Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 8 Issue 3(2016), Pages 35-48.

# BEST PROXIMITY POINT SOLUTIONS FOR CERTAIN CLASSES OF CYCLIC CONTRACTIONS IN ORDERED METRIC SPACES

## (COMMUNICATED BY NASEER SHAHZAD)

MOOSA GABELEH, OLIVIER OLELA OTAFUDU

Abstract. In this work we survey some best proximity point theorems for various classes of cyclic contractions by using a geometric notion of monotone proximally property on a nonempty pair of subsets in a metric space equipped with a partially ordered relation. We also extend and improve the main results of Sadiq Basha [S. Sadiq Basha, Discrete optimization in partially ordered sets, J. Global Optim. 54, 511-517, (2012)]. Examples are given to support our main conclusions.

## 1. Introduction and Preliminaries

In [15] Kirk et al. established an interesting extension of Banach contraction principle as follows:

**Theorem 1.1.** *Let $A$ and $B$ be two nonempty closed subsets of a complete metric space $(X, d)$. Suppose that $T : A \cup B \to A \cup B$ is a cyclic mapping, that is $T(A) \subseteq B$ and $T(B) \subseteq A$, such that $d(Tx, Ty) \leq \alpha\, d(x, y)$ for some $\alpha \in (0, 1)$and for all $x \in A, y \in B$. Then $T$ has a unique fixed point in $A \cap B$.*

It is interesting to find out about what happens when $A \cap B = \emptyset$ in Theorem 1.1. The answer to this question is clear that $T$ has no fixed point. Indeed the notion of best proximity point for cyclic mappings was derived from this observation.

**Definition 1.1.** Let $A, B$ be two nonempty subsets of a metric space $(X, d)$ and $T : A \cup B \to A \cup B$ be a cyclic mapping. A point $p \in A \cup B$ is called a best proximity point of $T$ if $d(p, Tp) = \operatorname{dist}(A, B)$, where $\operatorname{dist}(A, B) := \inf\{d(x, y) : x \in A, y \in B\}$.

Indeed best proximity point theorems have been studied to find necessary conditions such that the minimization problem $\min_{x \in A \cup B} d(x, Tx)$ has at least one solution.

---

2000 *Mathematics Subject Classification.* 47H10, 47A16.

*Key words and phrases.* Ordered metric space; best proximity point; cyclic contraction mapping; monotone proximally property.



Submitted May 10, 2016. Published July 22, 2016.

Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$. We set

$$d(x,A)=\mathrm{dist}(\{x\},A),\quad \forall x\in X,$$

$$\mathcal{P}_A(z)=\{x\in A: d(x,z)=d(z,A)\},\quad \forall z\in X,$$

$$A_0:=\{x\in A: d(x,y)=\mathrm{dist}(A,B)\quad \text{for some}\quad y\in B\},$$

$$B_0:=\{y\in B: d(x,y)=\mathrm{dist}(A,B)\quad \text{for some}\quad x\in A\}.$$

Note that if $(A,B)$ is a nonempty, weakly compact and convex pair of subsets of a Banach space $X$, then $A_0$ and $B_0$ are also nonempty, closed and convex subsets of $X$.

**Definition 1.2.** Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. We say that $A$ is Chebyshev set w.r.t. $B$ provided that $\mathcal{P}_A(x)$ is singleton for any $x\in B$.

For instance, if $A$ and $B$ are two nonempty, weakly compact and convex sets in a strictly convex Banach space $X$, then $A$ is Chebyshev set w.r.t. $B$ and $B$ is also Chebyshev set w.r.t. $A$.

In [30] Suzuki et al. introduced a notion of property UC on metric spaces as follows.

**Definition 1.3.**([30]) Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. Then $(A,B)$ is said to satisfy the property UC provided if $\{x_n\}$ and $\{z_n\}$ are sequences in $A$ and $\{y_n\}$ is a sequence in $B$ such that $\lim_n d(x_n,y_n)=\mathrm{dist}(A,B)$ and $\lim_n d(z_n,y_n)=\mathrm{dist}(A,B)$, then $\lim_n d(x_n,z_n)=0$.

**Example 1.3.**([7]) Let $A$ and $B$ be nonempty subsets of a uniformly convex Banach space $X$. Assume that $A$ is convex. Then $(A,B)$ satisfies the property UC.

Next lemma was proved in [30] which will be used in the sequel.

**Lemma 1.2.** *Let $A$ and $B$ be two nonempty subsets of a metric space $(X,d)$ such that $(A,B)$ satisfies the property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequences in $A$ and $B$, respectively, such that either of the following holds:*

$$\lim_{m\to\infty}\sup_{n\geq m} d(x_m,y_n)=dist(A,B)\ \text{or}\ \lim_{n\to\infty}\sup_{m\geq n} d(x_m,y_n)=dist(A,B).$$

*Then $\{x_n\}$ is a Cauchy sequence.*

Very recently, a weaker notion of property UC was introduced in [8] as follows.

**Definition 1.4.** Let $A$ and $B$ be nonempty subsets of a metric space $(X,d)$. The pair $(A,B)$ is said to satisfies the property WUC if for any sequence $\{x_n\}$ in $A$ such that for every $\varepsilon>0$ there exists $y\in B$ satisfying that $d(x_n,y)\leq \mathrm{dist}(A,B)+\varepsilon$ for $n\geq N$, then it is the case that $\{x_n\}$ is convergent.

It was proved that if $A$ and $B$ are two nonempty subsets of a *complete pointwise uniformly convex geodesic metric space* $(X,d)$ *with monotone modulus of convexity* such that $A$ is *convex*, then $(A,B)$ satisfies the property WUC (see Proposition 3.15 of [8]).

**Definition 1.5.** Let $(X,\preceq)$ be a partially ordered set. A self mapping $T:X\to X$ is said to be monotone nondecreasing iff $T(x)\preceq T(y)$ whenever $x,y\in X, x\preceq y$.

Here, we state the main result of [1] which establishes the existence and convergence of best proximity points for cyclic contraction type mappings in partially ordered metric spaces.

**Theorem 1.3.** *([1]) Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$. Let $A$, $B$ be two nonempty subsets of $X$ such that $(A, B)$ satisfies the property UC, and $A$ is complete. Assume that $X$ satisfies the condition*

$$\text{if a nondecreasing sequence } x_n \to x \in X, \text{ then } x_n \preceq x \ \forall n. \tag{1.1}$$

*Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that $T$ and $T^2$ are nondecreasing on $A$. Moreover,*

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + (1-\alpha)\mathrm{dist}(A, B),$$

*and*

$$d(T\acute{y}, T^2y) \leq \alpha d(\acute{y}, Ty) + (1-\alpha)\mathrm{dist}(A, B),$$

*for some $\alpha \in ]0, 1[$ and for all $(x, \acute{x}) \in A \times A, (y, \acute{y}) \in B \times B$ with $x \preceq \acute{x}, y \preceq \acute{y}$. If there exists $x_0 \in A$ such that $x_0 \preceq T^2x_0$ and $x_{n+1} = Tx_n$, then $T$ has a best proximity point $p \in A$ and $x_{2n} \to p$.*

We refer to [2, 18] for some generalizations of Theorem 1.3 and [3, 4, 9, 10, 20, 26, 27] for more information to the same problem. Some of recent results related to existence of fixed points for cyclic mappings can be found in [6, 11, 12, 13, 14, 21, 22, 23, 24].

This paper is organized as follows: in Section 2 we introduce a geometric notion of *monotone proximally property* on a nonempty pair of subsets in partially ordered metric which is an extension of the property WUC ([8]) and generalize and improve the main results of [1]. In Section 3, we study the existence of best proximity points for a class of cyclic mappings which are contractions in the sense of Meir-Keeler ([17]) in the metric spaces equipped a partially ordered relation and so extend the results of [5]. Finally, in Section 4 we introduce a new class on non-self mappings, called *generalized ordered proximal contractions* in ordered metric spaces, and prove a best proximity point theorem for this class of non-self mappings. Thereby, we improve and extend the main conclusions of [25]. Example are also given to useability of our main results.

## 2. Cyclic contractions

In this section we consider generalized cyclic contraction type mappings and study existence and convergence of best proximity points for this class of mappings which contains the class of mappings in Theorem 1.6 as a subclass. We begin with the following auxiliary lemma.

**Lemma 2.1.** *Let $A, B$ be nonempty subsets of a metric space $(X, d)$ and "$\preceq$" be a partially ordered relation on $A$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that $T^2$ is nondecreasing on $A$ and*

$$d(T\acute{x}, T^2x) \leq \alpha \max\{d(\acute{x}, Tx), d(\acute{x}, T\acute{x}), d(T^2x, Tx)\} + (1-\alpha)\mathrm{dist}(A, B),$$

*for some $\alpha \in (0, 1)$ and for all $x, \acute{x} \in A$ with $x \preceq \acute{x}$. If there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$ and $x_{n+1} = Tx_n$, then $d(x_n, x_{n+1}) \to \mathrm{dist}(A, B)$.*

*Proof.* Since $T^2$ is nondecreasing on $A$ and $x_0 \preceq T^2x_0$,

$$x_0 \preceq T^2x_0 \preceq \cdots \preceq T^{2n}x_0 \preceq \cdots .$$

Put $r_n = d(x_n, x_{n+1})$. We have

$$\begin{aligned} r_{2n} &= d(x_{2n}, x_{2n+1}) = d(T(x_{2n}), T^2(x_{2n-2})) \\ &\le \alpha \max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n-1})\} + (1-\alpha)\mathrm{dist}(A, B) \\ &= \alpha \max\{r_{2n-1}, r_{2n}\} + (1-\alpha)\mathrm{dist}(A, B). \end{aligned}$$

If there exists $n_0 \in \mathbb{N}$ such that $r_{2n_0-1} \le r_{2n_0}$, then we obtain

$$r_{2n_0} \le \alpha \max\{r_{2n_0-1}, r_{2n_0}\} + (1-\alpha)\mathrm{dist}(A, B) = \alpha r_{2n_0} + (1-\alpha)\mathrm{dist}(A, B).$$

Thus $r_{2n_0} = \mathrm{dist}(A, B)$. Besides,

$$\begin{aligned} r_{2n_0+1} &= d(x_{2n_0+1}, x_{2n_0+2}) = d(T(x_{2n_0}), T^2(x_{2n_0})) \\ &\le \alpha \max\{d(x_{2n_0}, x_{2n_0+1}), d(x_{2n_0}, x_{2n_0+1}), d(x_{2n_0+2}, x_{2n_0+1})\} + (1-\alpha)\mathrm{dist}(A, B) \\ &= \alpha r_{2n_0+1} + (1-\alpha)\mathrm{dist}(A, B). \end{aligned}$$

Therefore, $r_{2n_0+1} = \mathrm{dist}(A, B)$. Analogously, we conclude that $r_n = \mathrm{dist}(A, B)$, for all $n \ge n_0$. Similar argument implies that if there exists $n_0 \in \mathbb{N}$ such that $r_{2n_0-2} \le r_{2n_0-1}$ then $r_n = \mathrm{dist}(A, B)$, for all $n \ge n_0$ and hence $r_n \to \mathrm{dist}(A, B)$. Let $r_{2n} < r_{2n-1}$ and $r_{2n-1} < r_{2n-2}$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} r_{2n} &\le \alpha \max\{r_{2n-1}, r_{2n}\} + (1-\alpha)\mathrm{dist}(A, B) \\ &= \alpha r_{2n-1} + (1-\alpha)\mathrm{dist}(A, B) = \alpha d(T(x_{2n-2}), T^2(x_{2n-2})) + (1-\alpha)\mathrm{dist}(A, B) \\ &\le \alpha[\alpha \max\{d(x_{2n-2}, x_{2n-1}), d(x_{2n-2}, x_{2n-1}), d(x_{2n}, x_{2n-1})\} + (1-\alpha)\mathrm{dist}(A, B)] + (1-\alpha)\mathrm{dist}(A, B) \\ &= \alpha^2 r_{2n-2} + (1-\alpha^2)\mathrm{dist}(A, B) \le ... \le \alpha^{2n} r_0 + (1-\alpha^{2n})\mathrm{dist}(A, B). \end{aligned}$$

Now if $n \to \infty$, we obtain $r_{2n} \to \mathrm{dist}(A, B)$. Similarly, we see that $r_{2n-1} \to \mathrm{dist}(A, B)$. Hence, $r_n = d(x_n, x_{n+1}) \to \mathrm{dist}(A, B)$. □

We now establish the following existence theorem.

**Theorem 2.2.** *Let $A, B$ be nonempty closed subsets of a metric space $(X, d)$ and let "$\preceq$" be a partially ordered relation on $A$. Assume that $T$ is a cyclic mapping on $A \cup B$ such that $T^2$ is nondecreasing on $A$ and*

$$d(T\acute{x}, T^2x) \le \alpha \max\{d(\acute{x}, Tx), d(\acute{x}, T\acute{x}), d(T^2x, Tx)\} + (1-\alpha)\mathrm{dist}(A, B),$$

*for some $\alpha \in (0, 1)$ and for all $x, \acute{x} \in A$ with $x \preceq \acute{x}$. Suppose that there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$ and define $x_{n+1} = Tx_n$. If $A$ satisfies the Condition (1) and either $A$ or $B$ is boundedly compact, then $T$ has a best proximity point.*

*Proof.* We assume that $A$ is boundedly compact. Then there exists a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$ converging to some $p \in A$. Thus

$$\mathrm{dist}(A, B) \le d(p, x_{2n_k-1}) \le d(p, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}).$$

Now if $k \to \infty$, then by Lemma 2.1 we have $d(p, x_{2n_k-1}) \to \mathrm{dist}(A, B)$. Since $T^2$ is nondecreasing and the Condition (1) holds,

$$\begin{aligned} d(x_{2n_k}, Tp) &= d(Tp, T^2x_{2n_k-2}) \\ &\le \alpha \max\{d(p, x_{2n_k-1}), d(p, Tp), d(x_{2n_k}, x_{2n_k-1})\} + (1-\alpha)\mathrm{dist}(A, B). \end{aligned}$$

Letting $k \to \infty$, we obtain $d(p, Tp) = \mathrm{dist}(A, B)$. □

**Remark.** *It is worthwhile to note that if in the above theorem $\mathcal{P}_A(y)$ is singleton for any $y \in B_0$, then $T^2p = p$, that is, $p$ is a fixed point of $T^2$.*

*Proof.* Since $d(p, Tp) = \mathrm{dist}(A, B)$, we have $\mathcal{P}_A(Tp) = \{p\}$. Now from the contractive condition on $T$ we obtain

$$\begin{aligned} d(Tp, T^2p) &\le \alpha \max\{d(p, Tp), d(p, Tp), d(T^2p, Tp)\} + (1-\alpha)\mathrm{dist}(A, B) \\ &= \alpha d(T^2p, Tp) + (1-\alpha)\mathrm{dist}(A, B), \end{aligned}$$

which implies that $d(Tp, T^2p) = \mathrm{dist}(A, B)$. Thereby, $T^2p \in \mathcal{P}_A(Tp)$ and so $T^2p = p$.

□

Motivated by Definition 1.4, we introduce the concept of *monotone proximally property* in partially ordered metric spaces.

**Definition 2.1.** Let $A$ and $B$ be nonempty subsets of a partially metric space $(X, d)$. The pair $(A, B)$ is said to have monotone proximally property if for any increasing sequence $\{x_n\}$ in $A$ such that for every $\varepsilon > 0$ there exist $y \in B$ and $N \in \mathbb{N}$ satisfying that $d(x_n, y) \le \mathrm{dist}(A, B) + \varepsilon$ for $n \ge N$, then it is the case that $\{x_n\}$ is convergent.

Let us illustrate the above notion with the following examples.

**Example 2.1.** Consider $X = \mathbb{R}$ with the usual metric and with the natural partially ordered relation $\le$. Suppose $A = [-1, 0] \cup \{2\}$ and $B = \{1\}$. Then $\mathrm{dist}(A, B) = 1$ and $A_0 = \{0, 2\},\ B_0 = \{1\}$. Let $\varepsilon > 0$ be given. We just have the following two cases:
**Case 1.** Consider the nondecreasing sequence $\{x_n\}$ defined with $x_n = -\frac{1}{n}$. Then for $y = 1$ we have $\lim_{n\to\infty} d(x_n, y) = 1$ and so, there exists $N \in \mathbb{N}$ such that $d(x_n, y) \le \mathrm{dist}(A, B) + \varepsilon$ for all $n \ge N$. We note that $x_n \to 0$.
**Case 2.** Let $\{x_n\}$ be a sequence in $A$ for which $x_n = 2$ for all $n \in \mathbb{N}$ except perhaps finite numbers. In this case we have $x_n \to 2$ and that $\lim_{n\to\infty} d(x_n, y) = 1$.
Therefore, $(A, B)$ has the monotone proximally property. We now claim that $(A, B)$ does not satisfy the property WUC. To prove it, let us the sequence $\{z_n\}$ in $A$ as

$$z_n = \begin{cases} -\frac{1}{n} & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Then we have $\lim_{n\to\infty} d(z_n, y) = \mathrm{dist}(A, B)$ but the sequence $\{z_n\}$ is not convergent.

The following lemma will be used in the sequel. We omit the proof since it follows similar patterns to those given by the proof of Proposition 3.3 of [7].

**Lemma 2.3.** *Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that $T^2$ is nondecreasing on $A$ and*

$$d(T\acute{x}, T^2x) \le \alpha d(\acute{x}, Tx) + (1-\alpha)\mathrm{dist}(A, B),$$

*for some $\alpha \in (0, 1)$ and for all $x, \acute{x} \in A$ with $x \preceq \acute{x}$. If there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$ and if $x_{n+1} = Tx_n$, then the sequences $\{x_{2n-1}\}$ and $\{x_{2n}\}$ are bounded.*

Next theorem improves and extends Theorem 1.3.

**Theorem 2.4.** *Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$. Suppose that $(A, B)$ is a nonempty pair of subsets of $X$ such that $(A, B)$ has the monotone proximally property and $A$ is complete. Assume that the condition* (1) *holds on $X$ and $T : A \cup B \to A \cup B$ is a cyclic mapping such that $T$ and $T^2$ are nondecreasing on $A$ and*

$$d(T\acute{x}, T^2x) \leq \alpha d(\acute{x}, Tx) + (1-\alpha)\mathrm{dist}(A, B),$$

*for some $\alpha \in (0,1)$ and for all $(x, \acute{x}) \in A \times A$ and $(x, \acute{x}) \in B \times B$ with $x \preceq \acute{x}$. If there exists $x_0 \in A$ such that $x_0 \preceq T^2x_0$ and $x_{n+1} = Tx_n$, then $T$ has a best proximity point $p \in A$ and $x_{2n} \to p$.*

*Proof.* Consider $n \in \mathbb{N}$. Then for all $k \in \mathbb{N}$ by the fact that the sequences $\{x_{2n-1}\}$ and $\{x_{2n}\}$ are nondecreasing and bounded, we have

$$\begin{aligned}
d(x_{2n+2k}, x_{2n+1}) &= d(T^{2n+2k}x_0, T^{2n+1}x_0) = d(T(x_{2n+2k-1}), T^2(x_{2n-1})) \\
&\leq \alpha d(x_{2n+2k-1}, x_{2n}) + (1-\alpha)\mathrm{dist}(A,B) \quad (\text{since } x_{2n-1} \preceq x_{2n+2k-1}) \\
&= \alpha d(Tx_{2n+2k-2}, T^2x_{2n-2}) + (1-\alpha)\mathrm{dist}(A,B) \\
&\leq \alpha^2 d(x_{2n+2k-2}, x_{2n-1}) + (1-\alpha^2)\mathrm{dist}(A,B) \quad (\text{since } x_{2n-2} \preceq x_{2n+2k-2}) \\
&\leq \ldots \leq \alpha^{2n} d(x_{2k}, x_1) + (1-\alpha^{2n})\mathrm{dist}(A,B) \leq \alpha^{2n}M + (1-\alpha^{2n})\mathrm{dist}(A,B),
\end{aligned}$$

where $M := \sup\{d(x_{2k}, x_1) : k \in \mathbb{N}\}$. Thus for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_{2n_0+2k}, x_{2n_0+1}) \leq \varepsilon + \mathrm{dist}(A,B)$ which implies that

$$x_{2n_0+2k} \in \mathcal{B}(x_{2n_0+1}, \varepsilon + \mathrm{dist}(A,B)), \quad \forall k \in \mathbb{N}.$$

Since $(A, B)$ has the monotone proximally property, we conclude that $\{x_{2n}\}$ is convergent to a point such as $p \in A$. It now follows from a similar argument of Theorem 2.2 that $p$ is a best proximity point of $T$ and this completes the proof. □

**Example 2.2.** Suppose $X = \mathbb{R}^2$ and define the metric $d$ on $X$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}, \quad \forall\ (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2.$$

We know that $X$ is not strictly convex. Consider the partially ordered relation on $X$ with

$$(x_1, y_1) \preceq (x_2, y_2) \Leftrightarrow x_1 \leq x_2,\ y_1 \leq y_2.$$

Let

$$A = \{(0, 1 - \frac{1}{2n}) : n \in \mathbb{N}\} \cup \{(0, n) : n \in \mathbb{N}\}, \quad B = \{(2, y) : 0 \leq y \leq 1\}.$$

Then $\mathrm{dist}(A, B) = 2$ and $A_0 = \{(0, 1 - \frac{1}{2n}) : n \in \mathbb{N}\} \cup \{(0, n) : n \in \{1, 2, 3\}\}, B_0 = B$. Also, it is easy to see that $(A, B)$ has the monotone proximally property. Define the mapping $T : A \cup B \to A \cup B$ with

$$T(0, 1 - \frac{1}{2n}) = (2, \frac{1}{2n}), \quad T(0, n) = (2, 0) \text{ and } T(2, y) = (0, 1).$$

Now for any $(\mathbf{x}, \acute{\mathbf{x}}) \in A \times A \cup B \times B$ with $\mathbf{x} \preceq \acute{\mathbf{x}}$ and $\alpha \in (0,1)$ we have

$$d(T\acute{\mathbf{x}}, T^2\mathbf{x}) = 2 \leq \alpha d(\acute{\mathbf{x}}, T\mathbf{x}) + (1-\alpha)\mathrm{dist}(A, B).$$

Moreover, for any $\mathbf{x}_0 \in \{(0, 1 - \frac{1}{2n}) : n \in \mathbb{N}\}$ we have $\mathbf{x}_0 \preceq T^2\mathbf{x}_0$. It now follows from Theorem 2.4 that $T$ has a best proximity point in $A$ and if we define $\mathbf{x}_{n+1} = T\mathbf{x}_n$, then $\mathbf{x}_{2n}$ converges to the best proximity point of $T$ in $A$. It is worth noticing that any point of the set $\{(0, 1 - \frac{1}{2n}) : n \in \mathbb{N}\}$ is a best proximity point of $T$.

## 3. Cyclic Meir-Keeler contractions

The class of *cyclic Meir-Keeler contractions* were introduced by Di Bari et al. in [5] in order to study of existence and convergence of best proximity points in uniformly convex Banach spaces (see Theorem 2 of [5]).

Before we state the main result of them, we recall the following essentials.

**Definition 3.1.**(Lim [16]) A function $\varphi : [0,\infty) \to [0,\infty)$ is called an L-function if $\varphi(0) = 0, \varphi(s) > 0$ for $s \in (0,\infty)$, and for every $s > 0$ there exists $\delta > 0$ such that $\varphi(t) \le s$ for all $t \in [s, s+\delta]$.

**Lemma 3.1.** *(*[16, 28]*) Let $Y$ be a nonempty set and let $f$ and $g$ be functions from $Y$ into $[0,\infty)$. Then the following equivalent.*
*$(i)$ For each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$x \in Y, \quad f(x) < \varepsilon + \delta \quad \textit{implies} \quad g(x) < \varepsilon.$$

*$(ii)$ There exists a (nondecreasing, continuous) L-function $\varphi$ such that*

$$x \in Y, \quad f(x) > 0 \quad \textit{implies} \quad g(x) < \varphi(f(x))$$

*and*

$$x \in Y, \quad f(x) = 0 \quad \textit{implies} \quad g(x) = 0.$$

**Lemma 3.2.** *(*[28]*) Let $\varphi$ be an L-function. Let $\{s_n\}$ be a nondecreasing sequence of nonnegative real numbers. Suppose $s_{n+1} < \varphi(s_n)$ for all $n \in \mathbb{N}$ with $s_n > 0$. Then $\lim_n s_n = 0$.*

Next theorem is the main result of [5].

**Theorem 3.3.** *(Theorem 4 of* [5]*) Let $X$ be a uniformly convex Banach space and let $A$ and $B$ be nonempty subsets of $X$. Suppose that $A$ is closed and convex. Let $T : A \cup B \to A \cup B$ be a cyclic mapping so that for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|x - y\| < \operatorname{dist}(A,B) + \delta + \varepsilon \quad \textit{implies} \quad \|Tx - Ty\| < \operatorname{dist}(A,B) + \varepsilon,$$

*and*

$$\|Tx - Ty\| < \|x - y\| \quad \textit{whenever} \quad \|x - y\| > \operatorname{dist}(A,B),$$

*for any $(x,y) \in A \times B$. Then $T$ has a best proximity point in $A$ and for any $x_0 \in A$ if we define $x_{n+1} = Tx_n$, then $x_{2n}$ converges to the best proximity point of $T$.*

At the end of this section, we attempt to generalize Theorem 2.4 to metric spaces equipped with the partially ordered relation. We begin with the following lemma.

**Lemma 3.4.** *Let $A, B$ be nonempty subsets of a metric space $(X,d)$ and "$\preceq$" be a partially ordered relation on $A$. Let $T : A \cup B \to A \cup B$ be a cyclic mapping such that $T^2$ is nondecreasing on $A$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d^*(\acute{x}, Tx) < \delta + \varepsilon \quad \textit{implies} \quad d^*(T\acute{x}, T^2x) < \varepsilon,$$

*and*

$$d(T\acute{x}, T^2x) < d(\acute{x}, Tx) \quad \textit{whenever} \quad d^*(\acute{x}, Tx) > 0,$$

*for all $x, \acute{x} \in A$ with $x \preceq \acute{x}$, where $d^*(a,b) := d(a,b) - \operatorname{dist}(A,B)$ for all $(a,b) \in A \times B$. If there exists $x_0 \in A$ with $x_0 \preceq T^2x_0$ and if $x_{n+1} = Tx_n$, then $d(x_n, x_{n+1}) \to \operatorname{dist}(A,B)$.*

*Proof.* By Lemma 3.1 we have

$$d^*(\acute{x}, Tx) > 0 \quad \text{implies} \quad d^*(T\acute{x}, T^2x) < \varphi(d^*(\acute{x}, Tx))$$

and

$$d^*(\acute{x}, Tx) = 0 \quad \text{implies} \quad d^*(T\acute{x}, T^2x) = 0,$$

for all $x, \acute{x} \in A$ with $x \preceq \acute{x}$. According to the fact that $T^2$ is nondecreasing on $A$, $\{x_{2n}\}$ is a nondecreasing sequence. Set $s_n = d^*(x_n, x_{n+1})$. If there exists $n_0 \in \mathbb{N}$ such that $s_{n_0} = 0$, then we conclude that $d(x_n, Tx_n) = \operatorname{dist}(A, B)$ for all $n \geq n_0$ and we are finished. Now, suppose $s_n > 0$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} s_{2n} = d^*(x_{2n}, x_{2n+1}) &= d^*(Tx_{2n}, T^2x_{2n-2}) \quad (x_{2n-2} \preceq x_{2n}) \\ &< \varphi(d^*(x_{2n}, Tx_{2n-2})) = \varphi(s_{2n-1}) \leq s_{2n-1} \\ = d^*(x_{2n-1}, x_{2n}) &= d^*(Tx_{2n-2}, T^2x_{2n-2}) < \varphi(d^*(x_{2n-2}, x_{2n-1})) = \varphi(s_{2n-2}). \end{aligned}$$

It now follows from Lemma 3.2 that $s_{2n} \to 0$ and so $s_{2n-1} \to 0$. Therefore, $s_n \to 0$. □

**Theorem 3.5.** *Under the assumptions of Lemma 3.4 if $A, B$ are closed, $A$ satisfies the Condition (1) and either $A$ or $B$ is boundedly compact, then $T$ has a best proximity point.*

*Proof.* Let $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converging to some $p \in A$. Similar argument of Theorem 2.2 implies that $d^*(p, x_{2n_k-1}) \to 0$. Hence

$$\begin{aligned} d^*(x_{2n_k}, Tp) &= d^*(Tp, T^2(x_{2n_k-2})) \quad (x_{2n_k-2} \preceq p) \\ &< \varphi(d^*(p, x_{2n_k-1})) \leq d^*(p, x_{2n_k-1}). \end{aligned}$$

Letting $k \to \infty$, we obtain $d^*(x_{2n_k}, Tp) \to 0$. Then $d(p, Tp) = \operatorname{dist}(A, B)$. □

Next result is another extension of Theorem 1.3.

**Theorem 3.6.** *Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$. Suppose that $(A, B)$ is a nonempty pair of subsets of $X$ such that $(A, B)$ satisfies the property* UC *and $A$ is complete. Assume that the condition* (1) *holds on $A$ and $T : A \cup B \to A \cup B$ is a cyclic mapping such that $T$ and $T^2$ are nondecreasing on $A$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d^*(\acute{x}, Tx) < \delta + \varepsilon \quad \textit{implies} \quad d^*(T\acute{x}, T^2x) < \varepsilon,$$

*and*

$$d(T\acute{x}, T^2x) < d(\acute{x}, Tx) \quad \textit{whenever} \quad d^*(\acute{x}, Tx) > 0,$$

*for all $(x, \acute{x}) \in A \times A$ and $(x, \acute{x}) \in B \times B$ with $x \preceq \acute{x}$. If there exists $x_0 \in A$ such that $x_0 \preceq T^2x_0$ and $x_{n+1} = Tx_n$, then $T$ has a best proximity point $p \in A$ and $x_{2n} \to p$.*

*Proof.* By Lemma 3.4 for the nondecreasing sequences $\{x_{2n}\}$ and $\{x_{2n-1}\}$ we have

$$d^*(x_{2n}, x_{2n+1}) \to 0, \quad d^*(x_{2n+2}, x_{2n+1}) \to 0.$$

Since $(A, B)$ satisfies the property UC, $d(x_{2m}, x_{2m+2}) \to 0$. Let $\varepsilon > 0$ be given and choose $\delta \in (0, 1)$ satisfying $\varphi(\varepsilon + \delta) \leq 2\varepsilon$. Let $k \in \mathbb{N}$ be such that $d^*(x_{2m}, x_{2m+1}) < \varepsilon$ , $d^*(x_{2m+2}, x_{2m+1}) < \varepsilon$ and $d(x_{2m}, x_{2m+2}) < \delta$, for all $m \geq k$. Fix $m \in \mathbb{N}$ with $m \geq k$. We now prove that

$$d^*(x_{2m}, x_{2n+1}) < 3\varepsilon \tag{3.1}$$

for all $n \geq m$. If $n = m$, then (2) holds. Assume that (2) holds for $n \geq m$. We have

$$\begin{aligned} d^*(x_{2m+2}, x_{2n+3}) &\leq d^*(T(x_{2n+2}), T^2(x_{2m})) \quad (x_{2m} \preceq x_{2n+2}) \\ &\leq \varphi(d^*(x_{2n+2}, x_{2m+1})) \leq d^*(x_{2n+2}, x_{2m+1}) \\ &= d^*(T(x_{2n+1}), T^2(x_{2m-1})) \leq \varphi(d^*(x_{2m}, x_{2n+1})) \quad (x_{2m-1} \preceq x_{2n+1}) \\ &\leq \varphi(\varepsilon + \delta) \leq 2\varepsilon. \end{aligned}$$

This implies that

$$\begin{aligned} d^*(x_{2m}, x_{2n+3}) &\leq d(x_{2m}, x_{2m+2}) + d^*(x_{2m+2}, x_{2n+3}) \\ &\leq \delta + 2\varepsilon < 3\varepsilon, \end{aligned}$$

that is (2) holds. Therefore $\lim_{m\to\infty} \sup_{n\geq m} d^*(x_{2m}, x_{2n+1}) = 0$. It follows from Lemma 1.2 that $\{x_{2n}\}$ is a Cauchy sequence and by the completeness of the set $A$, there exists $p \in A$ such that $x_{2n} \to p$. Since the condition (1) holds on $A$, we have $x_{2n} \preceq p$, for all $n \in \mathbb{N}$. Thus

$$\begin{aligned} d^*(p, Tp) &= \lim_{n\to\infty} d^*(x_{2n}, Tp) = \lim_{n\to\infty} d^*(Tp, T^2(x_{2n-2})) \quad (x_{2n-2} \preceq p) \\ &\leq \lim_{n\to\infty} \varphi(d^*(p, x_{2n-1})) \leq \lim_{n\to\infty} d^*(p, x_{2n-1}) \\ &\leq \lim_{n\to\infty} (d(p, x_{2n}) + d^*(x_{2n}, x_{2n-1})) = 0. \end{aligned}$$

Hence $d(p, Tp) = \text{dist}(A, B)$ which completes the proof. □

We finish this section by raising the next problem.

**Question 3.1.** It is interesting to find out if Theorem 3.6 still holds whenever the pair $(A, B)$ has the monotone proximally property.

## 4. Generalized ordered proximal contractions

In the last section of the current work, we are going to extend the main results of [25]. We start by recalling the following notions.

**Definition 4.1.**([25]) Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$ and assume that $A, B$ are nonempty subsets of $X$. A mapping $T : A \to B$ is said to be a proximally increasing if it satisfies the condition that

$$\begin{cases} x \preceq y, \\ d(u, Tx) = \text{dist}(A, B), & \Rightarrow u \preceq v, \\ d(v, Ty) = \text{dist}(A, B), \end{cases}$$

for all $x, y, u, v \in A$.

**Definition 4.2.**([25]) A non-self mapping $T : A \to B$ is said to be an ordered proximal contraction if for all $u, v, x, y \in A$ with

$$x \preceq y, \quad d(u, Tx) = \text{dist}(A, B) \quad \text{and} \quad d(v, Ty) = \text{dist}(A, B),$$

we have

$$d(u, v) \leq r d(x, y). \tag{4.1}$$

Here, we introduce the concept of *generalized ordered proximal contractions* as below.

**Definition 4.3.** Consider a strictly decreasing function $\eta$ from $[0,1)$ onto $(\frac{1}{2},1]$ by $\eta(r)=\frac{1}{1+r}$. The non-self mapping $T:A\to B$ is said to be a generalized ordered proximal contraction if for all $u,v,x,y\in A$ with

$$x\preceq y,\quad d(u,Tx)=\mathrm{dist}(A,B)\quad\text{and}\quad d(v,Ty)=\mathrm{dist}(A,B),$$

we have

$$\eta(r)d^*(x,Tx)\le d(x,y)\ \text{ implies }\ d(u,v)\le rd(x,y). \tag{4.2}$$

**Remark.** *It is clear that the class of generalized ordered proximal contractions contains the class of ordered proximal contraction as a subclass.*

**Example 4.1.** Consider $X=\mathbb{R}^2$ with the partially ordered relation and the metric as in Example 2.2. Assume that $A:=\{(0,0),(4,5),(5,5)\}$ and $B:=\{(0,0),(4,5),(4,0)\}$. Let $T:A\to B$ be a mapping defined as

$$T(\mathbf{x})=\begin{cases}(4,5) & \text{if } \ \mathbf{x}=(5,5),\\ (0,0) & \text{if } \ \mathbf{x}\neq(5,5).\end{cases}$$

Note that $\mathrm{dist}(A,B)=0$. Then $T$ is generalized ordered proximal contraction for each $\frac{1}{5}\le r<1$. To this end, we consider two following cases.
**Case 1.** Let $(\mathbf{u},\mathbf{x})=((0,0),(0,0))$ and $(\mathbf{v},\mathbf{y})=((4,5),(5,5))$. Then we have $\mathbf{x}\preceq\mathbf{y}$ and $d(\mathbf{u},T\mathbf{x})=d(\mathbf{v},T\mathbf{y})=\mathrm{dist}(A,B)$. Also,

$$d(\mathbf{u},\mathbf{v})=1\le r\times 5=rd(\mathbf{x},\mathbf{y}).$$

**Case 2.** Let $(\mathbf{u},\mathbf{x})=((0,0),(4,5))$ and $(\mathbf{v},\mathbf{y})=((4,5),(5,5))$. Then we have $\mathbf{x}\preceq\mathbf{y}$ and $d(\mathbf{u},T\mathbf{x})=d(\mathbf{v},T\mathbf{y})=\mathrm{dist}(A,B)$. Also,

$$\eta(r)d^*(\mathbf{x},T\mathbf{x})=\frac{5}{1+r}>1=d(\mathbf{x},\mathbf{y}).$$

Hence, $T$ is a generalized ordered proximal contraction mapping. It is interesting to note that $T$ is not ordered proximal contraction. Indeed, in Case 2 we have

$$d(\mathbf{u},\mathbf{v})=5>r\times 1=rd(\mathbf{x},\mathbf{y}).$$

We now establish the following existences theorem.

**Theorem 4.1.** *(Compare with Theorem 3.1 of* [25]*) Let $(X,\preceq)$ be a partially ordered set and $d$ be a metric on $X$. Assume that $A,B$ are nonempty subsets of $X$ such that $A_0$ is nonempty, closed and the condition* (1) *holds on $A$. Let $T:A\to B$ be a non-self mapping satisfies the following conditions.*
*(i) $T$ is a proximally increasing and generalized ordered proximal contraction and $T(A_0)\subseteq B_0$.*
*(ii) There exist elements $x_0,x_1\in A_0$ such that*

$$d(x_1,Tx_0)=\mathrm{dist}(A,B)\quad\textit{and}\quad x_0\preceq x_1.$$

*Then $T$ has a best proximity point.*

*Proof.* By condition (ii) there exist $x_0,x_1\in A_0$ such that $x_0\preceq x_1$ and $d(x_1,Tx_0)=\mathrm{dist}(A,B)$. Since $Tx_1\in B_0$, there exists $x_2\in A_0$ such that $d(x_2,Tx_1)=\mathrm{dist}(A,B)$. By the fact that $T$ is proximally increasing we conclude that $x_1\preceq x_2$. Continuing this process, we can find a sequence $\{x_n\}$ in $A_0$ such that

$$d(x_{n+1},Tx_n)=\mathrm{dist}(A,B),\quad x_n\preceq x_{n+1},\ \text{for all }\ n\in\mathbb{N}\cup\{0\}. \tag{4.3}$$

For all $n \in \mathbb{N} \cup \{0\}$ we have

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + \operatorname{dist}(A, B).$$

Note that $\eta(r) \leq 1$, thus

$$x_n \preceq x_{n+1}, \quad \begin{cases} d(x_{n+1}, Tx_n) = \operatorname{dist}(A, B), \\ d(x_{n+2}, Tx_{n+1}) = \operatorname{dist}(A, B), \end{cases} \quad \& \quad \eta(r) d^*(x_n, Tx_n) \leq d(x_n, x_{n+1}).$$

By the fact that $T$ is generalized ordered proximal contraction,

$$d(x_{n+1}, x_{n+2}) \leq r d(x_n, x_{n+1}), \ \forall n \in \mathbb{N} \cup \{0\}.$$

By induction we conclude that

$$d(x_n, x_{n+1}) \leq r^n d(x_0, x_1),$$

which implies that

$$\Sigma_{n=1}^{\infty} d(x_n, x_{n+1}) \leq \Sigma_{n=1}^{\infty} r^n d(x_0, x_1) < \infty.$$

Thus, $\{x_n\}$ is a Cauchy and increasing sequence $A_0$. Since $A_0$ is closed and $X$ is complete, we deduce that $\{x_n\}$ is a convergent sequence. Let $p \in A_0$ be such that $x_n \to p$. Since the condition (1) holds on $A$, $x_n \preceq p$ for each $n \in \mathbb{N} \cup \{0\}$. We prove that

$$d^*(p, Tx) \leq r d(p, x), \quad \forall x \in A_0 \quad \text{with} \quad x_n \preceq x \precneqq p, \forall n \in \mathbb{N} \cup \{0\}. \tag{4.4}$$

Let $x \in A_0$ be such that $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$ and $x \precneqq p$. By the fact that $T(A_0) \subseteq B_0$, there exists $y \in A_0$ such that $d(y, Tx) = \operatorname{dist}(A, B)$. Since $x_n \to p$, there exists $N_1 \in \mathbb{N}$ such that

$$d(x_n, p) \leq \frac{1}{3} d(x, p), \quad \forall n \geq N_1.$$

Now, for each $n \geq N_1$ we have

$$\begin{aligned} \eta(r) d^*(x_n, Tx_n) &\leq d^*(x_n, Tx_n) \\ &\leq d(x_n, p) + d(p, x_{n+1}) + d^*(x_{n+1}, Tx_n) \\ &= d(x_n, p) + d(p, x_{n+1}) \leq \frac{2}{3} d(x, p) \\ &= d(x, p) - \frac{1}{3} d(x, p) \leq d(x, p) - d(x_n, p) \\ &\leq d(x_n, x). \end{aligned}$$

Thereby, for each $n \geq N_1$

$$x_n \preceq x, \quad \begin{cases} d(x_{n+1}, Tx_n) = \operatorname{dist}(A, B), \\ d(y, Tx) = \operatorname{dist}(A, B), \end{cases} \quad \& \quad \eta(r) d^*(x_n, Tx_n) \leq d(x_n, x).$$

Again, since $T$ is generalized ordered proximal contraction,

$$d(x_{n+1}, y) \leq r d(x_n, x).$$

Thus,

$$\begin{aligned} d(p, Tx) &= \lim_{n\to\infty} d(x_{n+1}, Tx) \leq \lim_{n\to\infty} [d(x_{n+1}, y) + d(y, Tx)] \\ &\leq \lim_{n\to\infty} [r d(x_n, x) + d(y, Tx)] = r d(p, x) + \operatorname{dist}(A, B) \end{aligned}$$

Hence,

$$d^*(p, Tx) \leq r d(p, x), \quad \forall x \in A_0, \quad \text{with} \quad x_n \preceq x \precneqq p, \forall n \in \mathbb{N} \cup \{0\},$$

which implies that, (6) holds. We now have

$$\begin{aligned} d^*(x_n, Tx_n) &\le d(x_n, p) + d^*(p, Tx_n) \\ &\le d(x_n, p) + rd(p, x_n) = (1+r)d(p, x_n). \end{aligned}$$

So,

$$\eta(r)d^*(x_n, Tx_n) \le d(p, x_n). \tag{4.5}$$

On the other hand, since $p \in A_0$ and $T(A_0) \subseteq B_0$, there exists $q \in A_0$ such that $d(q, Tp) = \operatorname{dist}(A, B)$. Therefore,

$$x_n \preceq p, \quad \begin{cases} d(x_{n+1}, Tx_n) = \operatorname{dist}(A, B), \\ d(q, Tp) = \operatorname{dist}(A, B), \end{cases} \qquad \& \quad \eta(r)d^*(x_n, Tx_n) \le d(x_n, p),$$

which deduces that

$$d(x_{n+1}, q) \le rd(x_n, p).$$

Since $x_n \to p$, by the above relation we must have $x_n \to q$. This implies that $p = q$ and so,

$$d(p, Tp) = \operatorname{dist}(A, B),$$

that is, $p$ is a best proximity point of $T$ and the proof completes.

□

The following result is an extension of *Suzuki's fixed point theorem* ([29]) in partially ordered metric spaces.

**Corollary 4.2.** *Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$ such that $(X, d)$ is complete and the condition* (1) *holds on $X$. Let $T : X \to X$ be a nondecreasing mapping for which*

$$\eta(r)d(x, Tx) \le d(x, y) \quad \textit{implies} \quad d(Tx, Ty) \le rd(x, y),$$

*for every $x, y \in X$ with $x \preceq y$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

Finally, we obtain the following fixed point theorem due to Nieto and Rodriguez-Lopez ([19]).

**Corollary 4.3.** *Let $(X, \preceq)$ be a partially ordered set and $d$ be a metric on $X$ such that $(X, d)$ is complete and the condition* (1) *holds on $X$. Let $T : X \to X$ be a nondecreasing mapping for which*

$$d(Tx, Ty) \le rd(x, y),$$

*for every $x, y \in X$ with $x \preceq y$ and for some $r \in (0, 1)$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

## References

[1] A. Abkar, M. Gabeleh, *Best proximity points for cyclic mappings in ordered metric spaces*, J. Optim. Theory Appl., **150** (2011) 188–193.
[2] A. Abkar, M. Gabeleh, *Generalized cyclic contractions in partially ordered metric spaces*, Optim. Lett. **6** (2012) 1819–1830.
[3] M.A. Al-Thagafi, N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Analysis, **70** (2009) 3665–3671.
[4] M. Derafshpour, Sh. Rezapour, N. Shahzad, *Best Proximity Points of cyclic $\varphi$-contractions in ordered metric spaces*, Topological Methods in Nonlinear Analysis, **37** (2011) 193–202.

[5] C. Di Bari, T. Suzuki, C. Vetro, *Best proximity points for cyclic Meir- Keeler contractions*, Nonlinear Analysis, **69** (2008) 3790–3794.
[6] N.V. Dung, V. Hang, *Remarks on cyclic contractions in b-metric spaces and applications to integral equations*, RACSAM, DOI 10.1007/s13398-016-0291-5.
[7] A. Anthony Eldred, P, Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl, **323** (2006) 1001–1006.
[8] R. Espnola, A. Fernndez-Len, *On best proximity points in metric and Banach spaces*, Canad. Math. Bull. **63** (2011) 533–550.
[9] M. Gabeleh, *Best proximity point theorems via proximal non-self mappings*, J. Optim. Theory, Appl., **164** (2015) 565–576.
[10] M. Gabeleh, N. Shahzad, *Best proximity points, cyclic Kannan maps and geodesic metric spaces*, J. Fixed Point Theory Appl. **18** (2016) 167–188.
[11] H. Huang, S. Radenovic, T.A. Lampert, *Remarks on common fixed point results for cyclic contractions in ordered bmetric spaces*, J. Computational Anal. Appl., **22** (2017) 538–545.
[12] Z. Kadelbur, S. Radenovic, *A note on some recent best proximity point results for non-self mappings*, Gulf Journal of Mathematics **1** (2013) 36–41.
[13] Z. Kadelbur, S. Radenovic, *A note on Pata-type cyclic contractions*, Sarajevo J. Math., **11** (2015) 1–11.
[14] Z. Kadelbur, S. Radenovic, J. Vujakovic, *A note on the paper "Fixed point theorems for cyclic weak contractions in compact metric spaces"*, Fixed Point Theory Appl. (2016) 2016:46.
[15] W.A. Kirk, P.S. Srinivasan, P. Veeramani, *Fixed points for mappings satisfying cyclic contractive conditions*, Fixed Point Theory, **4** (2003) 79–86.
[16] T.C. Lim *On characterization of Meir-Keeler contractive maps*, Nonlinear Analysis, **46** (2001) 113–120.
[17] A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969) 326–329.
[18] M. Mongkolkeha, P. Kumam, *Best proximity point theorems for generalized cyclic contractions in ordered metric spaces*, J. Optim. Theory Appl., **155** (2012) 215–226.
[19] J.J. Nieto, R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005) 223-229.
[20] V. Pragadeeswarar, M. Marudai, *Best proximity points: approximation and optimization in partially ordered metric spaces*, Optim. Lett., **7** (2013) 1883–1892.
[21] S. Radenovic, *A note on fixed point theory for cyclic weaker MeirKeeler function in complete metric spaces*, International J. Analysis Appl., **7** (2015) 16–21.
[22] S. Radenovic, *Some remarks on mappings satisfying cyclical contractive conditions*, Afr. Math. DOI. 10.1007/s13370-015-0327-6.
[23] S. Radenovic, *Classical fixed point results in 0-complete partial metric spaces via cyclic-type extension*, The Allahabad Math. Soc., (to appear).
[24] S. Radenovic, T. Doenovic, T.A. Lampert, Z. Golubovic, *A note on some recent fixed point results for cyclic contractions in b-metric spaces and an application to integral equations*, App. Math. Computation **273** (2016) 155–164.
[25] S. Sadiq Basha, *Discrete optimization in partially ordered sets*, J. Glob Optim, **54** (2012) 511–517.
[26] S. Sadiq Basha, N. Shahzad, R. Jeyaraj, *Best proximity points: approximation and optimization*, Optim. Lett. **7** (2013), 145–155.
[27] S. Sadiq Basha, N. Shahzad, R. Jeyaraj, *Best proximity point theorems for reckoning optimal approximate solutions*, Fixed Point Theory Appl. 2012, 2012:202.
[28] T. Suzuki. *Some notes on Meir-Keeler contractions and L-functions*, Bull. Kyushu Inst. Technol **53** (2006) 1–13.
[29] T. Suzuki, *A generalized Banach contraction principle which characterizes metric completeness*, Proc. Amer. Math. Soc., **136** (2008) 1861–1869.
[30] T. Suzuki, M. Kikkawa, C. Vetro, *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Analysis, **71** (2009) 2918–2926.

Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
*E-mail address*: gab.moo@gmail.com

School of Mathematical Sciences, North-West University (Mafikeng campus) Mmabatho 2735, South Africa
*E-mail address*: olivier.olelaotafudu@nwu.ac.za