$\Omega_L = \Omega_{L_{-1}} \oplus \Omega_{L_0} \oplus \Omega_{L_1} = \sum_{k=0}^{4} \Omega_L^{(k)} = \sum_{k=0}^{4}\sum_{i=0}^{k} \Omega_L^{(i,k-i)}.$$

A *complex projective vector field* v (infinitesimal complex projective transformation) on a locally projective complex manifold M^4 is a field whose local expression in projective coordinates in $U(z^1, z^2)$ is of the form

$$v|_U = \sum_{i=1}^{2}(\alpha^i + \sum_{r=1}^{2} \alpha^i_r z^r + z^i \sum_{r=1}^{2} \alpha_r z^r)\frac{\partial}{\partial z^i} \quad ,$$

where $(\alpha^i, \alpha^i_j, \alpha_i)$ are 8 arbitrary constants. We call them *projective Killing vector fields* and denote $K_{hproj}(M^4)$ the sheaf of germs of holomorphic projective Killing vector fields.
The operator $\pi : \tilde{\Omega}^{(0,k)}_{L_{-1}} \to \tilde{\Omega}^{(1,k)}_{L_0}$ is locally defined as follows: given $\omega^{(0,k)}_{-1}$, express it in local coordinates $U(z^1, z^2)$ with

$$\omega^{(0,k)}_{-1}|_U = \sum_{i,J,|J|=k} v^i_J d\bar{z}^J \otimes \frac{\partial}{\partial z^i} = v.$$

Consider then the "trace-less ∂-hessian" of the associated vectorial form with components

$$v^i_{J,jk} = \frac{\partial^2 v^i_J}{\partial z^j \partial z^k} - \frac{\delta^i_j}{3}\sum_s \frac{\partial^2 v^s_J}{\partial z^s \partial z^k} - \frac{\delta^i_k}{3}\sum_s \frac{\partial^2 v^s_J}{\partial z^s \partial z^j}, \tag{7}$$

and define a local $(1, k)$-form with values in $End^0 T$ using

$$\pi v = \sum_{i,j} (\pi v)^i_j \otimes dz^j \otimes \frac{\partial}{\partial z^i}, \qquad (\pi v)^i_j = \sum_{k,J} v^i_{J,jk} dz^k \otimes d\bar{z}^J. \tag{8}$$

Proposition 1. *Local $(1, k)-$ forms defined through the formulae (7) and (8) with respect to all projective coordinate charts match together into an element of $\tilde{\Omega}^{(1,k)}_{L_0}$, denoted by $\pi\omega_{-1}(0, k)$. In other words, (8) is the local expression of $\pi\omega$.*

Operator $\Upsilon : \tilde{\Omega}^{(0,m)}_{L_0} \to \tilde{\Omega}^{(1,m)}_{L_0}$ is locally defined as follows. Express first an element $\omega_0^{(0,m)}$ in local projective coordinates as

$$\omega_0^{(0,m)}|_U = \sum_{i,j} h^i_j \otimes dz^j \otimes \frac{\partial}{\partial z^i} = h$$

and define a local $(1,m)$-form with values in $End^0 T$ through

$$\Upsilon h = \sum_{i,j} (\Upsilon h)^i_j \otimes dz^j \otimes \frac{\partial}{\partial z^i}, \qquad (\Upsilon h)^i_j = \partial h^i_j - \frac{1}{2} dz^i \wedge \sum_s \frac{\partial}{\partial z^s} \lrcorner\, \partial h^s_j. \quad (9)$$

Proposition 2. *The matrix-form* $\left((\Upsilon h)^i_j\right)$, *locally defined by (9), is independent of the projective coordinate chart, and defines an element of* $\tilde{\Omega}^{(1,m)}_{L_0}$ *denoted as* $\Upsilon \omega_0^{(0,m)}$.

Operator $\partial \circ \operatorname{div} : \tilde{\Omega}^{(1,m)}_{L_0} \to \tilde{\Omega}^{(2,m)}_{L_1}$ is locally defined as follows. Again, express first an element $\omega^{(1,m)}$ in local projective coordinates as

$$\omega_0^{(1,m)}|_U = \sum_{i,j} h^i_j \otimes dz^j \otimes \frac{\partial}{\partial z^i} = h,$$

where h^i_j are differential forms of type $(n-1,m)$, and define a local (n,m)-form with values in T^* using

$$\partial \circ \operatorname{div} h = \sum_j (\partial \circ \operatorname{div} h)_j \otimes dz^j, \qquad (\partial \circ \operatorname{div} h)_j = \partial \left(\sum_s \frac{\partial}{\partial z^s} \lrcorner\, \partial h^s_j \right).$$

Proposition 3. *The previous differential form is the local expression of an element of* $\tilde{\Omega}^{(2,m)}_{L_1}$ *denoted* $(\partial \circ \operatorname{div})(\omega_0^{(1,m)})$.

We define now the morphisms D_k which appear in sequence (1) as follows:

$$D_k | \tilde{\Omega}^{(0,k)}_{L_{-1}} \stackrel{\text{def}}{=} \begin{cases} \bar{\partial} + \pi, & 0 \le k < 2 \\ \pi, & k = 2; \end{cases}$$

$$D_k | \tilde{\Omega}^{(p,k-p)}_{L_0} \stackrel{\text{def}}{=} \begin{cases} \bar{\partial} + \partial \operatorname{div}, & p = 1, \ \ 0 \le k-p < 2; \\ \bar{\partial}, & p = 2, \ \ 0 \le k-p \le 2 \\ \partial \operatorname{div}, & p = 1, \ \ k-p = 2; \end{cases}$$

From previous definitions it follows that

$$
D_k \stackrel{\text{def}}{=} \begin{cases} (\bar{\partial}+\pi)\oplus\bar{\partial}, \quad k=0; \\ (\bar{\partial}+\pi)\oplus(\bar{\partial}+\partial\,\mathrm{div}\,), \quad k=1; \\ \pi\oplus(\bar{\partial}+\partial\,\mathrm{div}\,)\oplus\bar{\partial}, \quad k=2; \\ \partial\,\mathrm{div}\,\oplus\bar{\partial}, \quad k=3; \\ 0, \quad k=4; \end{cases} \tag{10}
$$

Theorem 1. *Let $(M^4,\mathcal{P})$, be a locally projective two dimensional complex manifold, and $K_{hproj}(M^4)$ be the sheaf of germs of holomorphic projective vector fields. The fine resolution of the sheaf $K_{hproj}(M^4,\mathcal{P})$ is of the form (1), where operators D_k are defined by equation (10). The following sequence of sheaves and homomorphisms is the fine resolution of the sheaf $K_{hproj}(M^4,\mathcal{P})$: For $n=2$,*

$$
0\to K_{hproj}(M^4)\xrightarrow{i}\mathcal{X}(M^4)\xrightarrow{\bar{\partial}+\pi}\tilde{\Omega}^{(0,1)}_{L_{-1}}\oplus\tilde{\Omega}^{(1,0)}_{L_0}\xrightarrow{(\bar{\partial}+\pi)\oplus(\bar{\partial}+\partial\,\mathrm{div}\,)}
$$
$$
\tilde{\Omega}^{(0,2)}_{L_{-1}}\oplus\tilde{\Omega}^{(1,1)}_{L_0}\oplus\tilde{\Omega}^{(2,0)}_{L_1}\xrightarrow{\pi\oplus(\bar{\partial}+\partial\,\mathrm{div}\,)\oplus\bar{\partial}}\tilde{\Omega}^{(1,2)}_{L_0}\oplus\tilde{\Omega}^{(2,1)}_{L_1}\xrightarrow{\partial\,\mathrm{div}\,\oplus\bar{\partial}}\tilde{\Omega}^{(2,2)}_{L_1}\xrightarrow{0}0.
$$

In order to more obviously represent "rules" of how operators in given resolutions look like we are giving another "representation" of resolutions on two dimensional manifold:

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & K_{hproj}(M^4) \arrow[d] & & \\
 & & \mathcal{X}(M^4) \arrow[dl, "\bar{\partial}"'] \arrow[dr, "\pi"] & & \\
 & \tilde{\Omega}^{(0,1)}_{L_{-1}} \arrow[dl, "\bar{\partial}"'] \arrow[dr, "\pi"] & \oplus & \tilde{\Omega}^{(1,0)}_{L_0} \arrow[dl, "\bar{\partial}"'] \arrow[dr, "\partial\,\mathrm{div}"] & \\
\tilde{\Omega}^{(0,2)}_{L_{-1}} \arrow[dr, "\pi"'] & \oplus & \tilde{\Omega}^{(1,1)}_{L_0} \arrow[dl, "\bar{\partial}"'] \arrow[dr, "\partial\,\mathrm{div}"] & \oplus & \tilde{\Omega}^{(2,0)}_{L_1} \arrow[dl, "\bar{\partial}"] \\
 & \tilde{\Omega}^{(1,2)}_{L_0} \arrow[dr, "\partial\,\mathrm{div}"'] & \oplus & \tilde{\Omega}^{(2,1)}_{L_1} \arrow[dl, "\bar{\partial}"] & \\
 & & \tilde{\Omega}^{(2,2)}_{L_1} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}
$$

It is not difficult to prove the following lemmas.

Lemma 1. *Projective differential* $(2,i)$*-forms,* $0 \leq i \leq 2$ *satisfying conditions (3) on* M^4 *are identically equal to zero.*

The next three lemmas assertions are needed in the proof of previous theorem:

Lemma 2. *With appropriate domains of operators the following equalities hold:*

$$(i) \quad \bar{\partial}\pi = -\pi\bar{\partial}; \quad (ii) \quad \bar{\partial}\Upsilon = -\Upsilon\bar{\partial}; \quad (iii) \quad \bar{\partial}(\partial \operatorname{div}) = -(\partial \operatorname{div})\bar{\partial};$$

where Ker *denotes the kernel.*

Lemma 3. *With appropriate domains of operators the following equalities hold:*

$$(i) \quad \operatorname{Im}\pi = \operatorname{Ker}(\partial \operatorname{div}); \quad (ii) \quad \operatorname{Im}(\partial \operatorname{div}) = \Omega_L^{(2,k)};$$

where Im *denotes the image and index* i, j *and* k *depend on the domain of* $\partial \operatorname{div}$.

For the proof of this lemma we refer to [13]. Actually, we consider here smooth vector fields and use the Dolbeault lemma [3] while in [13] is used the Poincare lemma.

Lemma 4. *With appropriate domains of operators the following equalities hold:*

$$\operatorname{Ker}(\bar{\partial}\pi) = \{S + T | S \in \operatorname{Ker}\bar{\partial}, T \in \operatorname{Ker}\pi\}.$$

Lemma 5. $D_{k+1} \circ D_k = 0, \ \ 0 \leq k < 4.$

---

[3]Generalization of the Dolbeault lemma is (see [7]): If on a neighborhood $U \subseteq \mathbf{C}^n$ of z_0 C^∞ (p,q)–form φ, $q \geq 1$ satisfies $\bar{\partial}\varphi = 0$, then there exists C^∞ $(p, q-1)$- form Υ on W with $z_0 \in W \subseteq U$ (W open) such that $\varphi = \bar{\partial}\Upsilon$ on W. ¿From this Lemma it follows that the same statement holds not just for the operator $\bar{\partial}$, but also for ∂ : If on a neighborhood $U \subseteq \mathbf{C}^n$ of z_0 C^∞ (p,q)–form φ, $p \geq 1$ satisfies $\partial\varphi = 0$, then there exists C^∞ $(p-1, q)$ -form Υ on W with $z_0 \in W \subseteq U$ (W open) such that $\varphi = \partial\Upsilon$ on W. In the sequel we will use the second version of the Dolbeault lemma.

Lemma 5 can be proved using lemmas 2 and 3:

$$D_{k+1}\circ D_k\,|\tilde{\Omega}^{(p,k-p)}_{L_0} = \begin{cases} \bar\partial\bar\partial + \underbrace{\pi\bar\partial+\bar\partial\pi}_{0} + \underbrace{\partial\,\mathrm{div}\,\pi}_{0} = 0, & p=0,\ k=0;\\ \underbrace{\pi\bar\partial+\bar\partial\pi}_{0} + \underbrace{\partial\,\mathrm{div}\,\pi}_{0} = 0, & p=0,\ k=1;\\ \bar\partial\bar\partial + \underbrace{\partial\,\mathrm{div}\,\bar\partial + \bar\partial\partial\,\mathrm{div}}_{0} = 0, & p=1,\ k=1;\\ \underbrace{\partial\,\mathrm{div}\,\pi}_{0} = 0, & p=0,\ k=2;\\ \underbrace{\partial\,\mathrm{div}\,\bar\partial + \bar\partial\partial\,\mathrm{div}}_{0} = 0, & p=1,\ k=2;\\ \bar\partial\bar\partial = 0, & p=2,\ k=2; \end{cases}$$

Since from the definition of operator D_0 it follows that $\mathrm{Ker}\,D_0 = K_{hproj}(M^4)$ proof of Theorem 1 consists of the proofs of

$$\mathrm{Im}\,D_{k-1} = \mathrm{Ker}\,D_k, \quad 1\le k\le 2n.$$

¿From lemma 5 it follows that $\mathrm{Im}\,D_{k-1}\subseteq \mathrm{Ker}\,D_k$, $\quad 1\le k\le 2n$, so it is enough to prove that

$$\mathrm{Im}\,D_{k-1}\supseteq \mathrm{Ker}\,D_k, \quad 1\le k\le 4. \tag{11}$$

The proofs of (11) for various k, $1\le k\le 4$ can be grouped so that calculations involved in them have some parts in common. We have done it in the following way:

We will prove the exactness of the sequence of sheaves and homomorphisms for $n=2$. It will follow from the proof of (11) for $k=1,2,3,4$.

1. Proof of (11) for $k=1$, e.g. $\mathrm{Im}\,(\bar\partial+\pi)\supseteq \mathrm{Ker}\,((\bar\partial+\pi)\oplus(\bar\partial+\partial\,\mathrm{div}\,))$:

   Let $A^{01}\in\tilde{\Omega}^{(0,1)}_{L_{-1}}$, $A^{10}\in\tilde{\Omega}^{(1,0)}_{L_0}$ and $((\bar\partial+\pi)\oplus(\bar\partial+\partial\,\mathrm{div}\,))(A^{01},A^{10}) = (0,0,0)$, i.e.

$$\begin{aligned} \bar\partial A^{01} &= 0,\\ \pi A^{01}+\bar\partial A^{10} &= 0,\\ (\partial\,\mathrm{div}\,)(A^{10}) &= 0. \end{aligned}$$

   By lemma 3 from $\partial\,\mathrm{div}\,A^{10}=0$ it follows that there exists a smooth vector field v_1 such that $\pi(v_1)=A^{10}$. By the Dolbeault lemma, from

$\bar\partial A^{01}=0$, it follows that there exists a smooth vector field v_2 such that $\bar\partial v_2 = A^{01}$. We have

$$\begin{aligned} 0 &= \bar\partial A^{10} + \pi A^{01} \\ &= \bar\partial \pi v_1 + \pi \bar\partial v_2 \\ &= \bar\partial \pi v_1 - \bar\partial \pi v_2 \\ &= \bar\partial \pi (v_1 - v_2) \end{aligned}$$

By lemma 4 we have that $v_1 - v_2 = S + T$, $S \in \operatorname{Ker}\bar\partial, T \in \operatorname{Ker}\pi$. Let $v_3 = v_1 - T$ and $v_4 = v_2 + S$. Then $v_3 = v_4$,

$$\pi v_3 = \pi(v_1 - T) = \pi v_1 - \pi T = A^{10} - 0 = A^{10}$$

and

$$\bar\partial v_4 = \bar\partial(v_2 + S) = \bar\partial v_2 + \bar\partial S = A^{01} + 0 = A^{01}.$$

Vector field v_3 ($v_3 = v_4$) is such that fulfills $(\bar\partial + \pi)(v_3) = (A^{01}, A^{10})$. Therefore the sequence

$$\mathfrak{X}(M^4) \xrightarrow{\bar\partial+\pi} \tilde\Omega^{(0,1)}_{L_{-1}} \oplus \tilde\Omega^{(1,0)}_{L_0} \xrightarrow{(\bar\partial+\pi)\oplus(\bar\partial+\partial\operatorname{div})} \tilde\Omega^{(0,2)}_{L_{-1}} \oplus \tilde\Omega^{(1,1)}_{L_0} \oplus \tilde\Omega^{(2,0)}_{L_1}$$

is exact.

2. Proof of (11) for $k=2$, e.g. $\operatorname{Im}((\bar\partial+\pi)\oplus(\bar\partial+\partial\operatorname{div})) \supseteq \operatorname{Ker}(\pi \oplus (\bar\partial + \partial\operatorname{div}) \oplus \bar\partial)$:

   Let $(A^{02}, A^{11}, A^{20}) \in \operatorname{Ker}(\pi \oplus (\bar\partial+\partial\operatorname{div}) \oplus \bar\partial) \subset \tilde\Omega^{(0,2)}_{L_{-1}} \oplus \tilde\Omega^{(1,1)}_{L_0} \oplus \tilde\Omega^{(2,0)}_{L_1}$. Then

$$\begin{aligned} \pi A^{02} + \bar\partial A^{11} &= 0, &(12)\\ \partial \operatorname{div} A^{11} + \bar\partial A^{20} &= 0. &(13) \end{aligned}$$

   We should find $(B^{01}, B^{10}) \in \tilde\Omega^{(0,1)}_{L_{-1}} \oplus \tilde\Omega^{(1,0)}_{L_0}$ that satisfies

$$((\bar\partial+\pi)\oplus(\bar\partial+\partial\operatorname{div}))(B^{01}, B^{10}) = (A^{02}, A^{11}, A^{20}),$$

   i.e., the solution of the following system of equations:

$$\begin{aligned} \bar\partial B^{01} &= A^{02}, &(14)\\ \pi B^{01} + \bar\partial B^{10} &= A^{11}, &(15)\\ \partial \operatorname{div} B^{10} &= A^{20}. &(16) \end{aligned}$$

Since operator $\partial\,\mathrm{div}$ is surjective (lemma 3 (iii)), there exists $\tilde{B}^{10} \in \tilde{\Omega}^{(1,0)}_{L_0}$ such that

$$\partial\,\mathrm{div}\,\tilde{B}^{10} = A^{20}.$$

By the Dolbeault lemma and $\bar{\partial}A^{02} = 0$ it follows that there exists $\tilde{B}^{01} \in \tilde{\Omega}^{(0,1)}_{L_{-1}}$ such that

$$\bar{\partial}\tilde{B}^{01} = A^{02}.$$

System (14), (16) has solution $(\tilde{B}^{01}, \tilde{B}^{10})$, but that is not necessarily the solution of equation (15).

Let

$$\tilde{A}^{11} = \pi\tilde{B}^{01} + \bar{\partial}\tilde{B}^{10}.$$

If $\tilde{A}^{11} = A^{11}$ then $(B^{01}, B^{10}) = (\tilde{B}^{01}, \tilde{B}^{10})$ is the solution of system (14), (15), (16).

If $\tilde{A}^{11} \neq A^{11}$ then $(\tilde{B}^{01}, \tilde{B}^{10})$ is not the solution of equation (15). We will find a projective matrix form $C \in \tilde{\Omega}^{(1,0)}_{L_0}$ such that $\bar{\partial}C = A^{11} - \tilde{A}^{11}$ and $\partial\,\mathrm{div}\,C = 0$. With this, we have

$$\begin{aligned}
\bar{\partial}\tilde{B}^{01} &= A^{02}, \\
\bar{\partial}(C + \tilde{B}^{10}) + \pi\tilde{B}^{01} &= \underbrace{\bar{\partial}C}_{A^{11}-\tilde{A}^{11}} + \underbrace{\bar{\partial}\tilde{B}^{10} + \pi\tilde{A}^{01}}_{\tilde{A}^{11}} = A^{11}, \\
\partial\,\mathrm{div}\,(C + \tilde{B}^{10}) &= \underbrace{\partial\,\mathrm{div}\,C}_{0} + \underbrace{\partial\,\mathrm{div}\,\tilde{B}^{10}}_{A^{20}} = A^{20},
\end{aligned}$$

and $(B^{01}, B^{10}) = (\tilde{B}^{01}, C + \tilde{B}^{10})$ is the solution of system (14), (15) and (16).

Proof of existence of $C \in \tilde{\Omega}^{(1,0)}_{L_0}$ such that $\bar{\partial}C = A^{11} - \tilde{A}^{11}$ and $\partial\,\mathrm{div}\,C = 0$:

Since

$$\begin{aligned}
\bar{\partial}(A^{11} - \tilde{A}^{11}) &= \bar{\partial}A^{11} - \bar{\partial}(\bar{\partial}\tilde{A}^{10} + \pi\tilde{A}^{01}) \\
&= -\pi A^{02} - \underbrace{\bar{\partial}\pi}_{-\pi\bar{\partial}}\tilde{A}^{01} \\
&= -\pi A^{02} + \pi A^{02} \\
&= 0,
\end{aligned}$$

by the Dolbeault lemma it follows that there exists $C \in \tilde{\Omega}^{(1,0)}_{L_0}$ such that

$$\bar{\partial} C = A^{11} - \tilde{A}^{11} \neq 0. \tag{17}$$

Since

$$\begin{aligned} \partial\,\mathrm{div}\,(A^{11} - \tilde{A}^{11}) &= \underbrace{\partial\,\mathrm{div}\,A^{11}}_{-\bar{\partial}A^{20}} - \partial\,\mathrm{div}\,(\bar{\partial}\tilde{B}^{10} + \pi\tilde{X}^{01}) \\ &= -\bar{\partial}A^{20} - \underbrace{\partial\,\mathrm{div}\,\bar{\partial}}_{-\bar{\partial}\partial\,\mathrm{div}}\,\tilde{B}^{10} - \underbrace{\partial\,\mathrm{div}\,\pi}_{0}\,\tilde{X}^{01} \\ &= -\bar{\partial}A^{20} + \bar{\partial}\,\underbrace{\partial\,\mathrm{div}\,\tilde{B}^{10}}_{A^{20}} \qquad (18) \\ &= 0 \end{aligned}$$

and $\partial\,\mathrm{div}\,(A^{11} - \tilde{A}^{11}) = \partial\,\mathrm{div}\,\bar{\partial}C = -\bar{\partial}\partial\,\mathrm{div}\,C$ it follows that $\bar{\partial}\partial\,\mathrm{div}\,C = 0$. Therefore coefficients of $\partial\,\mathrm{div}\,C \in \tilde{\Omega}^{(2,0)}_{L_1}$ do not depend on $\bar{z}^1, \bar{z}^2$. So $\partial\,\mathrm{div}\,C \neq 0$ implies that the coefficients of C do not depend on $\bar{z}^1, \bar{z}^2$ because the operator $\partial\,\mathrm{div}$ acts only on variables z^1, z^2. Therefore $\bar{\partial}C = 0$ and this is in contradiction with (17). So it follows that $\partial\,\mathrm{div}\,C = 0$.

By this it is proved that the sequence

$$\tilde{\Omega}^{(0,1)}_{L_{-1}} \oplus \tilde{\Omega}^{(1,0)}_{L_0} \xrightarrow{(\bar{\partial}+\pi)\oplus(\bar{\partial}+\partial\,\mathrm{div}\,)} \tilde{\Omega}^{(0,2)}_{L_{-1}} \oplus \tilde{\Omega}^{(1,1)}_{L_0} \oplus \tilde{\Omega}^{(2,0)}_{L_1} \xrightarrow{\pi\oplus(\bar{\partial}+\partial\,\mathrm{div}\,)\oplus\bar{\partial}} \tilde{\Omega}^{(1,2)}_{L_0} \oplus \tilde{\Omega}^{(2,1)}_{L_1}$$

is exact.

3. Proof of (11) for $k = 3$, e.g. $\mathrm{Im}\,(\pi \oplus (\bar{\partial} + \partial\,\mathrm{div}\,) \oplus \bar{\partial}) \supseteq \mathrm{Ker}\,(\partial\,\mathrm{div} \oplus \bar{\partial})$:
   Let $(A^{12}, A^{21}) \in \mathrm{Ker}\,(\partial\,\mathrm{div}\,\oplus \bar{\partial})$. Then

$$\partial\,\mathrm{div}\,A^{12} + \bar{\partial}A^{21} = 0 \tag{19}$$

   We should find $(B^{02}, B^{11}, B^{20}) \in \tilde{\Omega}^{(0,2)}_{L_{-1}} \oplus \tilde{\Omega}^{(1,1)}_{L_0} \oplus \tilde{\Omega}^{(2,0)}_{L_1}$ so that

$$(\pi \oplus (\bar{\partial} + \partial\,\mathrm{div}\,) \oplus \bar{\partial})(B^{02}, B^{11}, B^{20}) = (A^{12}, A^{21}),$$

   i.e., the solution of the following system of equations:

$$\pi B^{02} + \bar{\partial}B^{11} = A^{12}, \tag{20}$$
$$\partial\,\mathrm{div}\,B^{11} + \bar{\partial}B^{20} = A^{21} \tag{21}$$

Lets take any $\tilde{B}^{02} \in \tilde{\Omega}^{(0,2)}_{L_{-1}}$ and define M as $M = A^{12} - \pi\tilde{B}^{02}$. Since $M \in \tilde{\Omega}^{(1,2)}_{L_0}$ it follows that $\bar{\partial}M = 0$ and by the Dolbeault lemma we have that there exists $\tilde{B}^{11} \in \tilde{\Omega}^{(1,1)}_{L_0}$ such that $\bar{\partial}\tilde{B}^{11} = M = A^{12} - \pi\tilde{B}^{02}$. Let $N = A^{21} - \partial \operatorname{div} \tilde{B}^{11}$. Then

$$\begin{aligned}
\bar{\partial}N &= \bar{\partial}A^{21} - \bar{\partial}\partial \operatorname{div} \tilde{B}^{11} \\
&= -\partial \operatorname{div} A^{12} + \partial \operatorname{div} \bar{\partial}\tilde{B}^{11} \\
&= -\partial \operatorname{div} (\pi\tilde{B}^{02} + \bar{\partial}\tilde{B}^{11}) + \partial \operatorname{div} \bar{\partial}\tilde{B}^{11} \\
&= \underbrace{-\partial \operatorname{div} \pi}_{0} \tilde{B}^{02} = 0
\end{aligned}$$

and therefore by the Dolbeault lemma we have that there exists $\tilde{B}^{20} \in \tilde{\Omega}^{(2,0)}_{L_1}$ such that $\bar{\partial}\tilde{B}^{20} = N$.

System (20), (21) has solution $(\tilde{B}^{02}, \tilde{B}^{11}, \tilde{B}^{20})$.

By this it is proved that the sequence

$$\tilde{\Omega}^{(0,2)}_{L_{-1}} \oplus \tilde{\Omega}^{(1,1)}_{L_0} \oplus \tilde{\Omega}^{(2,0)}_{L_1} \xrightarrow{\pi \oplus (\bar{\partial} + \partial \operatorname{div}) \oplus \bar{\partial}} \tilde{\Omega}^{(1,2)}_{L_0} \oplus \tilde{\Omega}^{(2,1)}_{L_1} \xrightarrow{\partial \operatorname{div} \oplus \bar{\partial}} \tilde{\Omega}^{(2,2)}_{L_1}$$

is exact.

4. Proof of (11) for $k = 4$, e.g. We should prove that $\operatorname{Im}(\partial \operatorname{div} \oplus \bar{\partial}) = \tilde{\Omega}^{(2,2)}_{L_1}$.

   Let $(A^{2,2}) \in \tilde{\Omega}^{(2,2)}_{L_1}$. We should find $(B^{1,2}, B^{2,1}) \in \tilde{\Omega}^{(1,2)}_{L_0} \oplus \tilde{\Omega}^{(2,1)}_{L_1}$ so that

$$(\partial \operatorname{div} \oplus \bar{\partial})(B^{1,2}, B^{2,1}) = A^{2,2},$$

   i.e., the solution of the equation:

$$\partial \operatorname{div} B^{1,2} + \bar{\partial}B^{2,1} = A^{2,2}. \tag{22}$$

   Since $\bar{\partial}A^{2,2} = 0$ by the Dolbeault lemma we have that there exists $B^{2,1} \in \tilde{\Omega}^{(2,1)}_{L_1}$ such that $\bar{\partial}B^{2,1} = A^{2,2}$. Therefore $(0, B^{2,1})$ is the solution of equation (22) and the following sequence is exact

$$\tilde{\Omega}^{(1,2)}_{L_0} \oplus \tilde{\Omega}^{(2,1)}_{L_1} \xrightarrow{\partial \operatorname{div} \oplus \bar{\partial}} \tilde{\Omega}^{(2,2)}_{L_1} \longrightarrow 0.$$

   By this the proof of the main Theorem 1 is finished.

P r o o f of Lemma 4 for two dimensional complex manifold M^4:
Let $\mathcal{K}_i$ be the set of the functions of the form

$$\alpha^i = a^i(\bar z) + \sum_{j=1}^{2} a^i_j(\bar z) z^j + z^i \sum_{s=1}^{2} a_s(\bar z) z^s \quad (\bar z = (\bar z^1, \bar z^2)),$$

where $a^i(\bar z), a^i_j(\bar z), a_s(\bar z)$ are smooth functions of $\bar z$ and $1 \le i, j, s \le 2$. In general, smooth functions f with $\partial f = 0$ are called antiholomorphic ones.

Let $\tilde{\mathcal{K}}_i$ be the set of the (0,1)–forms of the form

$$\alpha^i = a^i(\bar z) + \sum_{j=1}^{2} a^i_j(\bar z) z^j + z^i \sum_{s=1}^{2} a_s(\bar z) z^s,$$

where $a^i(\bar z), a^i_j(\bar z), a_s(\bar z)$ are (0,1)–forms whose coefficients are antiholomorphic functions.

We will use the fact that $\operatorname{Ker}\pi = \{a\frac{\partial}{\partial z^1} + b\frac{\partial}{\partial z^2} \mid a \in \mathcal{K}_1, b \in \mathcal{K}_2\}$. ¿From the definition of the operator $\tilde\pi$ it follows that $\operatorname{Ker}\tilde\pi = \{a\frac{\partial}{\partial z^1} + b\frac{\partial}{\partial z^2} \mid a \in \tilde{\mathcal{K}}_1, b \in \tilde{\mathcal{K}}_2\}$.

Since by lemma 2 $\bar\partial\pi = -\tilde\pi\bar\partial$, it follows that $\operatorname{Ker}\bar\partial\pi = \operatorname{Ker}\tilde\pi\bar\partial$. Let $v \in \operatorname{Ker}\tilde\pi\bar\partial \subset \mathfrak{X}(M^4)$, $\quad v = \sum_{i=1}^{2} v^i \frac{\partial}{\partial z^i}$. Then $0 = \tilde\pi\bar\partial(v) = \tilde\pi\left(\sum_{i=1}^{2} \bar\partial v^i \otimes \frac{\partial}{\partial z^i}\right)$ and therefore

$$\tilde\pi\left(\sum_{i=1}^{2} \frac{\partial v^i}{\partial \bar z^j} d\bar z^i \otimes \frac{\partial}{\partial z^i}\right) = 0 \quad 1 \le j \le 2.$$

Since $\bar\partial v^i \in \tilde{\mathcal{K}}_i$, $\quad 1 \le i \le 2$ it follows that $\frac{\partial v^i}{\partial \bar z^1}, \frac{\partial v^i}{\partial \bar z^2} \in \mathcal{K}_i$, $\quad 1 \le i \le 2$. Therefore they can be written in the following way

$$\begin{aligned} \frac{\partial v^i}{\partial \bar z^1} &= a^i(\bar z) + \sum_j a^i_j(\bar z) z^j + z^i \sum_s a_s(\bar z) z^s = \alpha^i, \quad 1 \le i \le 2, \quad \alpha^i \in \mathcal{K}_i, \\ \frac{\partial v^i}{\partial \bar z^2} &= b^i(\bar z) + \sum_j b^i_j(\bar z) z^j + z^i \sum_s b_s(\bar z) z^s = \beta^i, \quad 1 \le i \le 2, \quad \beta^i \in \mathcal{K}_i. \end{aligned} \tag{23}$$

¿From (23), for $i = 1$, it follows that

$$\begin{aligned} v^1 &= \int_{\bar z_0^1}^{\bar z^1} \alpha^1 \, d\bar z^1 + f_1(\bar z^2, z), \\ v^1 &= \int_{\bar z_0^2}^{\bar z^2} \beta^1 \, d\bar z^2 + f_2(\bar z^1, z), \end{aligned} \tag{24}$$

where f_1, resp., f_2, resp. are smooth functions of variables $z^1, z^2, \bar{z}^2$, resp., $z^1, \bar{z}^1, z^2$, resp., considered as independent variables. Integrals in (24) are elements of $\mathcal{K}_1$ so

$$v^1 = m_1 + f_1(\bar{z}^2, z) = m_2 + f_2(\bar{z}^1, z),\ m_1, m_2 \in \mathcal{K}_1. \tag{25}$$

We have

$$f_1(\bar{z}^2, z) - f_2(\bar{z}^1, z) = m_2 - m_1 = t^1(\bar{z}) + \sum_{j=1}^{2} t^1_j(\bar{z}) z^j + z^1 \sum_{s=1}^{2} t_s(\bar{z}) z^s., \tag{26}$$

We shall prove that (26) implies

$$t^1(\bar{z}) = p^1(\bar{z}^1) - q^1(\bar{z}^2), \quad t^1_j(\bar{z}) = p^1_j(\bar{z}^1) - q^1_j(\bar{z}^2),\ 1 \le j \le 2,$$
$$t_s(\bar{z}) = p_s(\bar{z}^1) - q_s(\bar{z}^2),\ 1 \le s \le 2,\ z = (z^1, z^2), \tag{27}$$

where on the right hand side appear appropriate antiholomorphic functions. Applying $\dfrac{\partial}{\partial z^1}$ and then $\dfrac{\partial}{\partial z^s}$ on both sides of (26), we obtain that

$$t_s(\bar{z}) = \frac{\partial}{\partial z^1}\frac{\partial}{\partial z^s}(f(\bar{z}^2, z) - g(\bar{z}^1, z)),\ 1 \le s \le 3. \tag{28}$$

This gives

$$f(\bar{z}^2, z) - g(\bar{z}^1, z) - z^1 \sum_{s=1}^{2} z^s \frac{\partial}{\partial z^1}\frac{\partial}{\partial z^s}\Big(f(\bar{z}^2, z) - g(\bar{z}^1, z)\Big) \tag{29}$$
$$= t^1(\bar{z}) + \sum_{j=1}^{2} t^1_j(\bar{z}) z^j.$$

Now apply $\frac{\partial}{\partial z^j}$ on both sides of (29). This implies ($1 \le j \le 3$)

$$t^1_j(\bar{z}) = \frac{\partial}{\partial z^j}\Big(f_1(\bar{z}^2, z) - f_2(\bar{z}^1, z) - z^1 \sum_{s=1}^{2} z^s \frac{\partial}{\partial z^1}\frac{\partial}{\partial z^s}(f_1(\bar{z}^2, z) - f_2(\bar{z}^1, z))\Big), \tag{30}$$

$$t^1(\bar{z}) = f_1(\bar{z}^2, z) - f_2(\bar{z}^1, z) - z^1 \sum_{s=1}^{2} z^s \frac{\partial}{\partial z^1}\frac{\partial}{\partial z^s}\Big(f_1(\bar{z}^2, z) - f_2(\bar{z}^1, z)\Big) \tag{31}$$
$$- \sum_{j=1}^{2} \frac{\partial}{\partial z^j}\Big(f_1(\bar{z}^2, z) - f_2(\bar{z}^1, z) - z^1 \sum_{s=1}^{2} z^s \frac{\partial}{\partial z^1}\frac{\partial}{\partial z^s}(f_1(\bar{z}^2, z) - f_2(\bar{z}^1, z))\Big).$$

We have $(1 \le j, s \le 2)$

$$\begin{aligned} t^1(\bar z) &= F^1(\bar z^2, z) - \tilde F^1(\bar z^1, z), \\ t_s(\bar z) &= F_s(\bar z^2, z) - \tilde F_s(\bar z^1, z), \\ t^1_j(\bar z) &= F^1_j(\bar z^2, z) - \tilde F^1_j(\bar z^1, z), \end{aligned}$$

where $F_s, \tilde F_s$ $F^1, \tilde F^1$ and $F^1_j, \tilde F^1_j$ are smooth functions on the left side of (28), (30) and (31), of variables considered as independent ones.

This implies

$$\begin{gathered} F^1(\bar z^2, z) - \tilde F^1(\bar z^1, z) = F^1(\bar z^2, z_0) - \tilde F^1(\bar z^1, z_0) = p^1(\bar z^1) - q^1(\bar z^2), \\ F_s(\bar z^2, z) - \tilde F_s(\bar z^1, z) = F_s(\bar z^2, z_0) - \tilde F_s(\bar z^1, z_0) = p_s(\bar z^1) - q_s(\bar z^2), \\ F^1_j(\bar z^2, z) - \tilde F^1_j(\bar z^1, z) = F^1_j(\bar z^2, z_0) - \tilde F^1_j(\bar z^1, z_0) = p^1_j(\bar z^1) - q^1_j(\bar z^2), \\ 1 \le j, s \le 2. \end{gathered} \tag{32}$$

This proves (27).

Decomposition (27) and (26) imply

$$\begin{aligned} f_1(\bar z^2, z) - f_2(\bar z^1, z) &= p^1(\bar z^1) - q^1(\bar z^2) + \sum_{j=1}^{2}(p^1_j(\bar z^1) - q^1_j(\bar z^2))z^j + \\ & z^1 \sum_{s=1}^{2}(p_s(\bar z^1) - q_s(\bar z^2)z^s) = \tilde m_2(\bar z^1, z) - \tilde m_1(\bar z^2, z), \end{aligned}$$

where $\tilde m_1, \tilde m_2 \in \mathcal{K}_1$.

Equation (33) implies $f_1(\bar z^2, z) + \tilde m_1(\bar z^2, z) = A_1(z)$. It follows that A_1 does not depend on $\bar z^2$. This gives $f_1(\bar z^2, z) = \tilde A_1(z) - \tilde m(\bar z^2, z) \in \operatorname{Ker} \bar\partial + \mathcal{K}_1$.

This completes the proof of lemma 4 for two dimensional complex manifold.

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REFERENCES

[1] R. L. B r y a n t, S. S. C h e r n, R. B. G a r d n e r, H. G o l d s c h m i d t, P. A. G r i f f i t h s, *Exterior differential systems*, Mathematical Sciences Research Institute Publications, 18. Springer-Verlag, New York, 1991.

[2] H. G o l d s c h m i d t, *Duality theorems in deformation theory*, Trans. Amer. Math. Soc. 292 (1985), 1-49.

[3] J. G a s q u i, H. G o l d s c h m i d t, *Théorémes de dualit en gomtrie conforme*, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), 99–102.

[4] J. G a s q u i, H. G o l d s h m i d t, *Déformations infinitésimales des structures conformes plates*, Birkhäuser, 1984.

[5] P. G r i f f i t h s, *On the theory of variation of structures defined by transitive continuous pseudogroups,* Osaka J Math 1 (1964) 175-199.

[6] P. G r i f f i t h s, J. H a r r i s,*Principles of algebraic geometry*, John Wiley & Sons, Inc., New York, 1994.

[7] R. C. G u n n i n g, H. R o s s i, *Analytic Functions of Several Complex Variables*, Englewood Cliffs, N. J.: Prentice Hall, 1965.

[8] Th. H a n g a n, *Complexes différentiels associés aux pseudo-groupes projectif et conforme,* C. R. Acad. Sci. Paris, t. 312, Série I (1991).

[9] Th. H a n g a n, *La résolution de Calabi et ses analogies projective et conforme, Seminari di Geometria,* Giornate di Topologia e Geometria Delle Varieta, Bologna, 1992.

[10] K. K o d a i r a, *On deformations of some complex pseudogroup structures*, Ann. of Math. (2), 71 (1960), 224-302.

[11] A. K u m p e r a, *Invariants différentiels d'un pseudogroupe de Lie.* J. Differential Geometry 10 (1975), Part I 289–345.

[12] O. M i l e n k o v i ć, S. P i l i p o v i ć, *Fine Resolution of the Sheaf of Germs of Holomorphic Projective Vector Fields,* to appear in Rev. Roum. Math. Pure. Appl.

[13] O. M i l e n k o v i ć,*Two Technical Lemmas About Forms Over Finite Dimensional Manifolds,* Univ. Beograd, Publ. Elektrotehn. Fak, Ser. Mat. 9 (1998) 71-73.

[14] D. C. S p e n c e r, *Deformations of structures on manifolds defined by transitive,continuous pseudogrups*,I, II, Ann. of Math. 76(1962), 306-445.

Ericsson Systems Expertise
Beech Hill, Clonskeagh,
Dublin 4 (Ireland)
Olivera.Milenkovic@ericsson.com