

## SOME NEW PROPERTIES OF INNER PRODUCT QUASILINEAR SPACES

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**ABSTRACT.** In the present paper, we introduce over symmetric set on a inner product quasilinear spaces. We establish some new results related to this new concept. Further, we obtain new conclusions for orthogonal and orthonormal subspaces of inner product quasilinear spaces. These results generalize recent well known results in the linear inner product spaces. Also, some examples have been given which provide an important contribution to understand the structure of inner product quasilinear spaces.

### 1. INTRODUCTION

The theory of quasilinear analysis is one of the fundamental theories in nonlinear analysis which has various applications such as integral and differential equations, approximation theory and bifurcation theory. In [2], Aseev generalized the concept of linear spaces, using the partial order relation hence they have defined the quasilinear spaces. He also described the convergence of sequences and norm in quasilinear space. Further, he introduced the concept of  $\Omega$ -space which is only meaningful in normed quasilinear spaces.

Later, various authors introduce new results on quasilinear spaces ([1], [7], [8], [9], [5],[11] etc.). The recently, in [4], inner product quasilinear spaces have been investigated. Again in this paper, some new notions such as Hilbert quasilinear spaces, orthogonality, orthonormality, orthogonal complement in inner product quasilinear spaces etc. have been defined as a generalizations of inner product space in classical analysis. In recent time, some new algebraic properties of quasilinear spaces which is very significant to improvement of quasilinear algebra are given by [3] and [5].

In this paper, motivated by the work of Assev [2] and Markow [8], we introduce new concepts on inner product quasilinear spaces and prove some theorems related to these notions. Moreover, we investigate some algebraic properties of some subspace of inner product quasilinear spaces. Further, the aim of this paper is to

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extend the results in [4]. Our consequences generalize the some theorems which were given in linear inner product spaces.

## 2. QUASILINEAR SPACES AND HILBERT QUASILINEAR SPACES

Let us start this section by introducing the definition of a quasilinear spaces and some its basic properties given by Aseev [2].

**Definition 1.** *A set  $X$  is called a quasilinear space (QLS, for short), if a partial order relation “ $\leq$ ”, an algebraic sum operation, and an operation of multiplication by real numbers are defined in it in such way that the following conditions hold for any elements  $x, y, z, v \in X$  and any real numbers  $\alpha, \beta \in \mathbb{R}$ :*

- (1)  $x \leq x$ ;
- (2)  $x \leq z$  if  $x \leq y$  and  $y \leq z$ ,
- (3)  $x = y$  if  $x \leq y$  and  $y \leq x$ ,
- (4)  $x + y = y + x$ ,
- (5)  $x + (y + z) = (x + y) + z$ ,
- (6) there exists an element  $\theta \in X$  such that  $x + \theta = x$ ,
- (7)  $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$ ,
- (8)  $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ ,
- (9)  $1 \cdot x = x$ ,
- (10)  $0 \cdot x = \theta$ ,
- (11)  $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$ ,
- (12)  $x + z \leq y + v$  if  $x \leq y$  and  $z \leq v$ ,
- (13)  $\alpha \cdot x \leq \alpha \cdot y$  if  $x \leq y$ .

A linear space is a quasilinear space with the partial order relation “ $=$ ”. The most popular example which is not a linear space is the set of all closed intervals of real numbers with the inclusion relation “ $\subseteq$ ”, algebraic sum operation

$$A + B = \{a + b : a \in A, b \in B\}$$

and the real-scalar multiplication

$$\lambda A = \{\lambda a : a \in A\}.$$

We denote this set by  $\Omega_C(\mathbb{R})$ . Another one is  $\Omega(\mathbb{R})$ , the set of all compact subsets of real numbers. By a slight modification of algebraic sum operation (with closure) such as

$$A + B = \overline{\{a + b : a \in A, b \in B\}}$$

and by the same real-scalar multiplication defined above and by the inclusion relation we get the nonlinear QLS,  $\Omega_C(E)$  and  $\Omega(E)$ , the space of all nonempty closed bounded and convex closed bounded subsets of some normed linear space  $E$ , respectively.

**Lemma 1.** *Suppose that any element  $x$  in a QLS  $X$  has an inverse element  $x' \in X$ . Then the partial order in  $X$  is determined by equality, the distributivity conditions hold, and consequently,  $X$  is a linear space [2].*

Suppose that  $X$  is a QLS and  $Y \subseteq X$ . Then  $Y$  is called a *subspace* of  $X$  whenever  $Y$  is a QLS with the same partial order and the restriction to  $Y$  of the operations on  $X$ . One can easily prove the following theorem using the condition of to be a QLS. It is quite similar to its linear space analogue.

**Theorem 2.**  *$Y$  is a subspace of a QLS  $X$  if and only if, for every,  $x, y \in Y$  and  $\alpha, \beta \in \mathbb{R}$ ,  $\alpha x + \beta y \in Y$ .*

Let  $X$  be a QLS. An  $x \in X$  is said to be symmetric if  $(-1) \cdot x = -x = x$ , and  $X_d$  denotes the set of all such elements.  $\theta$  denotes the zero's, additive unit of  $X$  and it is minimal, i.e.,  $x = \theta$  if  $x \leq \theta$ . An element  $x'$  is called *inverse* of  $x$  if  $x + x' = \theta$ . The inverse is unique whenever it exists and  $x' = -x$  in this case. Sometimes  $x'$  may not be exist but  $-x$  is always meaningful in QLSs. An element  $x$  possessing an inverse is called *regular*, otherwise is called *singular*. For a singular element  $x$  we should note that  $x - x \neq 0$ . Now,  $X_r$  and  $X_s$  stand for the sets of all regular and singular elements in  $X$ , respectively. Further,  $X_r$ ,  $X_d$  and  $X_s \cup \{0\}$  are subspaces of  $X$  and they are called regular, symmetric and singular subspaces of  $X$ , respectively [11].

**Proposition 3.** *In a quasilinear space  $X$  every regular element is minimal [11].*

**Definition 2.** *Let  $X$  be a QLS. A real function  $\|\cdot\|_X : X \rightarrow \mathbb{R}$  is called a norm if the following conditions hold [2]:*

- (14)  $\|x\|_X > 0$  if  $x \neq 0$ ,
- (15)  $\|x + y\|_X \leq \|x\|_X + \|y\|_X$ ,
- (16)  $\|\alpha \cdot x\|_X = |\alpha| \cdot \|x\|_X$ ,
- (17) if  $x \leq y$ , then  $\|x\|_X \leq \|y\|_X$ ,
- (18) if for any  $\varepsilon > 0$  there exists an element  $x_\varepsilon \in X$  such that,  $x \leq y + x_\varepsilon$  and  $\|x_\varepsilon\|_X \leq \varepsilon$  then  $x \leq y$ .

A quasilinear space  $X$  with a norm defined on it is called normed quasilinear space (NQLS, for short). It follows from Lemma 1 that if any  $x \in X$  has an inverse element  $x' \in X$ , then the concept of a NQLS coincides with the concept of a real normed linear space.

Let  $X$  be a NQLS. Hausdorff or norm metric on  $X$  is defined by the equality

$$h_X(x, y) = \inf \{r \geq 0 : x \leq y + a_1^r, y \leq x + a_2^r, \|a_i^r\| \leq r\}.$$

Since  $x \leq y + (x - y)$  and  $y \leq x + (y - x)$ , the quantity  $h_X(x, y)$  is well-defined for any elements  $x, y \in X$ , and

$$h_X(x, y) \leq \|x - y\|_X. \quad (2.1)$$

It is not hard to see that this function  $h_X(x, y)$  satisfies all of the metric axioms.

**Lemma 4.** *The operations of algebraic sum and multiplication by real numbers are continuous with respect to the Hausdorff metric. The norm is continuous function respect to the Hausdorff metric [2].*

**Example 1.** *Let  $E$  be a Banach space. A norm on  $\Omega(E)$  is defined by*

$$\|A\|_{\Omega(E)} = \sup_{a \in A} \|a\|_E.$$

*Then  $\Omega(E)$  and  $\Omega_C(E)$  are normed quasilinear spaces. In this case the Hausdorff metric is defined as usual:*

$$h_{\Omega_C(E)}(A, B) = \inf \{r \geq 0 : A \subset B + S_r(\theta), B \subset A + S_r(\theta)\},$$

*where  $S_r(\theta)$  denotes a closed ball of radius  $r$  about  $\theta \in X$  [2].*

**Definition 3.** Let  $X$  be a quasilinear space. A mapping  $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{R}$  is called an inner product on  $X$  if for any  $x, y, z \in X$  and  $\alpha \in \mathbb{R}$  the following conditions are satisfied [4]:

- (19)  $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$
- (20)  $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
- (21)  $\langle x, y \rangle = \langle y, x \rangle$
- (22)  $\langle x, x \rangle \geq 0$  and  $\langle x, x \rangle = 0 \Leftrightarrow x = 0$
- (23) if  $x \leq y$  and  $u \leq v$  then  $\langle x, u \rangle \leq \langle y, v \rangle$
- (24) if for any  $\varepsilon > 0$  there exists an element  $x_\varepsilon \in X$  such that  $x \leq y + x_\varepsilon$  and  $\langle x_\varepsilon, x_\varepsilon \rangle \leq \varepsilon$  then  $x \leq y$ .

A quasilinear space with an inner product is called a inner product quasilinear space, briefly, IPQLS.

**Example 2.**  $\Omega_C(\mathbb{R})$  is a IPQLS with inner product defined by

$$\langle A, B \rangle = \sup\{ab : a \in A, b \in B\}.$$

Every IPQLS  $X$  is a NQLS with the norm defined by

$$\|x\| = \sqrt{\langle x, x \rangle}$$

for every  $x \in X$  [4].

**Proposition 5.** If in a IPQLS  $x_n \rightarrow x$  and  $y_n \rightarrow y$ , then  $\langle x_n, y_n \rangle \rightarrow \langle x, y \rangle$  [4].

A IPQLS is called *Hilbert QLS*, if it is complete according to the Hausdorff metric.

**Example 3.** Let  $E$  be an inner product space. Then we know that  $\Omega(E)$  is a IPQLS and it is complete with respect to Hausdorff metric. So,  $\Omega(E)$  is a Hilbert QLS [4].

**Definition 4.** (Orthogonality) An element  $x$  of a IPQLS  $X$  is said to be orthogonal to an element  $y \in X$  if

$$\langle x, y \rangle = 0.$$

We also say that  $x$  and  $y$  are orthogonal and we write  $x \perp y$ . Similarly, for subsets  $\alpha, \beta \subseteq X$  we write  $x \perp \alpha$  if  $x \perp z$  for all  $z \in \alpha$  and  $\alpha \perp \beta$  if  $a \perp b$  for all  $a \in \alpha$  and  $b \in \beta$  [4].

An orthonormal set  $M \subset X$  is an orthogonal set in  $X$  whose elements have norm 1, that is, for all  $x, y \in M$

$$\langle x, y \rangle = \begin{cases} 0, & x \neq y \\ 1, & x = y \end{cases}$$

**Definition 5.** Let  $A$  be a nonempty subset of an quasilinear inner product space  $X$ . An element  $x \in X$  is said to be orthogonal to  $A$ , denoted by  $x \perp A$ , if  $\langle x, y \rangle = 0$  for every  $y \in A$ .

**Theorem 6.** For any subset  $A$  of an IPQLS  $X$ ,  $A^\perp$  is a closed subspace of  $X$  [4].

**Definition 6.** Let  $(X, \leq)$  be a QLS,  $\{x_k\}_{k=1}^n \subset X$  and  $\{\lambda_k\}_{k=1}^n \subset \mathbb{R}$ . If

$$\theta \leq \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n$$

implies  $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ , then  $\{x_k\}_{k=1}^n$  is said to be quasilinear independent (ql-independent) otherwise  $\{x_k\}_{k=1}^n$  is said to be quasilinear dependent (ql-dependent) [3].

**Definition 7.** Let  $X$  be a QLS and  $y \in X$ . The set of all regular elements preceding from  $y$  is called floor of  $y$ , and  $F_y$  denotes the set of all such elements. Therefore,

$$F_y = \{z \in X_r : x \leq y\}.$$

The floor of any subset  $M$  of  $X$  is the union of floors of all elements in  $M$  and is denoted by  $F_M$  [5]. Note again that  $X_r$  is the regular subspace of  $X$ .

### 3. MAIN RESULTS

In this part, we define the over symmetric subset of any QLS which are new concept. Then we give new results related to new notion. Also, in this section, we give some new theorems with important treatment about orthogonality and orthonormality on IPQLS.

**Definition 8.** Let  $X$  be a QLS.  $y$  is called over symmetric element of  $X$  whenever there exist a  $x \in X_d$  such that  $x \leq y$ . All over symmetric elements set of  $X$  denoted by  $X_{od}$ .

**Proposition 7.** If  $X$  is an IPQLS,  $X_{od}$  is a subspace of  $X$ .

*Proof.* From Definition 8, we may find  $a, b \in X_d$  such that  $a \leq x$  and  $b \leq y$  for every  $x, y \in X_{od}$ . By (12) and (13), we get

$$\alpha a + \beta b \leq \alpha x + \beta y$$

for every  $\alpha, \beta \in \mathbb{R}$ . By Theorem 2, we obtain  $\alpha a + \beta b \in X_d$ . This shows that  $\alpha x + \beta y \in X_{od}$ . ■

**Proposition 8.** Set of all over symmetric elements of a Hilbert QLS  $X$  is closed.

*Proof.* Suppose that  $(x_n)$  is a sequence in  $X_{od}$  and  $(x_n) \rightarrow x \in X$  for  $n \rightarrow \infty$ . Then for every  $\epsilon > 0$  there exist a  $n_0 \in \mathbb{N}$  such that the condition

$$x_n \leq x + a_{1,n}^\epsilon, \quad x \leq x_n + a_{2,n}^\epsilon, \quad \|a_{i,n}^\epsilon\| \leq \epsilon$$

hold for every  $n > n_0$ . Otherwise, since  $(x_n)$  is a sequence in  $X_{od}$ , we find  $b_n \in X_d$  such that  $b_n \leq x_n$  for every  $n \in \mathbb{N}$ . From here, we have

$$b_n \leq x + a_{1,n}^\epsilon, \quad \|a_{1,n}^\epsilon\| \leq \epsilon$$

for every  $n \in \mathbb{N}$ . Since  $X$  is a Hilbert QLS, we find  $\epsilon^2 > 0$  such that  $\langle a_{1,n}^\epsilon, a_{1,n}^\epsilon \rangle \leq \epsilon^2 = \epsilon^2$ . So, we have

$$b_n \leq x + a_{1,n}^\epsilon, \quad \langle a_{1,n}^\epsilon, a_{1,n}^\epsilon \rangle \leq \epsilon^2.$$

From (24), we obtain  $b_n \leq x$  for every  $n \in \mathbb{N}$ . Then, we have  $x \in X_{od}$ , this proves the theorem. ■

**Theorem 9.**  $(\Omega_C(\mathbb{R}))_{od}$  is a  $ql$ -dependent subset of  $\Omega_C(\mathbb{R})$ .

*Proof.* Suppose

$$0 \leq \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n, \tag{3.1}$$

for some  $x_1, x_2, \dots, x_n \in (\Omega_C(\mathbb{R}))_{od}$  and  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ . By Definition 8, we have  $b_n \in X_d$  such that  $b_n \leq x_n$  for every  $n \in \mathbb{N}$ . From (12) and (13), we find

$$\lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n \leq \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n.$$

Since all elements of  $(\Omega_C(\mathbb{R}))_d$  contain 0, we get

$$0 \leq b_n$$

for all  $n$  in  $\mathbb{N}$ . From (12), for every  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$  scalars,  $0 \leq \lambda_1 b_1 + \lambda_2 b_2 + \dots + \lambda_n b_n$  inequality is satisfy. This implies that (3.1) is satisfy for every  $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ . ■

**Theorem 10.** *Let  $X$  be a IPQLS.  $X_d$  is a closed subspace of  $X$ .*

*Proof.* Let  $(x_n) \in X_d$  and  $x_n \rightarrow x \in X$  for  $n \rightarrow \infty$ . By Lemma 4, we say

$$x_n \rightarrow x \Rightarrow -x_n \rightarrow -x.$$

Since  $(x_n) \in X_d$ ,  $x_n = -x_n$  for every  $n \in \mathbb{N}$ . From here, we obtain

$$x_n \rightarrow -x$$

for all  $n \in \mathbb{N}$ . This is true if and only if  $x = -x$ . This proves that  $x \in X_d$ . We can easily show that  $X_d$  is a subspace of  $X$ . ■

**Example 4.** *Consider the set  $A = \{-1, 1\}$  in  $(\Omega(\mathbb{R}))_d$ . It is obvious that  $\{0\} \subseteq \alpha \cdot \{-1, 1\}$  if and only if  $\alpha = 0$  where  $\{0\}$  is the zeros of the QLS  $(\Omega(\mathbb{R}))_d$ . Therefore, the singleton  $A = \{-1, 1\}$  is ql-independent set in  $(\Omega(\mathbb{R}))_d$ . However, the singleton  $B = \{-1, 1, 0\}$  is ql-independent since  $\{0\} \subseteq \beta \cdot \{-1, 1, 0\}$  for  $\beta = 1 \neq 0$ .*

**Theorem 11.** *Any orthonormal subset of an IPQLS can not contain symmetric elements.*

*Proof.* Assume that  $x$  be a symmetric element of an orthonormal subset of inner product quasilinear space  $X$ . Since  $\|x\|^2 = 1$ , we get  $\langle x, x \rangle = 1$ . Otherwise,  $x = -x$  from  $x$  is a symmetric element. From here, we have

$$\langle x, x \rangle = \langle x, -x \rangle = 1.$$

Whereas, by (20), we find

$$\langle x, -x \rangle = -1 \cdot \langle x, x \rangle = -1.$$

This is a contradiction since both  $\langle x, x \rangle = 1$  and  $\langle x, x \rangle = -1$ . ■

**Theorem 12.** *Any orthonormal subset of an IPQLS can not contain over symmetric elements.*

*Proof.* Let  $x$  be an over symmetric element of orthonormal subset of inner product quasilinear space  $X$ . Since  $x$  is an over symmetric element, we find a  $y \in X_d$  such that  $y \leq x$ . Moreover, by (23),

$$\langle y, y \rangle \leq \langle x, x \rangle = 1.$$

This gives  $\|y\|^2 \leq 1$ . On the other hand, since  $y \in X_d$ , we obtain

$$\langle y, y \rangle = -\langle y, y \rangle \leq 1$$

which means that  $-\|y\|^2 \leq 1$ . Therefore,  $y$  is not a symmetric element of  $X$ . This complete the proof. ■

**Remark 1.** *We know that only zero element is a symmetric element in a classical linear space. But, in a QLS has symmetric element other than 0. Further, set of symmetric elements of a QLS may not include 0.*

**Example 5.** *Let  $X = \{-1, 1\} \in \Omega(\mathbb{R})$ . Since  $-X = \{-1, 1\}$ ,  $X$  is a symmetric element of  $\Omega(\mathbb{R})$ . But  $X$  does not contain  $\{0\}$ .*

**Example 6.** We consider  $\Omega_C(\mathbb{R})_{od}$  subspace of  $\Omega_C(\mathbb{R})$ . By Definition 7, in this subspace, since floors of all elements will be empty set, we get

$$\begin{aligned} F_{\Omega_C(\mathbb{R})_{od}} &= \bigcup_{y \in \Omega_C(\mathbb{R})_{od}} F_y \\ &= \emptyset \end{aligned}$$

For example, let  $[-1, 2] \in \Omega_C(\mathbb{R})_{od}$ . Also, since  $(\Omega_C(\mathbb{R})_{od})_r = \emptyset$ ,  $F_{[-1, 2]} = \emptyset$ . Similar to the  $\Omega_C(\mathbb{R})_{od}$ , floor of  $\Omega_C(\mathbb{R})_d$  is empty. We can easily show that floors of  $\Omega(\mathbb{R})_{od}$  and  $\Omega(\mathbb{R})_d$  are empty similar to the  $\Omega_C(\mathbb{R})_{od}$  and  $\Omega_C(\mathbb{R})_d$ .

**Theorem 13.** Let  $X$  be an IPQLS and  $(x_n)$  is any sequence in  $X$ . For every  $x, y \in X$  we have

$$y \perp x_n \text{ and } x_n \rightarrow x \text{ implies } y \perp x.$$

*Proof.* Let  $y \perp x_n$  and  $x_n \rightarrow x$  in IPQLS  $X$ . Then for all  $n \in \mathbb{N}$  we have

$$\langle y, x_n \rangle = 0$$

and for every  $\epsilon > 0$  there exists a  $n_0 \in \mathbb{N}$  such that the conditions

$$x_n \leq x + a_{1,n}^\epsilon, \quad x \leq x_n + a_{2,n}^\epsilon, \quad \|a_{i,n}^\epsilon\| \leq \epsilon$$

hold for every  $n > n_0$ . Since  $X$  is an IPQLS, by (24), we get

$$x_n \leq x \text{ and } x \leq x_n$$

for all  $n \in \mathbb{N}$ . Clearly,  $y \leq y$  is satisfy for every  $y \in X$  in an IPQLS  $X$ . Therefore, by (23), we have

$$\langle x_n, y \rangle \leq \langle x, y \rangle \text{ and } \langle x, y \rangle \leq \langle x_n, y \rangle.$$

This gives  $\langle x, y \rangle = 0$  since  $\langle x_n, y \rangle = 0$  for all  $n \in \mathbb{N}$ . ■

**Theorem 14.** Let  $X$  be an IPQLS and  $(x_n)$  is any sequence in  $X$ . For every  $x \in X$  we have

$$\|x_n\| \rightarrow \|x\| \text{ and } \langle x_n, x \rangle \rightarrow \langle x, x \rangle \text{ implies } x_n \rightarrow x.$$

*Proof.* Suppose that  $\|x_n\| \rightarrow \|x\|$  and  $\langle x_n, x \rangle \rightarrow \langle x, x \rangle$ . By (2.1), we get

$$\begin{aligned} h^2(x_n, x) &\leq \|x_n - x\|^2 \\ &= \langle x_n - x, x_n - x \rangle \\ &= \|x_n\|^2 - 2\langle x_n, x \rangle + \|x\|^2. \end{aligned}$$

By letting  $n \rightarrow \infty$ , we obtain  $h(x_n, x) \leq 0$ . This proves that the  $x_n$  is a convergent to  $x$  in an IPQLS  $X$ . ■

Although,  $x + y \perp x - y$  when  $\|x\| = \|y\|$  in a classical inner product space,  $x + y \perp x - y$  may not be true when  $\|x\| = \|y\|$  in an IPQLS.

**Example 7.** We know from [2],  $\Omega_C(\mathbb{R})$  is an NQLS with  $\|A\| = \sup_{a \in A} |a|$ . Let  $A = [-1, 1], B = [0, 1] \in \Omega_C(\mathbb{R})$ . We thus have  $\|A\| = \sup_{a \in [-1, 1]} |a| = 1$  and  $\|B\| = \sup_{a \in [0, 1]} |b| = 1$ . But, since

$$A + B = [-1, 2] \text{ and } A - B = [-2, 1]$$

we have

$$\begin{aligned} \langle A + B, A - B \rangle &= \sup \{a.b : a \in [-1, 2], b \in [-2, 1]\} \\ &= 2. \end{aligned}$$

**Theorem 15.** *If  $X$  be an IPQLS, then*

$$x \perp y \iff \|x - \alpha y\| = \|x + \alpha y\|$$

for every  $x, y \in X$ .

*Proof.* The proof of theorem is similar to classical linear counterpart. ■

Similar to the linear space, in an IPQLS  $X$ ,

**Theorem 16.** *Let  $X$  and  $Y$  be IPQLSs and  $T : X \rightarrow Y$  be a quasilinear operator. For every  $x, x_1, x_2 \in X$ ,*

$$\|x\| \leq \|Tx\|$$

if and only if

$$\langle x_1, x_2 \rangle \leq \frac{1}{2} (\langle T(x_1), T(x_1) \rangle + \langle T(x_2), T(x_2) \rangle).$$

*Proof.* Let  $\|x\| \leq \|Tx\|$  for every  $x \in X$ . Suppose there exists  $y_1, y_2 \in Y$  such that  $T(x_1) = y_1$  and  $T(x_2) = y_2$  for every  $x_1, x_2 \in X$ . Since  $T$  is a quasilinear operator, we get

$$T(x_1 + x_2) \leq T(x_1) + T(x_2) \text{ and } T(x_1 - x_2) \leq T(x_1) - T(x_2).$$

By (17), we have

$$\|T(x_1 + x_2)\| \leq \|y_1 + y_2\| \text{ and } \|T(x_1 - x_2)\| \leq \|y_1 - y_2\|.$$

Moreover, by the parallelogram law

$$\begin{aligned} 4 \langle x_1, x_2 \rangle &\leq \|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 \\ &\leq \|T(x_1 + x_2)\|^2 + \|T(x_1 - x_2)\|^2 \\ &\leq \|y_1 + y_2\|^2 + \|y_1 - y_2\|^2 \\ &= 2(\|y_1\|^2 + \|y_2\|^2). \end{aligned}$$

This gives  $\langle x_1, x_2 \rangle \leq \frac{1}{2} (\langle T(x_1), T(x_1) \rangle + \langle T(x_2), T(x_2) \rangle)$ .

On the other hand,  $\langle x_1, x_2 \rangle \leq \frac{1}{2} (\langle T(x_1), T(x_1) \rangle + \langle T(x_2), T(x_2) \rangle)$  inequality is satisfy for every  $x_1, x_2 \in X$ . If  $x = x_1 = x_2$ , we have

$$\begin{aligned} \langle x, x \rangle &= \langle x_1, x_2 \rangle \\ &\leq \frac{1}{2} (\langle T(x_1), T(x_1) \rangle + \langle T(x_2), T(x_2) \rangle) \\ &= \langle T(x), T(x) \rangle. \end{aligned}$$

So, we obtain  $\|x\| \leq \|Tx\|$ . ■

**Remark 2.** *If  $X$  and  $Y$  is a linear space in the above Theorem, then, since  $T$  is a linear operator, we have*

$$\|x\| \leq \|Tx\| \iff \langle x_1, x_2 \rangle = \langle T(x_1), T(x_2) \rangle$$

for every  $x, x_1, x_2 \in X$ .

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