Combinatorial Results for Certain Semigroups of Transformations Preserving Orientation and a Uniform Partition

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Abstract. Let $T_X$ be the full transformation semigroup on a set $X$ and $E$ be a non-trivial equivalence on $X$. The set $T_E(X) = \{f \in T_X : \forall (x,y) \in E, (f(x), f(y)) \in E\}$ is a subsemigroup of $T_X$. For a finite totally ordered set $X$ and a convex equivalence $E$ on $X$, the set of all the orientation-preserving transformations in $T_E(X)$ forms a subsemigroup of $T_E(X)$ denoted by $OP_E(X)$. In this paper, under the hypothesis that the totally ordered set $X$ is of cardinality $mn$ $(m, n \geq 2)$ and the equivalence $E$ has $m$ classes such that each $E$-class contains $n$ consecutive points, we calculate the cardinality of the semigroup $OP_E(X)$, and that of its idempotents.

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1. Introduction

Let $X = \{1, 2, \cdots, n\}$ with the usual order and let $\mathcal{P}_X$ and $\mathcal{T}_X$ denote the partial transformation semigroup and the full transformation semigroup on $X$, respectively. A map $f \in \mathcal{T}_X$ is said to be order-preserving if $x \leq y$ implies $f(x) \leq f(y)$ for $x, y \in X$. The collection of all the order-preserving maps on $X$ is denoted by $O_X$ in [6] (the symbol $O_X$ is replaced by $O_n$ in [2]). A sequence $A = (a_1, a_2, \cdots, a_n)$ is said to be cyclic if there exists no more than one subscript $i$ such that $a_i > a_{i+1}$. A map $f \in \mathcal{T}_X$ is said to be orientation-preserving, if $(f(1), f(2), \cdots, f(n))$ is cyclic, which implies that there exists some $j \in \{0, 1, \cdots, n-1\}$ such that

$$f(j+1) \leq f(j+2) \leq \cdots \leq f(n) \leq f(1) \leq \cdots \leq f(j)$$

(where we adopt the convention that $f(1) \leq f(2) \leq \cdots \leq f(n)$ if $j = 0$). Clearly, if $f$ is order-preserving, then it is also orientation-preserving. The collection of all the orientation-preserving maps on $X$ is denoted by $OP_n$ and has been investigated by Catarino and Higgins in [2]. Combinatorial results of various classes of transformation subsemigroups of $\mathcal{P}_X$ and

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The authors considered the subsemigroup of \( E \) under the supposition that all \( E \) are of the same size, the regularity and Green’s relations for the semigroup \( OP \) were described.

In this paper, as in [14], we always assume the totally ordered set \( X \) and the convex equivalence \( E \) on \( X \), the authors considered the subsemigroup of \( T_e(X) \)

\[
OP_e(X) = \{ f \in T_e(X) : f \text{ is orientation-preserving} \},
\]

and under the supposition that all \( E \)-classes were of the same size, the regularity and Green’s relations for the semigroup \( OP_e(X) \) were described.

In this paper, as in [14], we always assume the totally ordered set \( X = \{1 < 2 < \cdots < mn\} \) \((m, n \geq 2)\) and the equivalence \( E \) to be

\[
E = (A_1 \times A_1) \cup (A_2 \times A_2) \cup \cdots \cup (A_m \times A_m),
\]

where \( A_i = [(i-1)n + 1, in] \) for \( 1 \leq i \leq m \). We investigate combinatorial properties of the semigroup \( OP_e(X) \). The paper is organized as follows. In Section 2, we determine the cardinality of \( OP_e(X) \). In Section 3, we characterize the idempotents in the semigroup \( OP_e(X) \) and calculate their number.

Denote by \( X/E \) the quotient set of \( X \). The following result whose proof is routine describes an essential property of the transformations in the semigroup \( T_e(X) \) where \( X \) is an arbitrary set and \( E \) is an arbitrary equivalence on \( X \).

**Lemma 1.1.** Let \( f \in T_e(X) \), then for each \( B \in X/E \), there exists \( B' \in X/E \) such that \( f(B) \subseteq B' \). Consequently, for each \( A \in X/E \), the set \( f^{-1}(A) \) is either \( \emptyset \) or a union of some \( E \)-classes.

For each \( f \in T_e(X) \), let

\[
E(f) = \{ f^{-1}(A) : A \in X/E \text{ and } f^{-1}(A) \neq \emptyset \}.
\]

Then \( E(f) \) is a partition of \( X \). The following result shows that each orientation-preserving transformation induces a partition of convex subsets.
Lemma 1.2. Let $f \in \text{OP}_{E}(X)$. Then each $U \in E(f)$ is a convex subset of $X$.

2. The cardinality of $\text{OP}_{E}(X)$

In this section, we focus our attention on the cardinality of $\text{OP}_{E}(X)$. We notice that for each $f \in \text{OP}_{E}(X)$, there exists some $j$ such that $f(j+1) = \min f(X)$ and $f(j) = \max f(X)$ and $j$ is unique if $f$ is not constant. Therefore, there are two cases: $(j, j+1) \notin E$ or $(j, j+1) \in E$. We first consider subsets of $\text{OP}_{E}(X)$ consisting of those elements for which $(j, j+1) \notin E$. It is not hard to see that in this case, $j$ is the greatest number in some $E$-class $A_{i}$ while $j+1$ is the smallest number in the next $E$-class $A_{i+1}$. Define certain subsets $\mathcal{A}_{1}, \mathcal{A}_{2}, \ldots, \mathcal{A}_{m}$ of $\text{OP}_{E}(X)$ by:

$$\mathcal{A}_{1} = \{ f \in \text{OP}_{E}(X) : f(1) = \min f(X) \text{ and } f(mn) = \max f(X) \},$$

$$\mathcal{A}_{2} = \{ f \in \text{OP}_{E}(X) : f(n+1) = \min f(X) \text{ and } f(n) = \max f(X) \},$$

$$\ldots,$$

$$\mathcal{A}_{m} = \{ f \in \text{OP}_{E}(X) : f((m-1)n+1) = \min f(X) \text{ and } f((m-1)n) = \max f(X) \}.$$

Obviously, if $f \in \mathcal{A}_{i}(1 \leq i \leq m)$, then $|f(X)| \leq mn$ and

$$f((i-1)n+1) \leq f((i-1)n+2) \leq \cdots \leq f(mn) \leq f(1) \leq \cdots \leq f((i-1)n).$$

Next, we consider another subsets consisting of those elements for which $(j, j+1) \in E$ and $f(j+1) = \min f(X), f(j) = \max f(X)$. For each $1 \leq s \leq m$, define certain subsets $\mathcal{B}_{s,1}, \mathcal{B}_{s,2}, \ldots, \mathcal{B}_{s,n-1}$ of $\text{OP}_{E}(X)$ by:

$$\mathcal{B}_{s,1} = \{ f \in \text{OP}_{E}(X) : f((s-1)n+2) = \min f(X) \text{ and } f((s-1)n+1) = \max f(X) \},$$

$$\mathcal{B}_{s,2} = \{ f \in \text{OP}_{E}(X) : f((s-1)n+3) = \min f(X) \text{ and } f((s-1)n+2) = \max f(X) \},$$

$$\ldots,$$

$$\mathcal{B}_{s,n-1} = \{ f \in \text{OP}_{E}(X) : f(sn) = \min f(X) \text{ and } f(sn-1) = \max f(X) \}.$$n

If $f \in \mathcal{B}_{s,t}(1 \leq t \leq n-1)$, then $f$ maps all the elements of $X$ into some $E$-class and

$$f((s-1)n+t+1) \leq f((s-1)n+t+2) \leq \cdots \leq f(mn) \leq f(1) \leq \cdots \leq f((s-1)n+t).$$

Therefore,

$$\text{OP}_{E}(X) = \left( \bigcup_{s=1}^{m} \mathcal{A}_{s} \right) \bigcup \left( \bigcup_{s=1}^{m-1} \bigcup_{t=1}^{n-1} \mathcal{B}_{s,t} \right)$$

and for $s \neq s', t \neq t'$,

$$\mathcal{A}_{s} \cap \mathcal{A}_{s'} = \mathcal{B}_{s,t} \cap \mathcal{B}_{s',t'} = \mathcal{B}_{s,t} \cap \mathcal{A}_{t} = \{(1), (2), \ldots, (mn)\},$$

where $(x)$ denotes the constant map which maps all the elements of $X$ into $x$.

We give two properties for the subsets $\mathcal{A}_{1}, \mathcal{A}_{2}, \ldots, \mathcal{A}_{m}$ and $\mathcal{B}_{s,1}, \mathcal{B}_{s,2}, \ldots, \mathcal{B}_{s,n-1}(1 \leq s \leq m)$.

Lemma 2.1. Let $\mathcal{A}_{1}, \mathcal{A}_{2}, \ldots, \mathcal{A}_{m}$ be as defined above. Then

$$|\mathcal{A}_{1}| = |\mathcal{A}_{2}| = \cdots = |\mathcal{A}_{m}|.$$
Proof. For \( f \in \mathcal{A}_1 \), define \( \psi_1 : \mathcal{A}_1 \rightarrow \mathcal{A}_2 \) by \( \psi_1(f) = g \) where
\[
g(x) = \begin{cases} 
  f(mn + x - n) & 1 \leq x \leq n \\
  f(x - n) & \text{otherwise.}
\end{cases}
\]

Then \( \psi_1 \) is well defined. To see \( g \in \mathcal{A}_2 \), let \((x, y) \in E\), if \( x, y \in A_1 \), then \((mn + x - n, mn + y - n) \in E\) and \((g(x), g(y)) = (f(mn + x - n), f(mn + y - n)) \in E\). If \( x, y \notin A_1 \), then \((x - n, y - n) \in E\) and \((g(x), g(y)) = (f(x - n), f(y - n)) \in E\) which implies that \( g \in T_E(X) \). Moreover,
\[
g(n + 1) = f(1) \leq g(n + 2) = f(2) \leq \cdots \leq g(mn) = f(mn - n) \\
\leq g(1) = f(mn + 1 - n) \leq g(2) = f(mn + 2 - n) \cdots \leq g(n) = f(mn).
\]

So \( g \in \mathcal{A}_2 \). It is clear that \( \psi_1 \) is a bijection from \( \mathcal{A}_1 \) onto \( \mathcal{A}_2 \). Therefore, \( |\mathcal{A}_1| = |\mathcal{A}_2| \). Similarly, we can define \( \psi_2, \psi_3, \cdots, \psi_{m-1} \) and show that \( |\mathcal{A}_2| = |\mathcal{A}_3| = |\mathcal{A}_4|, \cdots, |\mathcal{A}_{m-1}| = |\mathcal{A}_m| \). Consequently, \( |\mathcal{A}_1| = |\mathcal{A}_2| = \cdots = |\mathcal{A}_m| \).

Lemma 2.2. For \( 1 \leq s \leq m \), let \( \mathcal{B}_{s,1}, \mathcal{B}_{s,2}, \cdots, \mathcal{B}_{s,n-1} \) be as defined above. Then
\[
(1) \; |\mathcal{B}_{s,1}| = |\mathcal{B}_{s,2}| = \cdots = |\mathcal{B}_{s,n-1}|, \\
(2) \; |\mathcal{B}_{s,l}| = |\mathcal{B}_{s,l}| \text{ for } 1 \leq s, s' \leq m \text{ and } 1 \leq l \leq n - 1.
\]

Proof. (1) For \( f \in \mathcal{B}_{s,t}(1 \leq t \leq n - 1) \), define \( \rho : \mathcal{B}_{s,t} \rightarrow \mathcal{B}_{s,t+1} \) by \( \rho(f) = g \) where
\[
g(x) = \begin{cases} 
  f(mn) & x = 1 \\
  f(x - 1) & \text{otherwise.}
\end{cases}
\]

Since \( f \) maps \( X \) into some \( E \)-class and \( g(X) = f(X) \), we have \( g \in T_E(X) \). Moreover,
\[
g((s - 1)n + t + 2) = f((s - 1)n + t + 1) \leq g((s - 1)n + t + 3) = f((s - 1)n + t + 2) \leq \cdots \leq g(mn) = f(mn - 1) \leq g(1) = f(mn) \leq \cdots \leq g((s - 1)n + t + 1) = f((s - 1)n + t).
\]

Thus \( g \in \mathcal{B}_{s,t+1} \). One easily verifies that \( \rho \) is a bijection from \( \mathcal{B}_{s,t} \) onto \( \mathcal{B}_{s,t+1} \). Hence \( |\mathcal{B}_{s,t}| = |\mathcal{B}_{s,t+1}| \) and \( |\mathcal{B}_{s,1}| = |\mathcal{B}_{s,2}| = \cdots = |\mathcal{B}_{s,n-1}| \).

(2) Similar to that of Lemma 2.1.

As we know, the number of \( r \)-combinations of \( k \) distinct objects each available in unlimited supply is \( \binom{r + k - 1}{r} \) (see [1, Theorem 3.5.1, p. 72]).

We now can state and prove the main result of this section.

Theorem 2.1.
\[
|OP_E(X)| = m \sum_{k_1 + k_2 + \cdots + k_m = m} \prod_{s = 1}^{m} \binom{(k_s + 1)n - 1}{k_s n} + m^2(n - 1) \binom{n(m + 1) - 1}{mn} - mn(mn - 1),
\]
where \( (k_1, k_2, \cdots, k_m) \) is any non-negative integer solution to the equation \( \sum_{s = 1}^{m} k_s = m \).

Proof. By Lemmas 2.1 and 2.2, in order to calculate \( |OP_E(X)| \), we need only consider \( |\mathcal{A}_1| \) and \( |\mathcal{B}_{1,1}| \). We first calculate \( |\mathcal{A}_1| \). Suppose that
\[
(2.1) \; f([A_1, A_{k_1}]) \subseteq A_1, f([A_{k_1+1}, A_{k_1+k_2}]) \subseteq A_2, \cdots, f([A_{k_1+k_2+\cdots+k_{m-1}+1}, A_m]) \subseteq A_m,
\]
where \( (k_1, k_2, \cdots, k_m) \) is one non-negative integer solution to the equation \( \sum_{s = 1}^{m} k_s = m \). Then the number of maps \( f \) satisfying (2.1) is \( \prod_{s = 1}^{m} \binom{(k_s + 1)n - 1}{k_s n} \). Thus,
\[
|\mathcal{A}_1| = \sum_{k_1 + k_2 + \cdots + k_m = m} \prod_{s = 1}^{m} \binom{(k_s + 1)n - 1}{k_s n},
\]
where \((k_1, k_2, \ldots, k_m)\) is any non-negative integer solution to the equation \(\sum_{s=1}^{m} k_s = m\).

Hence it follows from Lemma 2.1 that
\[
|\mathcal{A}_1| = |\mathcal{A}_2| = \cdots = |\mathcal{A}_m| = \sum_{k_1+k_2+\cdots+k_m=m,s=1}^{m} \binom{(k_s+1)n-1}{k_sn}.
\]

Notice that, for any distinct \(s\) and \(s'\),
\[
\mathcal{A}_s \cap \mathcal{A}_{s'} = \{\langle 1 \rangle, \langle 2 \rangle, \cdots, \langle mn \rangle\},
\]
so the number of distinct maps \(f \in \bigcup_{s=1}^{m} \mathcal{A}_s\) is
\[
m \sum_{k_1+k_2+\cdots+k_m=m,s=1}^{m} \prod_{k_s}^{1} \binom{(k_s+1)n-1}{k_sn} - mn(m-1).
\]

We now calculate \(|\mathcal{B}_{1,1}|\). If \(f \in \mathcal{B}_{1,1}\), then \(f(X) \subseteq A\) for some \(A \in X/E\). Set
\[
\mathcal{F}_i = \{f \in \mathcal{B}_{1,1} : f(X) \subseteq A_i\},
\]
where \(1 \leq i \leq m\). It follows that \(|\mathcal{F}_i| = \binom{n(n+1)-1}{mn}\) and so \(|\mathcal{B}_{1,1}| = |\bigcup_{i=1}^{m} \mathcal{F}_i| = m \binom{n(m+1)-1}{mn}\).

By virtue of Lemma 2.2, for \(1 \leq s \leq m\) and \(1 \leq t \leq n-1\), we have \(|\mathcal{B}_{s,t}| = m \binom{n(m+1)-1}{mn}\).

Since
\[
\mathcal{B}_{s,t} \cap \mathcal{B}_{s',t'} = \mathcal{B}_{s,t} \cap \mathcal{A}_{s'} = \{\langle 1 \rangle, \langle 2 \rangle, \cdots, \langle mn \rangle\},
\]
the number of distinct non-constant maps \(f \in \bigcup_{s=1}^{m} \bigcup_{t=1}^{n-1} \mathcal{B}_{s,t}\) is
\[
m^2(n-1) \binom{n(m+1)-1}{mn} - m^2n(n-1).
\]

Therefore,
\[
|OP_E(X)| = m \sum_{k_1+k_2+\cdots+k_m=m,s=1}^{m} \prod_{k_s}^{1} \binom{(k_s+1)n-1}{k_sn} + m^2(n-1) \binom{n(m+1)-1}{mn} - mn(mn-1),
\]
as required.

Earlier the authors [12] considered the class of transformation semigroups
\[
O_E(X) = \{f \in T_E(X) : \forall x, y \in X, x \leq y \Rightarrow f(x) \leq f(y)\},
\]
where the set \(X\) and the equivalence \(E\) are as defined in this paper. It is clear that \(O_E(X) \subseteq OP_E(X)\), and in fact, the semigroup \(O_E(X)\) whose cardinality is not known hitherto, is exactly \(|\mathcal{A}_1|\). Thus, an immediate consequence of Theorem 2.1 is the following corollary.

**Corollary 2.1.**
\[
|O_E(X)| = m \sum_{k_1+k_2+\cdots+k_m=m,s=1}^{m} \prod_{k_s}^{1} \binom{(k_s+1)n-1}{k_sn},
\]
where \((k_1, k_2, \ldots, k_m)\) is any non-negative integer solution to the equation \(\sum_{s=1}^{m} k_s = m\).

**Remark 2.1.** Recently I have been told that Fernandes and Quinteiro [4] had calculated the size of the semigroups \(OP_E(X)\) and \(O_E(X)\). However, the approach used differs greatly from that in this paper.

The following Tables 1 and 2 give the size of the semigroups \(OP_E(X)\) and \(O_E(X)\) for smaller \(m\) and \(n\), respectively.
Table 1. The cardinality of $\mathcal{O}_E(X)$

<table>
<thead>
<tr>
<th>$m \setminus n$</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
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Table 2. The cardinality of $O_E(X)$

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<th>4</th>
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</tr>
</tbody>
</table>

3. The number of idempotents in $\mathcal{O}_E(X)$

For a given subset $M$ of the semigroup $\mathcal{O}_E(X)$, we denote by $E(M)$ its set of idempotents. In this section, we aim to calculate the cardinality of $E(\mathcal{O}_E(X))$. Since the semigroup $\mathcal{O}_E(X)$ has been divided into some subsets $\mathcal{A}_1(=O_E(X)), \mathcal{A}_2, \cdots, \mathcal{A}_m, \mathcal{B}_{s,1}, \mathcal{B}_{s,2}, \cdots, \mathcal{B}_{s,n-1}(1 \leq s \leq m)$, that is,

$$\mathcal{O}_E(X) = \bigcup_{s=1}^{m} \mathcal{A}_s \bigcup \left( \bigcup_{s=1}^{m} \bigcup_{t=1}^{n-1} \mathcal{B}_{s,t} \right),$$

we need only calculate the cardinality of the sets $E(\mathcal{A}_1), E(\mathcal{A}_2), \cdots, E(\mathcal{A}_m), \bigcup_{t=1}^{n-1} E(\mathcal{B}_{s,t}) (1 \leq s \leq m)$, respectively.

We begin with considering the number of idempotents in the semigroup $O_E(X)$. Recall that, the Fibonacci numbers are recursively defined by

$$F_0 = 0, F_1 = 1, F_{k+1} = F_k + F_{k-1}, \quad k \geq 1.$$  

The following lemma which comes from [6, Theorem 2.3] was reproved in [2, Lemma 2.9].

Lemma 3.1. $|E(O_n)| = F_{2n}$.

Lemma 3.2. Let $f \in O_E(X)$ and $f^{-1}(A_j) = [A_{i+1}, A_{i+t}]$ for $1 \leq i, t \leq m-1, i+1 \leq j \leq i+t$. Then the restriction of $f$ to $[A_{i+1}, A_{i+t}]$

$$f|_{[A_{i+1}, A_{i+t}]} : [A_{i+1}, A_{i+t}] \to A_j$$

is an idempotent in $\mathcal{F}_{[A_{i+1}, A_{i+t}]}$ if and only if the restriction of $f$ to the E-class $A_j$

$$f|_{A_j} : A_j \to A_j$$

is an idempotent in $\mathcal{F}_{A_j}$ and $f([A_{i+1}, A_{j-1}]) = f(a), f([A_{j+1}, A_{i+t}]) = f(b)$ where $a = \min A_j = (j-1)n+1$ and $b = \max A_j = jn$.

Proof. It is immediate for an order-preserving transformation in $T_E(X)$. \[\square\]
**Remark 3.1.** From Lemma 3.2, in order to construct an idempotent

\[ f|_{[A_{i+1}, A_{i+t}]} : [A_{i+1}, A_{i+t}] \to A_j \]

in \( T[A_{i+1}, A_{i+t}] \), we go along the following line:

Step 1. Construct an idempotent \( f|_{A_j} : A_j \to A_j \) in \( T[A_j] \);
Step 2. Let \( f([A_{i+1}, A_{j-1}]) = f(a) \) and \( f([A_{j+1}, A_{i+t}]) = f(b) \) where \( a = \min A_j = (j - 1)n + 1 \) and \( b = \max A_j = jn \).

From Lemma 3.2 and Remark 3.1, we can deduce

**Lemma 3.3.** Let \( f \in O_E(X) \) and \( f^{-1}(A_j) = [A_{i+1}, A_{i+t}] \) for \( 1 \leq i, t \leq m - 1 \), \( i + 1 \leq j \leq i + t \). Then the number of idempotents

\[ f|_{[A_{i+1}, A_{i+t}]} : [A_{i+1}, A_{i+t}] \to A_j \]

in \( T[A_{i+1}, A_{i+t}] \) equals that of idempotents in \( T[A_j] \).

**Theorem 3.1.**

\[ |E(O_E(X))| = \sum_{t=1}^{m} \left( \sum_{k_1 + k_2 + \cdots + k_t = m} \prod_{i=1}^{t} k_i 2^{n} \right) \]

where \((k_1, k_2, \cdots, k_t)\) is any positive integer solution to the equation \( \sum_{i=1}^{t} k_i = m \).

**Proof.** Let \( f \in E(O_E(X)) \). Denote

\[ t = | \{ A \in X/E : A \cap f(X) \neq \emptyset \} | , \]

where \( 1 \leq t \leq m \). Suppose that

\[ f([A_1, A_{k_1}]) \subseteq A_{s_1}, f([A_{k_1+1}, A_{k_1+k_2}]) \subseteq A_{s_2}, \ldots, f([A_{k_1+k_2+\cdots+k_{i-1}+1}, A_m]) \subseteq A_{s_i} \]

where \( A_{s_i} \in X/E \) for \( 1 \leq i \leq t \), the subscript set \( \{s_1, s_2, \cdots, s_t\} \subseteq \{1, 2, \cdots, m\} \) and \((k_1, k_2, \cdots, k_t)\) is one positive integer solution to the equation \( \sum_{i=1}^{t} k_i = m \). Then, for each \( i \), there are \( k_i \) choices for \( A_{s_i} \). By Lemma 3.3, for the fixed positive integer solution \((k_1, k_2, \cdots, k_t)\) to the equation \( \sum_{i=1}^{t} k_i = m \), the number of idempotents \( f \) satisfying (3.1) is \( \prod_{i=1}^{t} k_i 2^{n} \). So the number of idempotents \( f \) satisfying (3.1) is \( \sum_{k_1 + k_2 + \cdots + k_t = m} \prod_{i=1}^{t} k_i 2^{n} \), where \((k_1, k_2, \cdots, k_t)\) is any positive integer solution to the equation \( \sum_{i=1}^{t} k_i = m \). Noting that \( 1 \leq t \leq m \), we have

\[ |E(O_E(X))| = \sum_{t=1}^{m} \left( \sum_{k_1 + k_2 + \cdots + k_t = m} \prod_{i=1}^{t} k_i 2^{n} \right) . \]

**Remark 3.2.** From Lemma 2.1, \( |\mathcal{A}_1| = |\mathcal{A}_2| = \cdots = |\mathcal{A}_m| \). However, in general, the number of idempotents in \( \mathcal{A}_1 \) doesn’t equal that of \( \mathcal{A}_j \) for \( j \neq 1 \). For example, let \( m = 2, n = 2 \), that is, \( A_1 = \{1, 2\}, A_2 = \{3, 4\} \). By Theorem 3.1, we have

\[ |E(\mathcal{A}_1)| = 2F_4 + F_4F_4 = 15 . \]

Denote by \((abcd)\) the map \( f \in OP_E(X) \) which maps \( 1, 2, 3, 4 \) into \( a, b, c, d \), respectively, and

\[ E(\mathcal{A}_1) = \{ (1), (1222), (2), (1133), (1134), (1144), (1233), (1234), (1244), (2233), (2234), (2244), (3), (3334), (4) \} . \]

However, there are only 6 idempotents in \( \mathcal{A}_2 \), and

\[ E(\mathcal{A}_2) = \{ (1), (1211), (2), (3), (4434), (4) \} . \]
Now we calculate the number of idempotents in $\mathcal{A}_l$ for $2 \leq l \leq m$.

**Lemma 3.4.** Let $f \in \mathcal{A}_l$ $(2 \leq l \leq m)$ and $f^{-1}(A_p) = [A_i,A_p]$ for some $E$-class $A_p$ with $p \leq l' < l$. Then the restriction of $f$ to $[A_i,A_p]$

$$f|_{[A_i,A_p]} : [A_i,A_p] \to A_p$$

is an idempotent in $\mathcal{T}_{[A_i,A_p]}$ if and only if the restriction of $f$ to $A_p$

$$f|_{A_p} : A_p \to A_p$$

is an idempotent in $\mathcal{T}_{A_p}$ and $f([A_i,A_{p-1}]) = f(a)$, $f([A_{p+1},A_p]) = f(b)$ where $a = \min A_p = (p-1)n+1$ and $b = \max A_p = pn$.

**Remark 3.3.** In Lemma 3.4, there are two special cases.

1. if $p = l' = 1$, then the restriction of $f$ to $[A_i,A_1]$

$$f|_{[A_i,A_1]} : [A_i,A_1] \to A_1$$

is an idempotent in $\mathcal{T}_{[A_i,A_1]}$ if and only if the restriction of $f$ to $A_1$

$$f|_{A_1} : A_1 \to A_1$$

is an idempotent in $\mathcal{T}_{A_1}$ and $f([A_i,A_{m}]) = f(1)$.

2. if $p = l' \geq 2$, then the restriction of $f$ to $[A_i,A_p]$

$$f|_{[A_i,A_p]} : [A_i,A_p] \to A_p$$

is an idempotent in $\mathcal{T}_{[A_i,A_p]}$ if and only if the restriction of $f$ to $A_p$

$$f|_{A_p} : A_p \to A_p$$

is an idempotent in $\mathcal{T}_{A_p}$ and $f([A_i,A_{p-1}]) = f((p-1)n+1)$.

To illustrate Lemma 3.4, let $m = 4$, $n = 2$ and $A_1 = \{1,2\}, A_2 = \{3,4\}, A_3 = \{5,6\}, A_4 = \{7,8\}$. Then $f_1 = (12111111) \in E(\mathcal{A}_2), f_2 = (12221111) \in E(\mathcal{A}_3)$ and $f_3 = (33344433) \in E(\mathcal{A}_4)$. Clearly, $f_1|_{A_1}$ is an idempotent in $\mathcal{T}_{A_1}, f_1([A_2,A_4]) = f_1(1)$, and $f_2|_{A_1}$ is an idempotent in $\mathcal{T}_{A_1}, f_2([A_3,A_4]) = f_2(1), f_2(A_2) = f_2(2)$, and $f_3|_{A_2}$ is an idempotent in $\mathcal{T}_{A_2}, f_3([A_4, A_1]) = f_3(3), f_3(A_3) = f_3(4)$.

**Lemma 3.5.** For $2 \leq l \leq m$,

$$|E(\mathcal{A}_l)| = \sum_{v=1}^{l-1} \sum_{j_1+j_2+\cdots+j_v=l-1} \prod_{w=1}^{v} j_w F_{2n} + \sum_{t=1}^{\frac{m-1}{2}+1} \sum_{k_1+k_2+\cdots+k_t=m} \left( \prod_{i=1}^{t-1} k_i F_{2n} \right) (k_t - l + 1) F_{2n},$$

where $(j_1,j_2,\cdots,j_v)$ is any positive integer solution to the equation $\sum_{w=1}^{v} j_w = l - 1$, and $(k_1,k_2,\cdots,k_t)$ is any positive integer solution to the equation $\sum_{i=1}^{t} k_i = m$ and the final positive integer $k_t \geq 1$.

**Proof.** Let $f \in E(\mathcal{A}_l)$. There are two cases to consider.

**Case 1.** $f(A_i) \subseteq A_p$ for $p \in \{1,2,\cdots,l-1\}$. Since $f$ is an idempotent, we can deduce that $f([A_i,A_{m}]) \subseteq A_p$. Let

$$v = |\{A \in X/E : A \cap f(X) \neq \emptyset\}|,$$
where \(1 \leq v \leq l - 1\). Suppose
\[
\left(3.2\right) \quad f([A_l, A_{j_1}]) \subseteq A_{s_1}, \quad f([A_{j_1} + 1, A_{j_1} + j_2]) \subseteq A_{s_2}, \ldots
\]
\[
f([A_{j_1} + 2 + \cdots + j_{s-1} + 1, A_{j_1} + j_2 + \cdots + j_{s-1} + j_s = A_{l-1}]) \subseteq A_{s_v},
\]
where \((j_1, j_2, \ldots, j_v)\) is one positive integer solution to the equation \(\sum_{w=1}^{v} j_w = l - 1\), the subscript set \(\{s_1, s_2, \ldots, s_v\} \subseteq \{1, 2, \ldots, l - 1\}\) and
\[
A_p = A_{s_1} < A_{s_2} < \cdots < A_{s_v} \leq A_{l-1}.
\]

If \(v = 1\), then \(f\) maps all the elements of \(X\) into \(A_p\), which has \(l - 1\) possible choices and so the number of \(f\) is \((l - 1)F_{2n}\). Suppose that \(v > 1\) and then, for each \(w (1 \leq w \leq v)\), there are \(j_w\) possible choices for \(A_{s_w}\). By Lemma 3.4, for the fixed positive integer solution \((j_1, j_2, \ldots, j_v)\) to the equation \(\sum_{w=1}^{v} j_w = l - 1\), the number of \(f\) satisfying \(3.2\) should be \(\prod_{w=1}^{v} j_w F_{2n}\). So the number of all \(f\) satisfying \(3.2\) is \(\sum_{v=1}^{l-1} \prod_{w=1}^{v} j_w F_{2n}\). Taking the sum from 2 to \(l - 1\), we obtain that the number of \(f\) satisfying \(3.2\) is \(\sum_{v=1}^{l-1} \prod_{w=1}^{v} j_w F_{2n}\). Therefore, the number of \(f\) satisfying the condition that \(f(A_l) \subseteq A_p\) for \(p \in \{1, 2, \ldots, l - 1\}\) is
\[
(l - 1)F_{2n} + \sum_{v=2}^{l-1} \prod_{w=1}^{v} j_w F_{2n} = \sum_{v=1}^{l-1} \prod_{w=1}^{v} j_w F_{2n}.
\]

**Case 2.** \(f(A_l) \subseteq A_p\) for \(p \in \{l, l + 1, \ldots, m\}\). Set
\[
t = |\{A \in X/E : A \cap f(X) \neq \emptyset\}|,
\]
where \(1 \leq t \leq m - (l - 1)\). Suppose
\[
\left(3.3\right) \quad f([A_l, A_{k_1}]) \subseteq A_{s_1}, \quad f([A_{k_1} + 1, A_{k_1} + k_2]) \subseteq A_{s_2}, \ldots
\]
\[
f([A_{k_1} + k_2 + \cdots + k_{s-1} + 1, A_{k_1} + k_2 + \cdots + k_{s-1} + k_t = A_{l-1}]) \subseteq A_{s_t},
\]
where \((k_1, k_2, \ldots, k_t)\) is any integer solution to the equation \(\sum_{i=1}^{t} k_i = m - 1\) and \(k_1 \geq 0, k_2 \geq 1, k_3 \geq 1, \ldots, k_{s-1} \geq 1, k_t \geq l\) (since \(f\) maps at least \(E\)-classes \(A_m, A_1, \ldots, A_{l-1}\) into \(A_{s_t}\)), the subscript set \(\{s_1, s_2, \ldots, s_t\} \subseteq \{l, l + 1, \ldots, m\}\) and
\[
A_l \leq A_p = A_{s_1} < A_{s_2} < \cdots < A_{s_t} \leq A_m.
\]

If \(t = 1\), it is clear that the number of \(f\) is \((m - l + 1)F_{2n}\). If \(t = 2\), then there are \((k_1 + 1)\) choices for \(A_{s_1}\) and \((k_2 - l + 1)\) choices for \(A_{s_2}\). Thus the number of \(f\) is
\[
\sum_{k_1 + k_2 = m-1} ((k_1 + 1)F_{2n})((k_2 - l + 1)F_{2n}).
\]

If \(3 \leq t \leq m - (l - 1)\), there are \((k_1 + 1)\) choices for \(A_{s_1}\), and, for each \(i (2 \leq i \leq t - 1)\), \(k_i\) choices for \(A_{s_i}\), and \((k_t - l + 1)\) choices for \(A_{s_t}\). So, for the fixed integer solution \((k_1, k_2, \ldots, k_t)\) to the equation \(\sum_{i=1}^{t} k_i = m - 1\), the number of \(f\) satisfying \(3.3\) is \((k_1 + 1)F_{2n} (\prod_{i=2}^{t-1} k_i F_{2n}) (k_t - l + 1)F_{2n}\). Thus, the number of all \(f\) satisfying \(3.3\) is
\[
\sum_{k_1 + k_2 + \cdots + k_t = m-1} ((k_1 + 1)F_{2n} (\prod_{i=2}^{t-1} k_i F_{2n}) (k_t - l + 1)F_{2n}).
\]
Taking the sum $t$ from 3 to $m - l + 1$ yields

$$
\sum_{t=3}^{m-l+1} \sum_{k_1+k_2+\ldots+k_t=m-1} ((k_1+1)F_{2n}\left(\prod_{i=2}^{t-1} k_i F_{2n}\right)(k_t-l+1)F_{2n})
$$

Therefore, the number of $f$ satisfying the condition that $f(A_i) \subseteq A_p$ for $p \in \{l,l+1,\ldots,m\}$ is

$$
(m-l+1)F_{2n} + \sum_{t=3}^{m-l+1} \sum_{k_1+k_2+\ldots+k_t=m-1} ((k_1+1)F_{2n}\left(\prod_{i=2}^{t-1} k_i F_{2n}\right)(k_t-l+1)F_{2n})
$$

$$
= \sum_{t=1}^{m-l+1} \sum_{k_1+k_2+\ldots+k_t=m-1} ((k_1+1)F_{2n}\left(\prod_{i=2}^{t-1} k_i F_{2n}\right)(k_t-l+1)F_{2n}),
$$

where $(k_1,k_2,\ldots,k_t)$ is any positive integer solution to the equation $\sum_{i=1}^t k_i = m$ and the final positive integer $k_t \geq l$. Consequently,

$$
|E(\mathcal{A}_t)| = \sum_{v=1}^{l-1} \sum_{j_1+j_2+\ldots+j_v=l-1} \prod_{i=1}^v j_i F_{2n} + \sum_{t=1}^{m-l+1} \sum_{k_1+k_2+\ldots+k_t=m} \left(\prod_{i=1}^{t-1} k_i F_{2n}\right)((k_t-l+1)F_{2n}).
$$

**Remark 3.4.** In Lemma 3.5, when $t = 1$, we have

$$
\sum_{k_1+k_2+\ldots+k_t=m} \left(\prod_{i=1}^{t-1} k_i F_{2n}\right)((k_t-l+1)F_{2n}) = (m-l+1)F_{2n}.
$$

**Example 3.1.** By virtue of Lemma 3.5, we calculate $|E(\mathcal{A}_2)|, |E(\mathcal{A}_3)|, |E(\mathcal{A}_4)|$ for $m = 4, n = 3$ and have

$$
|E(\mathcal{A}_2)| = F_6 + (3F_6 + 2F_6 F_6 + F_6(2F_6) + F_6 F_6 F_6) = 800,
$$

$$
|E(\mathcal{A}_3)| = (2F_6 + F_6 F_6) + (2F_6 + F_6 F_6) = 160
$$

and

$$
|E(\mathcal{A}_4)| = (3F_6 + F_6(2F_6) + 2F_6 F_6 + F_6 F_6 F_6) + F_6 = 800.
$$

Finally we consider the number of idempotents in $\bigcup_{1 \leq s \leq m} \mathcal{B}_{s,t}$. The following lemma comes from [2, Theorem 2.10].

**Lemma 3.6.** $|E(OP_t)| = F_{2n-1} + F_{2n+1} - (n^2 - n + 2)$.

**Lemma 3.7.** Let $f \in \mathcal{B}_{s,t}$ with $1 \leq s \leq m$ and $1 \leq t \leq n-1$.

1. If $f(X) \subseteq A_q$ for $q \neq s$, then $f: X \to A_q$ is an idempotent in $OP_E(X)$ if and only if $f|_{A_q}: A_q \to A_q$ is an idempotent in $\mathcal{B}_{A_q}$ and

$$
f([s-1]n + t + 1, A_{q-1}) = f(a), \quad f([A_{q+1}, (s-1)n + t]) = f(b),
$$

where $a = \min A_q = (q-1)n + 1$ and $b = \max A_q = qn$. 

(2) If \( f(X) \subseteq A_s \), then \( f : X \rightarrow A_s \) is an idempotent in \( OP_E(X) \) if and only if \( f|_{A_s} : A_s \rightarrow A_s \) is an idempotent in \( \mathcal{F}_{A_s} \), moreover, if \( f((s-1)n+t+1) \leq (s-1)n+t \), then \( f((s-1)n+t+1, f((s-1)n+t+1)) = f((s-1)n+t+1) \), and if \( f((s-1)n+t+1) > (s-1)n+t \), then \( f([A_{s+1}, (s-1)n+t] = f(sn) \).

**Proof.** Here we only show (2). Since \( f \in \mathcal{B}_{s,t} \), we have
\[
f((s-1)n+t+1) \leq f((s-1)n+t+2) \leq \cdots \leq f(mn) \leq f(1) \leq \cdots \leq f((s-1)n+t).
\]

We now suppose that \( f : X \rightarrow A_s \) is an idempotent in \( OP_E(X) \), then \( f|_{A_s} : A_s \rightarrow A_s \) is also an idempotent in \( \mathcal{F}_{A_s} \). Let \( c = f((s-1)n+t+1) \) and \( x \in [(s-1)n+t+1, c] \). If \( c \leq (s-1)n+t \), then \( f(x) \leq f(c) = c \) and \( f(x) \geq f((s-1)n+t+1) = c \). Thus \( f(x) = c \). If \( c > (s-1)n+t \) and \( x \in [A_{s+1}, (s-1)n+t] \), then \( f(sn) \leq f(x) \) and we can assert that \( f(sn) = f(x) \). Indeed, if \( f(sn) < f(x) \). Noting that \( f \) maps \( X \) into \( A_s \), we have \( f(x) \leq sn \) and \( f(x) = f^2(x) \leq f(sn) \), a contradiction. The sufficiency is clear and the proof is completed.

**Remark 3.5.** In Lemma 3.7(1), we consider two special cases.

1. If \( q = 1 \), then \( f : X \rightarrow A_1 \) is an idempotent in \( OP_E(X) \) if and only if \( f|_{A_1} : A_1 \rightarrow A_1 \) is an idempotent in \( \mathcal{F}_{A_1} \), and
\[
f(((s-1)n+t+1, A_m]) = f(1), f([A_2, (s-1)n+t]) = f(n).
\]

2. If \( q = m \), then \( f : X \rightarrow A_m \) is an idempotent in \( OP_E(X) \) if and only if \( f|_{A_m} : A_m \rightarrow A_m \) is an idempotent in \( \mathcal{F}_{A_m} \) and
\[
f(((s-1)n+t+1, A_{m-1}]) = f((m-1)n+1), f([A_1, (s-1)n+t]) = f(mn).
\]

To illustrate Lemma 3.7, let \( m = 3, n = 5 \) and \( A_1 = \{1, 2, 3, 4, 5\} \), \( A_2 = \{6, 7, 8, 9, 10\} \), \( A_3 = \{11, 12, 13, 14, 15\} \). Let
\[
g_1 = \left( \begin{array}{cccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 & 14 \end{array} \right) \in E(\mathcal{B}_{2,2}),
\]
\[
g_2 = \left( \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \end{array} \right) \in E(\mathcal{B}_{2,2}),
\]
and
\[
g_3 = \left( \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 & 10 \end{array} \right) \in E(\mathcal{B}_{2,2}).
\]

Clearly, \( g_1|_{A_3} \) is an idempotent in \( \mathcal{F}_{A_3} \), \( g_1([8, 10]) = g_1(11) \), \( g_1([A_1, 7]) = g_1(15) \), and \( g_2|_{A_2} \) is an idempotent in \( \mathcal{F}_{A_2} \), \( g_2([8, g_2(8)]) = g_2(8) \), and \( g_3|_{A_2} \) is an idempotent in \( \mathcal{F}_{A_2} \), \( g_3([A_3, 7]) = g_3(10) \).

**Lemma 3.8.** For \( 1 \leq s \leq m \),
\[
\sum_{t=1}^{n-1} |E(\mathcal{B}_{s,t})| = (n-1)(m-1)F_{2n} + 2(F_{2n-1} - 1).
\]

**Proof.** Let \( f \in E(\mathcal{B}_{s,t}) \) for \( 1 \leq t \leq n-1 \). Set
\[
M^s_{t} = \{ f \in E(\mathcal{B}_{s,t}) : f(X) \subseteq A_q \}.
\]

There are two cases to consider.
Case 1. \( q \neq s \). Then, by Lemmas 3.1 and 3.7 (1), \( |M_{q}^{E, s}| = F_{2n} \) since \( f \) is order-preserving on the \( E \)-class \( A_{q} \). Thus \( |\cup_{q \neq s} M_{q}^{E, s}| = (m - 1)F_{2n} \).

Case 2. \( q = s \). Then, by Lemma 3.7(2),

\[
|M_{s}^{E, s}| = |\{ f \in E(OP_{n}) : f(t + 1) \leq f(t) + 2 \leq \cdots \leq f(n - 1) \leq f(n) \leq f(1) \leq \cdots \leq f(t) \}|.
\]

Noting that in \( OP_{n} \), by Lemmas 3.6, the number of idempotents which are not order-preserving is \( F_{2n-1} + F_{2n+1} - (n^2 - n + 2) - F_{2n} \), we have

\[
\sum_{t=1}^{n-1} |M_{s}^{E, s}| = F_{2n-1} + F_{2n+1} - (n^2 - n + 2) - F_{2n} + (n - 1)n = 2(F_{2n-1} - 1).
\]

Consequently,

\[
\sum_{t=1}^{n-1} |E(B_{s, t})| = (n - 1)(m - 1)F_{2n} + \sum_{t=1}^{n-1} |M_{s}^{E, s}| = (n - 1)(m - 1)F_{2n} + 2(F_{2n-1} - 1).
\]

Observing that for \( 1 \leq s, s' \leq m, 1 \leq t, t' \leq n-1 \),

\[
E(A_{s}) \cap E(A_{s'}) = E(A_{s}) \cap E(A_{s'}) = E(A_{s}) \cap E(A_{s'}) = \{ \langle 1 \rangle, \langle 2 \rangle, \cdots, \langle mn \rangle \},
\]

and that the total number of idempotents \( \langle 1 \rangle, \langle 2 \rangle, \cdots, \langle mn \rangle \) in \( A_{2}, A_{3}, \cdots, A_{m}, \cup_{s=1}^{m-1} B_{s, t} \) \( (1 \leq s \leq m) \) is \( (m - 1)mn + (n - 1)m^2n \), by Theorem 3.1, Lemma 3.5 and Lemma 3.8, we obtain the main result in this section.

**Theorem 3.2.**

\[
|E(OP_{E}(X))| = \sum_{t=1}^{m} k_{t}F_{2n} + \sum_{i=1}^{m} \sum_{j_{1}+j_{2}+\cdots+j_{i}=m}^{l-1} \left\{ \sum_{v=1}^{l-1} \left( \prod_{i=1}^{v-1} k_{i}F_{2n} \right) \left( \prod_{i=1}^{l-1} k_{i}F_{2n} \right) \right\} \left( k_{l} - l + 1 \right) F_{2n}
\]

\[
+ m ((n - 1)(m - 1)F_{2n} + 2(F_{2n-1} - 1)) - (m - 1)mn + (n - 1)m^2n,
\]

where \( k_{1}, k_{2}, \cdots, k_{l} \) is any positive integer solution to the equation \( \sum_{i=1}^{l} k_{i} = m \), and \( j_{1}, j_{2}, \cdots, j_{v} \) is any positive integer solution to the equation \( \sum_{w=1}^{v} j_{w} = l - 1 \), and \( k_{1}', k_{2}', \cdots, k_{l}' \) is any positive integer solution to the equation \( \sum_{i=1}^{l} k_{i}' = m \) and the final positive integer \( k_{l}' \geq l \).

**Example 3.2.** Let \( m = 4, n = 3 \). By Theorem 3.1,

\[
|E(A_{4})| = 4F_{6} + (F_{6}(3F_{6}) + (2F_{6})(2F_{6}) + 3F_{6}(F_{6})) + (F_{6}F_{6}(2F_{6}) + F_{6}(2F_{6})F_{6} + (2F_{6})F_{6}F_{6}) + F_{6}F_{6}F_{6}F_{6} = 7840.
\]

From Example 3.1, we know \( |E(A_{2})| = 800, |E(A_{3})| = 160 \) and \( |E(A_{4})| = 800 \). It follows from Lemma 3.8 that \( \sum_{t=1}^{2} |E(B_{s, t})| = 6F_{6} + 2(F_{5} - 1) = 56 \) for \( 1 \leq s \leq 4 \). Thus,

\[
|E(OP_{E}(X))| = 7840 + 800 + 160 + 800 + 56 \times 4 - (36 + 96) = 9692.
\]
To conclude this section, we give the following Tables 3 and 4 providing the number of idempotents in $OP_E(X)$ and $O_E(X)$ for smaller $m, n$, respectively.

Table 3. The number of idempotents in $OP_E(X)$

<table>
<thead>
<tr>
<th>$m \backslash n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>643</td>
<td>3727</td>
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<tr>
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<td>1016</td>
<td>12414</td>
<td>186328</td>
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Table 4. The number of idempotents in $O_E(X)$

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References


