Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 7 Issue 2(2015), Pages 28-36.

APPROXIMATION OF FUNCTION $f(x,y)$ BELONGING TO GENERALIZED LIPSCHITZ CLASS BY (N,p_m,q_n) METHOD OF THE DOUBLE FOURIER SERIES

(COMMUNICATED BY HUSEYIN BOR)

SHYAM LAL AND ABHISHEK MISHRA

Abstract. In this paper, a new estimate for the degree of approximation of a function $f(x,y)$ belonging to generalized Lipschitz class $Lip(\xi_1,\xi_2)$ by (N,p_m,q_n) summability method of its double Fourier series has been determined.

1. Introduction

Alexits [1] proved:

Theorem A *If periodic function $f \in Lip\alpha,\ 0<\alpha\leq 1$, then its degree of approximation by (C,δ)-means $\sigma_n^\delta(f)$ of its Fourier series is*

$$\left\|f-\sigma_n^\delta(f)\right\|_C = O\left(\frac{1}{n^\alpha}\right) \quad for \ \ 0<\alpha<\delta\leq 1$$

and

$$\left\|f-\sigma_n^\delta(f)\right\|_C = O\left(\frac{ln\ n}{n^\alpha}\right) \quad for \ \ 0<\alpha\leq\delta\leq 1.$$

Generalizing the result of Alexits [1], the degree of approximation by Voronoi-Nörlund means of a single variable function $f \in Lip\alpha$ is estimated by Siddiqi [10], Chandra [3] and Qureshi [8]. The similar problems, in particular, for one variable functions, are considered by Dzjadyk [2], Korneichuk [7], Stepantes [14],[13] and others. Working in slight different direction for function of two and several variables, Stepantes [12], P.V. Zaderei, N. M. Zaderei and others (see also [2], [14]) discussed the degree of approximation for the class of functions that are defined by using the Lipschitz condition and indicated the procedure to get estimates for two variables functions from corresponding estimates for one variable functions. As regards the first arithmetic means of Fourier series in the one-dimensional case, Yoshimitsu [6] generalized Theorem A in the two-dimensional case as follows:

1991 *Mathematics Subject Classification.* 42B05, 42B08.

Key words and phrases. Degree of approximation, Double Fourier series, $(C,1,1)$ means, (N,p_m,q_n) summability means, Functions of class $Lip(\xi_1,\xi_2)$.

Submitted December 26, 2014. Published April 20, 2015.

Theorem B *If a continuous function $f(x,y)$ of period 2π with respect to each x and y belongs to $Lip(\alpha,\beta)$, where $0<\alpha<1$ and $0<\beta<1$, then*

$$|\sigma_{m,n}(x,y)-f(x,y)|=O\left(m^{-\alpha}+n^{-\beta}\right)$$

uniformly in (x,y) as m and n tend to infinity independently of each other. If $\alpha=\beta=1$, then

$$|\sigma_{m,n}(x,y)-f(x,y)|=O(m^{-1}\log m+n^{-1}\log n)$$

uniformly in (x,y) as m and n tend to infinity independently of each other.

Motivated by such results, in this paper, a new estimate for the degree of approximation a function $f(x,y)$ belonging to the class $Lip(\xi_1,\xi_2)$ by (N,p_m,q_n) method of the double Fourier series has been determined.

2. Definitions and Notations

Let $f(x,y)$ be a Lebesgue integrable function of period 2π with respect to each variable x and y and summable in the fundamental square $Q:(-\pi,\pi)\times(-\pi,\pi)$. The double Fourier series of $f(x,y)$ is given by

$$\begin{aligned} f(x,y)\sim\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\lambda_{m,n}\,[a_{m,n}\cos mx\cos ny+b_{m,n}\sin mx\cos ny \\ +c_{m,n}\cos mx\sin ny+d_{m,n}\sin mx\cos ny]\end{aligned} \tag{2.1}$$

with $(m,n)^{th}$ partial sums $s_{m,n}(f;(x,y))$ where

$$\begin{aligned}\lambda_{m,n} &= \begin{cases}1/4, & for\ m=n=0\\ 1/2, & for\ m>0,\ n=0\ and\ m=0,\ n>0\\ 1\ \ , & for\ m>0,\ n>0.\end{cases}\\ a_{m,n} &= \pi^{-2}\iint_Q f(x,y)\cos mx\cos ny dxdy\end{aligned}$$

and similar expressions for $b_{m,n}, c_{m,n}$ and $d_{m,n}$.

Let p_m and q_n be two sequence of real constants and let

$$P_m=p_0+p_1+p_2+...+p_m=\sum_{j=0}^{m}p_j\neq 0,\quad P_{-1}=p_{-1}=0,\quad p_0>0$$

and

$$Q_n=q_0+q_1+q_2+...+q_n=\sum_{k=0}^{n}q_k\neq 0,\quad Q_{-1}=q_{-1}=0,\quad q_0>0.$$

The double series $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}u_{m,n}$ with the sequence of the partial sum

$$s_{m,n}=\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}u_{\mu,\nu}$$

is said to be summable by double Nörlund method or summable (N,p_m,q_n) if $t_{m,n}^{(N)}$ tends to a limit s as $m\to\infty,\ n\to\infty$ where the (N,p_m,q_n) mean $t_{m,n}^{(N)}$ is defined by

$$\begin{aligned} t_{m,n}^{(N)} &= \frac{1}{P_m Q_n}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} p_{m-\mu}q_{n-\nu}s_{\mu,\nu} \\ &= \frac{1}{P_m Q_n}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} p_{\mu}q_{\nu}s_{m-\mu,n-\nu}. \end{aligned}$$

The necessary and sufficient condition for the regularity of (N, p_m, q_n) method of summability are

$$p_m q_n = O(P_m Q_n)$$

and

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} p_\mu q_\nu = O\left(|P_m|\,|Q_n|\right),\ as\ (m,n)\to\infty$$

There are two important particular cases of (N, p_m, q_n) summability method:

(1) $(C, 1, 1)$ summability [4] if $p_m = 1\ \forall m$ and $q_n = 1\ \forall n$.
(2) (C, δ_1, δ_2) summability if $p_m = \binom{m+\delta_1-1}{\delta_1-1}$, $\delta_1 > 0$ and $q_n = \binom{n+\delta_2-1}{\delta_2-1}$, $\delta_2 > 0$.

A function $f : R^2 \to R$ of two variables x and y is said to belong to $Lip(\alpha,\beta)$ class [6] if

$$f(x+u, y+v) - f(x,y) = O(|u|^{\alpha} + |v|^{\beta}),\ 0<\alpha\le 1,\ 0<\beta\le 1.$$

$f(x,y) \in Lip(\xi_1, \xi_2)$ class if

$$f(x+u, y+v) - f(x,y) = O\left(\xi_1(u) + \xi_2(v)\right).$$

If $\xi_1(u) = u^{\alpha}$ and $\xi_2(v) = v^{\beta}$, $0<\alpha\le 1,\ 0<\beta\le 1$ then $Lip(\xi_1, \xi_2)$ class coincides to $Lip(\alpha,\beta)$ class.
If capital order 'O' is replaced by little 'o' in the above definition then $f(x,y)$ is said to belong to $lip(\alpha,\beta)$ and $lip(\xi_1,\xi_2)$.

The degree of approximation of a function $f : R^2 \to R$ by a trigonometric polynomial

$$\begin{aligned} t_{m,n}(x,y) = \sum_{j=0}^{m}\sum_{k=0}^{n} \lambda_{m,n}\,&[a_{j,k}\cos mx\cos ny + b_{j,k}\sin mx\cos ny \\ &+ c_{j,k}\cos mx\sin ny + d_{j,k}\sin mx\cos ny] \end{aligned}$$

of order $(m+n)$ is defined by

$$\|t_{m,n} - f\|_{\infty} = sup\left\{|t_{m,n}(x,y) - f(x,y)| : x,y\in R\right\}.$$

We write,

$$\begin{aligned} \Delta p_k &= p_k - p_{k+1},\ k\ge 0. \\ \phi(u,v) &= (1/4)[f(x+u,y+v) + f(x+u,y-v) \\ &\quad + f(x-u,y+v) + f(x-u,y-v) - 4f(x,y)] \\ M_m^{(N)}(u) &= \frac{1}{2\pi P_m}\sum_{j=0}^{m} p_m \frac{\sin\left(m-j+\frac{1}{2}\right)u}{\sin\frac{u}{2}}, \\ K_n^{(N)}(v) &= \frac{1}{2\pi Q_n}\sum_{k=0}^{n} q_n \frac{\sin\left(n-k+\frac{1}{2}\right)v}{\sin\frac{v}{2}}. \end{aligned}$$

3. Statement of the Result

The purpose of present paper is to extend Theorem B to establish a new estimate for the degree of approximation of a function $f(x,y) \in Lip(\xi_1, \xi_2)$ class by (N, p_m, q_n) summability method of its double Fourier series in the following form:
Theorem Let (N, p_n, q_n) be a regular double Nörlund method defined by positive sequences $\{p_n\}$ and $\{q_n\}$ such that

$$\sum_{j=0}^{m-1} |\Delta p_j| = O\left(\frac{P_m}{m+1}\right), \ (m+1)p_m = O(P_m). \tag{3.1}$$

$$\sum_{k=0}^{n-1} |\Delta q_k| = O\left(\frac{Q_n}{n+1}\right), \ (n+1)q_n = O(Q_n). \tag{3.2}$$

If $f(x,y)$ is a 2π periodic function with respect to each variable x and y, Lebesgue integrable in $(-\pi,\pi) \times (-\pi,\pi)$ and belongs to $Lip(\xi_1, \xi_2)$ class then the degree of approximation of $f(x,y)$ by (N, p_m, q_n) summability means

$$t_{m,n}^{(N)} = \frac{1}{P_m Q_n} \sum_{j=0}^{m} \sum_{k=0}^{n} p_j q_k s_{m-j,n-k}$$

of double Fourier series (2.1) satisfies

$$\left\| t_{m,n}^{(N)} - f \right\|_\infty = O\left(\xi_1\left(\tfrac{1}{m+1}\right)\log(m+1)\pi + \xi_2\left(\tfrac{1}{n+1}\right)\log(n+1)\pi\right)$$
$$\text{for } m = 0,1,2,... \ ; \ n = 0,1,2,...$$

provided that ξ_1 and ξ_2 are positive monotonic non-decreasing functions such that

$$\left(\frac{\xi_1(u)}{u}\right), \ \left(\frac{\xi_2(v)}{v}\right) \text{ are monotonic decreasing} \tag{3.3}$$

and $\xi_1\left(\frac{1}{m+1}\right)\log(m+1)\pi \to 0$ as $m \to \infty$, $\xi_2\left(\frac{1}{n+1}\right)\log(n+1)\pi \to 0$ as $n \to \infty$.

4. Lemmas

For the proof of our theorem, following lemmas are required:

Lemma 4.1. *The estimates*

$$\begin{aligned} M_m^{(N)}(u) &= O(m+1) \text{ for } 0 < u \le (m+1)^{-1} \\ \text{and } K_n^{(N)}(v) &= O(n+1), \quad \text{for } 0 < v \le (n+1)^{-1} \end{aligned}$$

hold true.

Proof. For $0 < u \le 1/(m+1)$, $\sin mu \le mu$ and Jordan's lemma, we have

$$\begin{aligned} \left| M_m^{(N)}(u) \right| &= \frac{1}{2\pi P_n} \sum_{j=0}^{m} p_m \frac{\sin\left(m-j+\frac{1}{2}\right)u}{\sin\frac{u}{2}} \\ &\le \frac{1}{2\pi P_n} \sum_{j=0}^{m} p_m \frac{\left(m-j+\frac{1}{2}\right)u}{u/\pi} \\ &\le \frac{(m+1)}{2P_n} \sum_{j=0}^{m} p_m \\ &= O(m+1). \end{aligned}$$

Similarly, $K_n^{(N)}(v) = O(n+1)$, for $0 < v \leq (n+1)^{-1}$.

Lemma 4.2. *The estimates*

$$\begin{aligned} M_m^{(N)}(u) &= O\left(\frac{1}{(m+1)u^2}\right), \textit{for } (m+1)^{-1} < u \leq \pi \\ \textit{and}\ \ K_n^{(N)}(v) &= O\left(\frac{1}{(n+1)v^2}\right)\ \textit{for } (n+1)^{-1} < v \leq \pi \end{aligned}$$

hold true.

Proof. For $(m+1)^{-1} < u \leq \pi$, using Jordan's lemma and Abel's lemma, we get

$$\begin{aligned} \left|M_m^{(N)}(u)\right| &= \left|\frac{1}{2\pi P_m}\sum_{j=0}^{m} p_j \frac{\sin(m-j+\frac{1}{2})u}{\sin\frac{u}{2}}\right| \\ &\leq \frac{1}{2\pi P_m}\left[\left|\sum_{j=0}^{m-1}(p_j - p_{j-1})\sum_{\nu=0}^{j}\frac{\sin\left(m-\nu+\frac{1}{2}\right)u}{\sin\frac{u}{2}}\right|\right. \\ &\qquad \left. + |p_m|\left|\sum_{j=0}^{m}\frac{\sin\left(m-j+\frac{1}{2}\right)u}{\sin\frac{u}{2}}\right|\right] \\ &\leq \frac{1}{2\pi P_m}\left[\left\{\sum_{j=0}^{m-1}|\Delta p_j| + |p_m|\right\}\max_{0\leq j\leq m}\left|\sum_{\nu=0}^{j}\frac{\sin\left(m-\nu+\frac{1}{2}\right)u}{\sin\frac{u}{2}}\right|\right] \\ &\leq \frac{1}{2\pi P_m}\left[\left\{\sum_{j=0}^{m-1}|\Delta p_j| + |p_m|\right\}\left(\frac{1}{u^2/\pi^2}\right)\right] \\ &= \frac{1}{2P_m}\left[O\left(\frac{P_m}{(n+1)u^2}\right)\right], \text{ by Eq. (3.1)}, \\ &= O\left(\frac{1}{(m+1)u^2}\right). \end{aligned}$$

Similarly, $K_n^{(N)}(v) = O\left(\frac{1}{(n+1)v^2}\right)$, for $(n+1)^{-1} < v \leq \pi$.

Lemma 4.3. *If $f(x,y) \in Lip(\xi_1,\xi_2)$ then $\phi(x,y) \in Lip(\xi_1,\xi_2)$ and*

$$|\phi(x,y)| = O(\xi_1(u) + \xi_2(v)).$$

Proof. Clearly,

$$\begin{aligned} |\phi(x+u,y+v) - \phi(x,y)| &\leq |f(t+x+u,w+y+v) - (t+x,w+y)| \\ &\quad + |f(t-x-u,w+y+v) - f(t-x,w+y)| \\ &\quad + |f(t+x+u,w-y-v) - f(t+x,w-y)| \\ &\quad + |f(t-x-u,w-y-v) - f(t-x,w-y)| \\ &= O(\xi_1(u) + \xi_2(v)). \end{aligned}$$

Then $f(x, y) \in Lip(\xi_1, \xi_2) \Rightarrow \phi(x, y) \in Lip(\xi_1, \xi_2)$.

$$\begin{aligned}
|\phi(x, y)| &= \frac{1}{4}\,|f(x+u, y+v) + f(x+u, y-v) + f(x-u, y+v) \\
&\quad + f(x-u, y-v) - 4f(x, y)| \\
&\leq \frac{1}{4}\,[|f(x+u, y+v) - f(x, y)| + |f(x+u, y-v) - f(x, y)| \\
&\quad + |f(x-u, y+v) - f(x, y)| + |f(x-u, y-v) - f(x, y)|] \\
&= O(\xi_1(u) + \xi_2(v)) + O(\xi_1(u) + \xi_2(v)) \\
&\quad + O(\xi_1(u) + \xi_2(v)) + O(\xi_1(u) + \xi_2(v)) \\
&= O(\xi_1(u) + \xi_2(v)).
\end{aligned}$$

5. Proof of Theorem

$(j, k)^{th}$ partial sum of the double Fourier series (2.1) is given by

$$s_{j,k}(f; (x, y)) - f(x, y) = \frac{1}{4\pi^2} \int_0^{\pi}\int_0^{\pi} \phi(u, v) \frac{\sin(j + \frac{1}{2})u}{\sin\frac{u}{2}} \frac{\sin(k + \frac{1}{2})v}{\sin\frac{v}{2}} dudv.$$

Denoting (N, p_m, q_n) transform of $s_{j,k}(f; (x, y))$ by $t^{(N)}_{m,n}(x, y)$, we get

$$\begin{aligned}
&\frac{1}{P_m Q_n} \sum_{j=o}^{m}\sum_{k=0}^{n} p_m q_n \left(s_{m-j,n-k}(f; (x, y)) - f(x, y)\right) \\
&\quad = \frac{1}{P_m Q_n} \sum_{j=0}^{m}\sum_{k=0}^{n} p_m q_n \left\{ \frac{1}{4\pi^2} \int_0^{\pi}\int_0^{\pi} \phi(u, v) \frac{\sin(m - j + \frac{1}{2})u \sin(n - k + \frac{1}{2})v}{\sin\frac{u}{2}\sin\frac{v}{2}} dudv \right\}.
\end{aligned}$$

$$\begin{aligned}
&t^{(N)}_{m,n}(x, y) - f(x, y) \\
&\quad = \frac{1}{4\pi^2 P_m Q_n} \int_0^{\pi}\int_0^{\pi} \phi(u, v) \sum_{j=0}^{m}\sum_{k=0}^{n} \frac{\sin(m - j + \frac{1}{2})u \sin(n - k + \frac{1}{2})v}{\sin\frac{u}{2}\sin\frac{v}{2}} dudv \\
&\quad = \int_0^{\pi}\int_0^{\pi} \phi(u, v) M^{(N)}_m(u) K^{(N)}_n(v) dudv \\
&\quad = \left(\int_0^{\frac{1}{m+1}}\int_0^{\frac{1}{n+1}} + \int_0^{\frac{1}{m+1}}\int_{\frac{1}{n+1}}^{\pi} + \int_{\frac{1}{m+1}}^{\pi}\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{m+1}}^{\pi}\int_{\frac{1}{n+1}}^{\pi} \right) \phi(u, v) M^{(N)}_m(u) K^{(N)}_n(v) dudv \\
&\quad = I_1 + I_2 + I_3 + I_4, \quad (say). \qquad (5.1)
\end{aligned}$$

Using Lemma 4.1 and Lemma 4.3, we have

$$\begin{aligned}
|I_1| &= \left| \int_0^{\frac{1}{m+1}} \int_0^{\frac{1}{n+1}} \phi(u,v) M_m^{(N)}(u) K_n^{(N)}(v)\, du dv \right| \\
&= O\left[(m+1)(n+1)\left(\int_0^{\frac{1}{m+1}} \int_0^{\frac{1}{n+1}} (\xi_1(u)+\xi_2(v))\, du dv\right)\right] \\
&= O\left[(m+1)(n+1)\left(\frac{1}{n+1}\int_0^{\frac{1}{m+1}} \xi_1(u) du + \frac{\xi_2\left(\frac{1}{n+1}\right)}{n+1}\int_0^{\frac{1}{n+1}} dv\right)\right] \\
&= O\left[(m+1)(n+1)\left(\frac{\xi_1\left(\frac{1}{m+1}\right)}{(m+1)(n+1)} + \frac{\xi_2\left(\frac{1}{n+1}\right)}{(m+1)(n+1)}\right)\right] \\
&= O\left(\xi_1\left(\frac{1}{m+1}\right) + \xi_2\left(\frac{1}{n+1}\right)\right). \qquad (5.2)
\end{aligned}$$

By Lemma 4.1, Lemma 4.2 and Lemma 4.3, we observe that

$$\begin{aligned}
|I_2| &= O\left[\int_0^{\frac{1}{m+1}} \int_{\frac{1}{n+1}}^{\pi} (\xi_1(u)+\xi_2(v)) \frac{(m+1)}{(n+1)v^2} du dv\right] \\
&= O\left[\left(\frac{m+1}{n+1}\right)\left(\int_0^{\frac{1}{m+1}} \int_{\frac{1}{n+1}}^{\pi} \frac{\xi_1(u)}{v^2} du dv + \int_0^{\frac{1}{m+1}} \int_{\frac{1}{n+1}}^{\pi} \frac{\xi_2(v)}{v^2} du dv\right)\right] \\
&= O\left[\left(\frac{m+1}{n+1}\right)\left(\int_0^{\frac{1}{m+1}} \xi_1(u)\left((n+1)-\frac{1}{\pi}\right) du\right.\right. \\
&\qquad \left.\left. +(n+1)\xi\left(\frac{1}{n+1}\right)\int_0^{\frac{1}{m+1}} \int_{\frac{1}{n+1}}^{\pi} \frac{1}{v} du dv\right)\right] \quad \text{by condition (3.3)} \\
&= O\left[\left(\frac{m+1}{n+1}\right)\left(\frac{(n+1)\xi_1\left(\frac{1}{m+1}\right)}{(m+1)} + \frac{(n+1)\xi_2\left(\frac{1}{n+1}\right)}{(m+1)} \log(n+1)\pi\right)\right] \\
&= O\left(\xi_1\left(\frac{1}{m+1}\right) + \xi_2\left(\frac{1}{n+1}\right)\log(n+1)\pi\right) \qquad (5.3)
\end{aligned}$$

Similarly,

$$\begin{aligned}|I_3| &= O\left[\int_{\frac{1}{m+1}}^{\pi}\int_{0}^{\frac{1}{n+1}}(\xi_1(u)+\xi_2(v))\frac{(n+1)}{(m+1)u^2}dudv\right]\\ &= O\left(\xi_1\left(\frac{1}{m+1}\right)\log(m+1)\pi+\xi_2\left(\frac{1}{n+1}\right)\right) \qquad (5.4)\end{aligned}$$

Also, using Lemma 4.2 and Lemma 4.3, we get

$$\begin{aligned}|I_4| &= O\left[\int_{\frac{1}{m+1}}^{\pi}\int_{\frac{1}{n+1}}^{\pi}(\xi_1+\xi_2)\frac{1}{(m+1)u^2}\frac{1}{(n+1)v^2}dudv\right]\\ &= O\left(\xi_1\left(\frac{1}{m+1}\right)\log(m+1)\pi+\xi_2\left(\frac{1}{n+1}\right)\log(n+1)\pi\right) \qquad (5.5)\end{aligned}$$

Combining equations (5.1) to (5.5), we have

$$\left\|t_{m,n}^{(N)}-f\right\|_{\infty}=O\left(\xi_1\left(\frac{1}{m+1}\right)\log(m+1)\pi+\xi_2\left(\frac{1}{n+1}\right)\log(n+1)\pi\right).$$

This completes the proof of the main Theorem.

6. Corollaries

Following corollaries can be derived from the main theorem:

Corollary 6.1. *If $p_m = 1\forall j$ and $q_n = 1\forall k$, then the degree of approximation of $f(x,y)\in Lip(\xi_1,\xi_2)$ by $(C,1,1)$ means $t_{m,n}^{(C)}=\frac{1}{(m+1)(n+1)}\sum_{j=1}^{m}\sum_{k=1}^{n}s_{j,k}$ of double Fourier series (2.1) is given by*

$$\left\|t_{m,n}^{(C)}-f\right\|_{\infty}=O\left(\xi_1\left(\frac{1}{m+1}\right)\log(m+1)\pi+\xi_2\left(\frac{1}{n+1}\right)\log(n+1)\pi\right)$$

for $m=0,1,2,\ldots$; $n=0,1,2,\ldots$.

Corollary 6.2. *If we take $\xi_1(u)=\frac{u^\alpha}{\left(\log\frac{\pi}{u}\right)^\gamma}$ for $0<\gamma<\alpha<1$, $0<u<\pi$ and $\xi_2(v)=\frac{v^\beta}{\left(\log\frac{\pi}{v}\right)^\delta}$, for $0<\delta<\beta<1$, $0<v<\pi$, then the estimate of the main theorem becomes*

$$\left\|t_{m,n}^{N}-f\right\|_{\infty}=O\left(\left(\frac{\log(m+1)\pi}{(m+1)^\alpha(\log(m+1)\pi)^\gamma}\right)+\left(\frac{\log(n+1)\pi}{(n+1)^\beta(\log(n+1)\pi)^\delta}\right)\right).$$

Corollary 6.3. *If $f(x,y)$ is a 2π periodic function with respect to each variable x and y, Lebesgue integrable in $(-\pi,\pi)\times(-\pi,\pi)$ and belongs to $Lip(\alpha,\beta)$ class then the degree of approximation of $f(x,y)$ by (N,p_m,q_n) summability means $t_{m,n}^{(N)}$ of double Fourier series (2.1), such that conditions (3.1) and (3.2) holds, satisfies*

$$\left\|t_{m,n}^{(N)}-f\right\|_{\infty}=\begin{cases}O\left((m+1)^{-\alpha}+(n+1)^{-\beta}\right) & ,0<\alpha<1,\ 0<\beta<1\\ O\left((m+1)^{-\alpha}+\frac{\log(n+1)\pi}{(n+1)}\right) & ,\ 0<\alpha<1,\ \beta=1\\ O\left(\frac{\log(m+1)\pi}{(m+1)}+(n+1)^{-\beta}\right) & ,\ \alpha=1,\ 0<\beta<1\\ O\left(\frac{\log(m+1)\pi}{(m+1)}+\frac{\log(n+1)\pi}{(n+1)}\right) & ,\ \alpha=\beta=1\end{cases}$$

Corollary 6.4. *If $f(x,y)$ is a 2π periodic function with respect to each variable x and y, Lebesgue integrable in $(-\pi,\pi)\times(-\pi,\pi)$ and is belonging to $Lip(\alpha,\beta)$, $0<\alpha\leq 1,\ 0<\beta\leq 1$, class then the degree of approximation of $f(x,y)$ by $(C,1,1)$ summability means $t^{(C)}_{m,n} = \frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n} s_{j,k}$ of double Fourier series (2.1), satisfies*

$$\left\|t^{(C)}_{m,n}-f\right\|_{\infty}=\begin{cases} O\left((m+1)^{-\alpha}+(n+1)^{-\beta}\right) & ,0<\alpha<1,\ 0<\beta<1 \\ O\left((m+1)^{-\alpha}+\frac{\log(n+1)\pi}{(n+1)}\right) & ,\ 0<\alpha<1,\ \beta=1 \\ O\left(\frac{\log(m+1)\pi}{(m+1)}+(n+1)^{-\beta}\right) & ,\ \alpha=1,\ 0<\beta<1 \\ O\left(\frac{\log(m+1)\pi}{(m+1)}+\frac{\log(n+1)\pi}{(n+1)}\right) & ,\ \alpha=\beta=1 \end{cases}$$

for $m=0,1,2,...\ ;\ n=0,1,2,...$.

Acknowledgment. The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

References

[1] Alexits, G., Convergence problems of orthogonal series, *Pergamon Press, London*, (1961).
[2] Dzjadyk, V. K., Introduction to the theory of uniform approximation of functions by polynomials (in Russian), Mascow, (1977), 511pp.
[3] Chandra, P., On the degree of approximation of function belonging to the Lipschitz class, Nanta Math, 8, (1975), 88-91.
[4] Chow Y. S., On the Cesàro summability of double Fourier series, Tohoku Math. J., 5 , (1935), 277-283.
[5] Hille, E. and Tamarkin, J. D., On the summability of Fourier series Trans, Amer.Math.Soc., 34, (1932), 757-783.
[6] Hasegawa, Yoshimitsu, On summabilities of double Fourier series, Kōdai Math. Sem. Rep., 15, (1963),226-238.
[7] Korneichuk, N. P., Extremal problems of approximation theory (in Russian), Moscow, (1976).
[8] Qureshi, K., On the degree of approximation of functions belonging to the class *Lip*α, Indian J. Pure Appl. Maths. 13, no., 8, (1982), 898-903.
[9] Sharma, P. L.. On the harmonic summability of double Fourier series Proc., Amer. Math. Soc 91, (1958), 979-986.
[10] Siddiqi, A. H., On the degree of approximation to a function by the Cesàro means of its Fourier series, Indian J. Pure Appl. Math, 2 , no.3, (1971), 367-373.
[11] Siddiqui, J. A., On the harmonic summability of Fourier series, Proc. Indian Acad. Sci., Sect. A., 28, (1948), 527-531.
[12] Stepantes, A. I., Approximation of some class of periodic two variables by linear summation methods Fourier series, Ukrainian Math. Journal, 26, No2, (1974), 205-215.
[13] Stepantes, A. I., Methods of approximation Theory, VSP: Leiden, Boston, (2005).
[14] Stepantes, A. I., Uniform approximation by trigonometric polynomials (in Russian), kiev, 1981. [see also Uniform approximation by trigonometric polynomials VSP: Utrecht, Boston, Koln, Tokyo, 2001.]
[15] Zygmund, A. , Trigonometric series, 2nd rev.ed., I, Cambridge Univ. Press, Cambridge, (1968).

Shyam Lal and Abhishek Mishra
Department of Mathematics, Faculty of Science Banaras Hindu University, Varanasi-221005, India
E-mail address: shyam_lal@gmail.com, abhi87mangal@gmail.com