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MORE OPERATOR INEQUALITIES FOR POSITIVE LINEAR MAPS

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Abstract. In this paper we present some new operator inequality for convex functions. We have obtained a number of Jensen's type inequalities for convex and operator convex functions of self-adjoint operators for positive linear maps. Some results are exemplified for power and logarithmic functions.

1. Introduction and Preliminaries

As is customary, we reserve M, m for scalars. Other capital letters are used to denote general elements of the C^*-algebra $\mathcal{B}(\mathscr{H})$ of all bounded linear operators acting on a Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$. An operator $A \in \mathcal{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$, and we then write $A \geq 0$. For self-adjoint operators $A, B \in \mathcal{B}(\mathscr{H})$ we say that $A \leq B$ if $B - A \geq 0$. The Gelfand map establishes an isometrically $*$-isomorphism Φ between the set $C(sp(A))$ of all continuous functions on the spectrum of A, denoted $sp(A)$, and the C^*-algebra generated by A and I (see for instance [13, p. 15]). For any $f, g \in C(sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

- $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
- $\Phi(fg) = \Phi(f)\Phi(g)$;
- $\|\Phi(f)\| = \|f\| := \sup\limits_{t \in sp(A)} |f(t)|$;
- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in sp(A)$.

With this notation we define $f(A) = \Phi(f)$ for all $f \in C(sp(A))$ and we call it the continuous functional calculus for a self-adjoint operator A. It is well known that, if A is a self-adjoint operator and $f \in C(sp(A))$, then $f(t) \geq 0$ for any $t \in sp(A)$ implies that $f(A) \geq 0$. It is extendible for two real valued functions on $sp(A)$. A linear map ϕ is positive if $\phi(A) \geq 0$ whenever $A \geq 0$. It said to be normalized if $\phi(I) = I$. For more studies in this direction, we refer to [2, 6]. As is known to all, in [8, Theorem 1.2], the authors presented the operator version for the Jensen inequality. They proved the inequality

$$f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle,$$

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where A is a self-adjoint operator with $sp(A) \subseteq [m, M]$ and $f(t)$ is a convex function on an interval $[m, M]$. Choi [3] and Davis [4] showed that if ϕ is a normalized positive linear map on $\mathcal{B}(\mathscr{H})$ and if f is an operator convex function on an interval $[m, M]$, then the so-called Choi-Davis-Jensen inequality

$$f(\phi(A)) \leq \phi(f(A)), \tag{1.1}$$

holds for every self adjoint operator A on $\mathscr{H}$ whose spectrum is contained in $[m, M]$. As a special case of inequality (1.1), the authors in [8, Theorem 1.21] established the following generalization of Jensen's inequality under the additional condition $p_i, (i \in \{1, \ldots, n\})$ are positive real numbers such that $\sum\limits_{i=1}^{n} p_i = 1$

$$f\left(\sum_{i=1}^{n} p_i \phi_i(A_i)\right) \leq \sum_{i=1}^{n} p_i \phi_i(f(A_i)).$$

The following variant of Jensen's operator inequality for a convex function $f \in C([m, M])$, self-adjoint operators $A_i \in \mathcal{B}(\mathscr{H})$ with spectra in $[m, M]$ and normalized positive linear maps ϕ_i on $\mathcal{B}(\mathscr{H})$ was proved in [11]

$$f\left((m+M)1_{\mathscr{H}} - \sum_{i=1}^{n} \phi_i(A_i)\right) \leq (f(m) + f(M))1_{\mathscr{H}} - \sum_{i=1}^{n} \phi_i(f(A_i)).$$

Moreover, in the same paper the following series of inequalities was proved

$$\begin{aligned}
&f(m+M)1_{\mathscr{H}} - \sum_{i=1}^{n} \phi_i(A_i) \\
&\leq \frac{M1_{\mathscr{H}} - \sum\limits_{i=1}^{n} \phi_i(A_i)}{M-m} f(M) + \frac{\sum\limits_{i=1}^{n} \phi_i(A_i) - m1_{\mathscr{H}}}{M-m} f(M) \\
&\leq (f(m) + f(M))1_{\mathscr{H}} - \sum_{i=1}^{n} \phi_i(f(A_i)).
\end{aligned}$$

There is considerable amount of literature devoting to the study of Jensen inequality, we refer to [9, 10, 12] for a recent survey and references therein.

A function $f : I \to (0, +\infty)$ is said to be log-convex if $log(f)$ is convex, or, equivalently, if for all $x, y \in I$ and $t \in [0, 1]$ one has the inequality

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}.$$

The function $f(x) = \frac{1}{x}$, $x \in (0, +\infty)$ is log-convex on $(0, +\infty)$.

This paper include but are not restricted to the positive linear map version of the Dragomir-Ionescu inequality, Slater's type inequalities for operators and its inverses, Jensen's inequality for differentiable functions, Jensen's type inequalities for log-convex functions and for differentiable log-convex functions.

2. **Inequalities for Differentiable and Convex Functions**

The following result provides an operator version of the Dragomir-Ionescu inequality (see [7]):

Theorem 2.1. *Let A be a self adjoint operator on the Hilbert space $\mathscr{H}$ with $sp(A) \subseteq I$ and ϕ be a normalized positive linear map on $\mathcal{B}(\mathscr{H})$ and let $f : I \to \mathbb{R}$ be a convex and differentiable function on interval I whose derivative f' is continuous on I. Then*

$$\begin{aligned} 0 &\leq \langle \phi(f(A))x, x\rangle - f(\langle \phi(A)x, x\rangle) \\ &\leq \langle \phi(f'(A)A)x, x\rangle - \langle \phi(A)x, x\rangle \langle \phi(f'(A))x, x\rangle \end{aligned} \tag{2.1}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

Proof. First we remark that, the first inequality in (2.1) has been proven for convex functions with no differentiability assumption in [5]. As in [5] and for the sake of completeness, we give here a short proof of the second inequality from (2.1). Since f is convex and differentiable, by the gradient inequality we have that

$$f(t) - f(s) \leq f'(t)(t - s),$$

for any $t, s \in [m, M]$. Since $sp(A) \subseteq [m, M]$ by substitute $s = \langle \phi(A)x, x\rangle \in [m, M]$, we have

$$f(t) - f(\langle \phi(A)x, x\rangle) \leq f'(t)(t - \langle \phi(A)x, x\rangle).$$

Now, by the functional calculus for $t \in [m, M]$ we deduce

$$f(A) - f(\langle \phi(A)x, x\rangle) 1_{\mathscr{H}} \leq f'(A)(A - \langle \phi(A)x, x\rangle 1_{\mathscr{H}})$$

and since ϕ is a normalized positive linear map we have

$$\begin{aligned} &\langle \phi(f(A))x, x\rangle - f(\langle \phi(A)x, x\rangle) \\ &\leq \langle \phi(f'(A)A)x, x\rangle - \langle \phi(A)x, x\rangle \langle \phi(f'(A))x, x\rangle \end{aligned}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$. □

Remark. *There are several examples of normalized positive linear maps. But for our application we consider among them $\phi : \mathscr{M}_n(\mathbb{C}) \to \mathscr{M}_n(\mathbb{C})$, which $\phi(A) = \left(\frac{1}{n} tr(A)\right) 1_{\mathscr{H}}$ for all Hermitian matrices $A \in \mathscr{M}_n(\mathbb{C})$. From inequality (2.1), we have*

$$0 \leq \frac{1}{n} tr(f(A)) - f\left(\frac{1}{n} tr(A)\right) \leq \frac{1}{n} tr(f'(A)A) - \frac{1}{n^2} tr(A)\, tr(f'(A)).$$

If we take $f(t) = t^2$, then

$$\frac{1}{n} tr^2(A) \leq tr\left(A^2\right).$$

In the following we give more interesting situations.

Corollary 2.2. *Let A_i be self-adjoint operators with $sp(A_i) \subseteq I$, $i \in \{1, \ldots, n\}$ and ϕ_i are normalized positive linear map on $\mathcal{B}(\mathscr{H})$ $(i \in \{1, \ldots, n\})$ and assume that f is as in the Theorem 2.1. Then*

$$\begin{aligned} 0 &\leq \left\langle \sum_{i=1}^{n} \phi_i(f(A_i))x, x \right\rangle - f\left\langle \sum_{i=1}^{n} \phi_i(A_i)x, x \right\rangle \\ &\leq \left\langle \sum_{i=1}^{n} \phi_i(f'(A_i)A_i)x, x \right\rangle - \left\langle \sum_{i=1}^{n} \phi_i(A_i)x, x \right\rangle \\ &\quad \times \left\langle \sum_{i=1}^{n} \phi_i(f'(A_i))x, x \right\rangle, \end{aligned} \tag{2.2}$$

for each $x \in \mathscr{H}$ *with* $\|x\| = 1$.

Proof. If we put

$$\widetilde{A} := \begin{pmatrix} A_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_n \end{pmatrix},$$

then we have $sp\left(\widetilde{A}\right) \subseteq [m, M] \subseteq \overset{o}{I}$, moreover, define

$$\phi : \mathcal{B}(\mathscr{H}) \oplus \ldots \oplus \mathcal{B}(\mathscr{H}) \to \mathcal{B}(\mathscr{H})$$

as

$$\phi(A_1 \oplus \ldots \oplus A_n) = \sum_{i=1}^{n} \phi_i(A_i)$$

in this case, we have

$$\phi\left(f\left(\widetilde{A}\right)\right) = \sum_{i=1}^{n} \phi_i(f(A_i)), \ \phi\left(\widetilde{A}\right) = \sum_{i=1}^{n} \phi_i(A_i).$$

Applying Theorem 2.1 for $\widetilde{A}$ we deduce the desired result (2.2). □

Corollary 2.3. *Notation as in above. If* p_i *are positive real numbers with* $\sum_{i=1}^{n} p_i = 1$, *then*

$$\begin{aligned} 0 &\le \left\langle \sum_{i=1}^{n} p_i f(\phi_i(A_i))x, x \right\rangle - f\left(\left\langle \sum_{i=1}^{n} p_i \phi_i(A_i)x, x \right\rangle\right) \\ &\le \left\langle \sum_{i=1}^{n} p_i f'(\phi_i(A_i)A_i)x, x \right\rangle - \left\langle \sum_{i=1}^{n} p_i(\phi_i(A_i))x, x \right\rangle \\ &\quad \times \left\langle \sum_{i=1}^{n} p_i f'(\phi_i(A_i))x, x \right\rangle, \end{aligned}$$

for each $x \in \mathscr{H}$ *with* $\|x\| = 1$.

Remark. *The following particular cases are of interest (see also* [5]*):*

(a) If $p \ge 1$ *and* A *is a positive operator on* $\mathscr{H}$, *then we have the inequality*

$$\begin{aligned} 0 &\le \langle \phi(A^p)x, x \rangle - \langle \phi(A)x, x \rangle^p \\ &\le p\left[\langle \phi(A^p)x, x \rangle - \langle \phi(A)x, x \rangle \left\langle \phi\left(A^{p-1}\right)x, x \right\rangle\right] \end{aligned} \tag{2.3}$$

for each $x \in \mathscr{H}$ *with* $\|x\| = 1$.

(b) If A *is a positive definite operator on* $\mathscr{H}$, *then the inequality* (2.3) *holds for* $p < 0$.

(c) If $0 < p < 1$ *and* A *is a positive definite operator on* $\mathscr{H}$, *then*

$$\begin{aligned} &\langle \phi(A^p)x, x \rangle - \langle \phi(A)x, x \rangle^p \\ &\ge p\left[\langle \phi(A^p)x, x \rangle - \langle \phi(A)x, x \rangle \left\langle \phi\left(A^{p-1}\right)x, x \right\rangle\right] \ge 0 \end{aligned}$$

for each $x \in \mathscr{H}$ *with* $\|x\| = 1$.

Consider the convex function $f : (0, \infty) \to \mathbb{R}$, $f(x) = -\ln x$. As a direct consequence of Theorem 2.1, we obtain the following corollary.

Corollary 2.4. *Let A be a strictly positive operator on the Hilbert space $\mathscr{H}$. Then*

$$0 \le \ln \langle \phi(A) x, x \rangle - \langle \phi(\ln A) x, x \rangle \le \langle \phi(A) x, x \rangle \left\langle \phi\left(A^{-1}\right) x, x \right\rangle - 1$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

Now, from a different view point we may state:

Theorem 2.5. *All assumptions as in Theorem 2.1. Then*

$$\begin{aligned} &\langle \phi(f(A) - f(B)) x, x \rangle \\ &\le \langle \phi(f'(A) A) x, x \rangle - \langle \phi(f'(A)) x, x \rangle \langle \phi(B) x, x \rangle . \end{aligned} \tag{2.4}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

Proof. Since f is convex and differentiable we have

$$f(t) - f(s) \le f'(t)(t - s)$$

for any $t, s \in I$. Fix $s \in I$ and apply the functional calculus for the operator A, then

$$f(A) - f(s) 1_{\mathscr{H}} \le f'(A)(A - s).$$

Since ϕ is normalized positive linear map we get

$$\phi(f(A)) - f(s) 1_{\mathscr{H}} \le \phi(f'(A) A) - s\phi(f'(A)) \tag{2.5}$$

therefore

$$\langle \phi(f(A)) x, x \rangle - f(s) \le \langle \phi(f'(A) A) x, x \rangle - s \langle \phi(f'(A)) x, x \rangle$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$. Apply again functional calculus for the operator B to obtain

$$\begin{aligned} &\langle \phi(f(A)) x, x \rangle 1_{\mathscr{H}} - f(B) \\ &\le \langle \phi(f'(A) A) x, x \rangle 1_{\mathscr{H}} - \langle \phi(f'(A)) x, x \rangle B. \end{aligned} \tag{2.6}$$

Since ϕ is normalized positive linear map we have

$$\begin{aligned} &\langle \phi(f(A)) x, x \rangle 1_{\mathscr{H}} - \phi(f(B)) \\ &\le \langle \phi(f'(A) A) x, x \rangle 1_{\mathscr{H}} - \langle \phi(f'(A)) x, x \rangle \phi(B) \end{aligned} \tag{2.7}$$

therefore

$$\begin{aligned} &\langle \phi(f(A)) x, x \rangle - \langle \phi(f(B)) y, y \rangle \\ &\le \langle \phi(f'(A) A) x, x \rangle - \langle \phi(f'(A)) x, x \rangle \langle \phi(B) y, y \rangle . \end{aligned} \tag{2.8}$$

for each $x, y \in \mathscr{H}$ with $\|x\| = \|y\| = 1$. This is an inequality of interest in itself. Finally, on making $y = x$ in (2.8) we deduce the desired result (2.4). □

By setting $\phi(A) = \left(\frac{1}{n} tr(A)\right) 1_{\mathscr{H}}$ in (2.4), we find the following result.

Remark. *Let $A, B \in \mathscr{M}_n(\mathbb{C})$ be two Hermitian matrices, then*

$$\frac{1}{n} tr(f(A) - f(B)) \le \frac{1}{n} tr(f'(A) A) - \frac{1}{n^2} tr(f'(A)) tr(B).$$

Remark. *If we choose $A = B$ in (2.4), we get*

$$\langle \phi(f'(A) A) x, x \rangle \ge \langle \phi(f'(A)) x, x \rangle \langle \phi(A) x, x \rangle \tag{2.9}$$

for each $x \in \mathscr{H}$ with $\|x\| = 1$.

The following result is known in the literature as the Holder-McCarthy inequality.

Remark. *If we take* $f(t)=t^2$ *in* (2.9), *we deduce*

$$\left\langle \phi\left(A^2\right)x,x\right\rangle \geq \left\langle \phi(A)x,x\right\rangle^2$$

for each $x\in\mathscr{H}$ *with* $\|x\|=1$.

The case of norm operator may be of interest and is embodied in the following remark.

Remark. *Let* A *be a positive operator on* $\mathcal{B}(\mathscr{H})$ *and* $f'\in C(sp(A))$. *By taking supremum over* $x\in\mathscr{H}$ *with* $\|x\|=1$, *and* $y\in\mathscr{H}$ *with* $\|y\|=1$ *in* (2.8) *respectively, we obtain*

$$\|\phi(f'(A)A)\|\geq\|\phi(f'(A))\|\,\|\phi(A)\|.$$

The following result concerning Slater type inequality holds:

Theorem 2.6. *Let* A *be a self-adjoint operator on the Hilbert space* $\mathscr{H}$ *with* $sp(A)\subseteq[m,M]\subseteq I$ *and* ϕ *be a normalized positive linear map on* $\mathcal{B}(\mathscr{H})$ *and let* $f:I\to\mathbb{R}$ *be a convex and differentiable function on interval* I *whose derivative* f' *is continuous and strictly positive on* I. *Then*

$$\begin{aligned}0&\leq f\left(\frac{\langle\phi(Af'(A))x,x\rangle}{\langle\phi(f'(A))x,x\rangle}\right)-\langle\phi(f(A))x,x\rangle\\&\leq f'\left(\frac{\langle\phi(Af'(A))x,x\rangle}{\langle\phi(f'(A))x,x\rangle}\right)\left[\frac{\langle\phi(Af'(A))x,x\rangle}{\langle\phi(f'(A))x,x\rangle}-\langle\phi(A)x,x\rangle\right]\end{aligned}\tag{2.10}$$

for each $x\in\mathscr{H}$ *with* $\|x\|=1$.

Proof. Since f is convex and differentiable on I, then we have that

$$f'(s)(t-s)\leq f(t)-f(s)\leq f'(t)(t-s)$$

for any $t,s\in[m,M]$. If we fix $t\in[m,M]$ and apply the functional calculus for the operator A, then we obtain

$$f'(A)(t-A)\leq f(t)1_{\mathscr{H}}-f(A)\leq f'(t)(t-A)\tag{2.11}$$

for any $t\in[m,M]$. Since ϕ is normalized positive linear map we have

$$t\phi(f'(A))-\phi(f'(A)A)\leq f(t)1_{\mathscr{H}}-\phi(f(A))\leq f'(t)t1_{\mathscr{H}}-f'(t)\phi(A)\tag{2.12}$$

for any $t\in[m,M]$. The inequality (2.12) is equivalent with

$$\begin{aligned}t\langle\phi(f'(A))x,x\rangle-\langle\phi(f'(A)A)x,x\rangle&\leq f(t)-\langle\phi(f(A))x,x\rangle\\&\leq f'(t)t-f'(t)\langle\phi(A)x,x\rangle\end{aligned}\tag{2.13}$$

for any $t\in[m,M]$. Now, since A is self-adjoint with $m1_{\mathscr{H}}\leq A\leq M1_{\mathscr{H}}$ and $f'(A)$ is strictly positive, then $mf'(A)\leq Af'(A)\leq Mf'(A)$. Also, ϕ is a unital positive linear map, therefore $m\phi(f'(A))\leq\phi(Af'(A))\leq M\phi(f'(A))$, i.e.,

$$m\langle\phi(f'(A))x,x\rangle\leq\langle\phi(Af'(A))x,x\rangle\leq M\langle\phi(f'(A))x,x\rangle,$$

hence $\langle\phi(Af'(A))x,x\rangle\langle\phi(f'(A))x,x\rangle^{-1}\in[m,M]$. If we put

$$t:=\frac{\langle\phi(Af'(A))x,x\rangle}{\langle\phi(f'(A))x,x\rangle}$$

in the equation (2.13), we get the desired result (2.10). □

The following results follows from the inequality (2.10) for the convex function $f(t) = t^p, p \in (-\infty, 0) \cup [1, \infty)$ by performing the required calculation. The details are omitted.

Remark. *Let ϕ be a positive linear map on $\mathcal{B}(\mathscr{H})$ and let A be a strictly positive operator on $\mathscr{H}$.*

(a) If $p \geq 1$, we have that

$$\begin{aligned} 0 &\leq \langle \phi(A^p)x, x\rangle^{p-1} - \left\langle \phi\left(A^{p-1}\right)x, x\right\rangle^p \\ &\leq p\langle \phi(A^p)x, x\rangle^{p-2}\left[\langle \phi(A^p)x, x\rangle - \langle \phi(A)x, x\rangle\left\langle \phi\left(A^{p-1}\right)x, x\right\rangle\right]. \end{aligned}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

(b) If $p < 0$, we have that

$$\begin{aligned} 0 &\leq \langle \phi(A^p)x, x\rangle^{p-1} - \left\langle \phi\left(A^{p-1}\right)x, x\right\rangle^p \\ &\leq (-p)\langle \phi(A^p)x, x\rangle^{p-2}\left[\langle \phi(A)x, x\rangle - \left\langle \phi\left(A^{p-1}\right)x, x\right\rangle\langle \phi(A^p)x, x\rangle\right]. \end{aligned}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Remark. *Notice that we can obtain other results by taking various functions. The details are omitted.*

3. **Some Inequalities Involving States**

In this section, we apply the continuous functional calculus to convex and differentiable functions and present some reverses Jensen's inequalities involving states on C^*-algebras. Our main result of this section reads as follows.

Theorem 3.1. *Let τ be a state on $\mathcal{B}(\mathscr{H})$ and $f : I \to \mathbb{R}$ be a convex and differentiable function on interval I whose derivative f' is continuous on I. Then*

$$0 \leq \tau(f(A)) - f(\tau(A)) \leq \tau(f'(A)A) - \tau(A)\tau(f'(A)).$$

Proof. Since f is convex and differentiable, we have that

$$f(t) - f(s) \leq f'(t)(t - s)$$

if we chose in this inequality $s = \tau(A)$ we obtain

$$f(t) - f(\tau(A)) \leq f'(t)(t - \tau(A)).$$

Applying functional calculus for the operator A

$$f(A) - f(\tau(A)) \leq f'(A)A - f'(A)\tau(A).$$

For the state τ we have

$$\tau(f(A)) - f(\tau(A)) \leq \tau(f'(A)A) - \tau(A)\tau(f'(A)).$$

□

Apply Theorem 3.1 to the state τ defined by $\tau(A) = \langle Ax, x\rangle$ for fixed unit vector $x \in \mathscr{H}$. We have the following result:

Corollary 3.2. *Let A be a self adjoint operator on the Hilbert space $\mathscr{H}$ with $sp(A) \subseteq I$ and let $f : I \to \mathbb{R}$ be a convex and differentiable function on interval I whose derivative f' is continuous on I. Then*

$$0 \leq \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A)Ax, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle.$$

This inequality was obtained by Dragomir (see [1, Theorem 5]).

By Theorem 2.5 and making use of a similar argument to the one in the proof of Theorem 3.1 we can state the following result as well. However, the details are nor provided here.

Theorem 3.3. *Let τ be a state on $\mathcal{B}(\mathscr{H})$ and $f : I \to \mathbb{R}$ be a convex and differentiable function on interval I whose derivative f' is continuous on I. Then*

$$\tau(f(A)) - \tau(f(B)) \le \tau(f'(A)A) - \tau(f'(A))\tau(B). \tag{3.1}$$

Remark. *If we choose $A = B$ in (3.1), we obtain*

$$\tau(f'(A))\tau(A) \le \tau(f'(A)A).$$

Remark. *Let τ be a state on $\mathcal{B}(\mathscr{H})$ and $f(t) = t^2$ in (3.1), then*

$$\tau(A)^2 \le \tau\left(A^2\right) \quad (A \ge 0).$$

In a similar fashion, for self-adjoint operators $A \in \mathcal{B}(\mathscr{H})$

$$\tau\left(e^{\alpha A}\right)^2 \le \tau\left(e^{2\alpha A}\right) \quad (\alpha \ge 0).$$

By applying Theorem 2.6 and using the same strategy as in the proof of Theorem 3.1 we get the next result.

Theorem 3.4. *Let τ be a state on $\mathcal{B}(\mathscr{H})$ and $f : I \to \mathbb{R}$ be a convex and differentiable function on interval I whose derivative f' is continuous and strictly positive on I. Then*

$$\begin{aligned} 0 &\le f\left(\frac{\tau(Af'(A))}{\tau(f'(A))}\right) - \tau(f(A)) \\ &\le f'\left(\frac{\tau(Af'(A))}{\tau(f'(A))}\right)\left[\frac{\tau(Af'(A))}{\tau(f'(A))} - \tau(A)\right]. \end{aligned}$$

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