Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 6 Issue 2(2014), Pages 1-22.

TRIPLE FIXED POINTS IN ORDERED UNIFORM SPACES

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Abstract. In this paper, we prove some tripled fixed point theorems for generalized contractive mappings in uniform spaces and apply them to study the existences-uniqueness problem for a class of nonlinear integral equations of with unbounded deviations. We also give some examples to show that our results are effective.

1. Introduction

Fixed point theory plays a crucial role not only in the existence theory of differential equations, integral equations, functional equations, partial differential equations, random differential equations and but also in computer science and economics. In 2006, Bhashkar and Lakshmikantham introduced the concepts of coupled fixed point and mixed monotone property for contractive mappings of the form $F : X \times X \to X$, where X is a partially ordered metric space, and established some interesting coupled fixed point theorems. Recently, Berinde and Borcut [9] introduced the concept of the triple fixed point and investigated some tripled fixed point theorems in partially ordered metric spaces. Later, various results on triple fixed points have been obtained, see e.g. [4], [5], [6].

The main purpose of our work is to present some results concerning the tripled fixed point theorems in uniform spaces as natural extensions of tripled fixed point theorems, which have been recently exposed by many authors (see [7], [8] and the references given therein) in metric spaces.

2. Preliminaries

Let X be a uniform space. The uniform topology on X is generated by a family of uniform continuous pseudometrics on X (see [13]). In this paper, by $(X, \mathcal{P})$ we mean a Hausdorff uniform space whose uniformity is generated by a saturated family of pseudometrics $\mathcal{P} = \{d_\alpha(x, y) : \alpha \in I\}$, where I is an index set. Note that, $(X, \mathcal{P})$ is Hausdorff if and only if $d_\alpha(x, y) = 0$ for all $\alpha \in I$ implies $x = y$.

2000 *Mathematics Subject Classification.* 46J10, 46J15, 47H10.
Key words and phrases. Triple fixed point, uniform spaces, contractive mappings.
Submitted April 4, 2014. Published Jun 3, 2014.

Definition 2.1. ([1]) Let $(X, \mathcal{P})$ be a Hausdorff uniform space.

1) The sequence $\{x_n\} \subset X$ is *Cauchy* if $d_\alpha(x_n, x_m) \to 0$ as $m, n \to \infty$ for every $\alpha \in I$.

2) X is said to be *sequentially complete* if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$.

Definition 2.2. ([1]) Let $j : I \to I$ be an arbitrary mapping of the index I into itself. The *iterations* of j can be defined inductively

$$j^0(\alpha) = \alpha, j^k(\alpha) = j\big(j^{k-1}(\alpha)\big), k = 1, 2, \ldots$$

The following concept was introduce by Vasile Berinde and Marin Borcut.

Definition 2.3. ([5]) Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to have the *mixed monotone property* if for any $x, y, z \in X$

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y, z) \leq F(x_2, y, z),$$
$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1, z) \geq F(x, y_2, z)$$

and

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow F(x, y, z_1) \leq F(x, y, z_2).$$

Definition 2.4. ([5]) Let $F : X \times X \times X \to X$. An element (x, y, z) is called a *triple fixed point* of F if

$$F(x, y, z) = x, F(y, x, y) = y \text{ and } F(z, y, x) = z.$$

Definition 2.5. Let X be a uniform space. A mapping $T : X \to X$ is said to be *ICS* if T is injective, continuous and has the property: for every net $\{x_\alpha\}$ in X, if net $\{Tx_\alpha\}$ is convergent then $\{x_\alpha\}$ is also convergent.

Now, we introduce the class of functions which plays a crucial role in the fixed point theory. Sometimes, they are called to be control functions.

Let $\Phi = \{\varphi_\alpha : \alpha \in I\}$ be a family of functions (which one call Φ-contractive) with the properties:

i) $\varphi_\alpha : [0, +\infty) \to [0, +\infty)$ is monotone non-decreasing;

ii) $0 < \varphi_\alpha(t) < t$ for all $t > 0$ and $\varphi_\alpha(0) = 0$.

Remark 2.6. If (X, d) is a metric space, then the uniform topology generated by the metric d coincides with the metric topology on X. More precisely, $d_\alpha(x, y) = d(x, y)$ for all $x, y \in X$ and $\alpha \in I$, where the family of pseudometrics $\mathcal{P} = \{d_\alpha : \alpha \in I\}$ generates a uniform structure of X. Therefore, as a corollary of our results, we obtain the tripled fixed point theorems in the metric space.

3. Triple fixed points in uniform spaces

From now on, we denote $X^3 = X \times X \times X$. We begin this section at giving a new triple fixed point theorem in ordered uniform spaces.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and $\mathcal{P} = \{d_\alpha(x, y) : \alpha \in I\}$ be a family of pseudometrics on X such that $(X, \mathcal{P})$ is a Hausdorff sequentially complete uniform space. Let $T : X \to X$ is an ICS mapping and $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Suppose that*

1) For every $\alpha \in I$ there exists $\varphi_\alpha \in \Phi$ such that

$$
\begin{aligned}
&d_\alpha\big(TF(x,y,z), TF(u,v,w)\big) \\
&\leq \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx,Tu), d_{j(\alpha)}(Ty,Tv), d_{j(\alpha)}(Tz,Tw)\big\}\Big),
\end{aligned}
\tag{3.1}
$$

for all $x \leq u, y \geq v$ and $z \leq w$;

2) For each $\alpha \in I$, there exists $\overline{\varphi}_\alpha \in \Phi$ such that

$$\sup\{\varphi_{j^n(\alpha)}(t) : n = 0, 1, \ldots\} \leq \overline{\varphi}_\alpha(t) \text{ for all } t > 0,$$

and $\dfrac{\overline{\varphi}_\alpha(t)}{t}$ is non-decreasing on $(0, +\infty)$;

3) There are $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0)$, $z_0 \leq F(z_0, y_0, x_0)$ and

$$
\begin{aligned}
\max\Big\{d_{j^n(\alpha)}\big(Tx_0, TF(x_0,y_0,z_0)\big),\ & d_{j^n(\alpha)}\big(Ty_0, TF(y_0,x_0,y_0)\big), \\
& d_{j^n(\alpha)}\big(Tz_0, TF(z_0,y_0,x_0)\big)\Big\} < p(\alpha) < \infty,
\end{aligned}
$$

for every $\alpha \in I, n \in \mathbb{N}$.

Also, assume either a) F is continuous, or

b) X has the property:

i) If a non-decreasing sequence $\{x_n\}$ in X converges to x then $x_n \leq x$ for all n;

ii) If a non-increasing sequence $\{y_n\}$ in X converges to y then $y_n \geq y$ for all n.

Then F has a triple fixed point, that is, there exists $x, y, z \in X$ such that

$$F(x,y,z) = x, F(y,x,y) = y, F(z,y,x) = z.$$

Proof. Let $x_0, y_0, z_0 \in X$ such that $x_0 \leq F(x_0, y_0, z_0), y_0 \geq F(y_0, x_0, y_0)$ and $z_0 \leq F(z_0, y_0, x_0)$. Put

$$x_1 = F(x_0, y_0, z_0), y_1 = F(y_0, x_0, y_0) \text{ and } z_1 = F(z_0, y_0, x_0).$$

Continuing this process, we can construct sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n, z_n), y_{n+1} = F(y_n, x_n, y_n) \text{ and } z_{n+1} = F(z_n, y_n, x_n), n = 0, 1, 2, \ldots \tag{3.2}$$

Since F has the mixed monotone property, then using a mathematical induction it is easy that

$$x_n \leq x_{n+1}, y_n \geq y_{n+1} \text{ and } z_n \leq z_{n+1} \text{ for all } n = 0, 1, 2, \ldots \tag{3.3}$$

Now, for each $n = 0, 1, 2, \ldots$ and $\alpha \in I$, we put

$$\delta_n^\alpha = \max\big\{d_\alpha(Tx_n, Tx_{n+1}), d_\alpha(Ty_n, Ty_{n+1}), d_\alpha(Tz_n, Tz_{n+1})\big\}.$$

By the assumption 3), we have

$$
\begin{aligned}
\delta_0^{j^n(\alpha)} = \max\Big\{&d_{j^n(\alpha)}\big(Tx_0, TF(x_0,y_0,z_0)\big), d_{j^n(\alpha)}\big(Ty_0, TF(y_0,x_0,y_0)\big), \\
&d_{j^n(\alpha)}\big(Tz_0, TF(z_0,y_0,x_0)\big)\Big\} < p(\alpha) < \infty.
\end{aligned}
\tag{3.4}
$$

Now we claim that $\delta_n^\alpha \le \varphi_\alpha\big(\delta_{n-1}^{j(\alpha)}\big)$ for every $\alpha \in I, n \in \mathbb{N}$. Indeed, in view of the condition 1) and since $x_{n-1} \le x_n, y_{n-1} \ge y_n$ and $z_{n-1} \le z_n$, we obtain

$$\begin{aligned} & d_\alpha(Tx_n, Tx_{n+1}) = d_\alpha\big(TF(x_{n-1}, y_{n-1}, z_{n-1}), TF(x_n, y_n, z_n)\big) \\ & \le \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx_{n-1}, Tx_n), d_{j(\alpha)}(Ty_{n-1}, Ty_n), d_{j(\alpha)}(Tz_{n-1}, Tz_n)\big\}\Big) \\ & = \varphi_\alpha\big(\delta_{n-1}^{j(\alpha)}\big). \end{aligned} \tag{3.5}$$

Similarly, we have

$$d_\alpha(Tz_n, Tz_{n+1}) \le \varphi_\alpha\big(\delta_{n-1}^{j(\alpha)}\big), \tag{3.6}$$

and

$$\begin{aligned} & d_\alpha(Ty_n, Ty_{n+1}) = d_\alpha\big(TF(y_{n-1}, x_{n-1}, y_{n-1}), TF(y_n, x_n, y_n)\big) \\ & \le \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Ty_{n-1}, Ty_n), d_{j(\alpha)}(Tx_{n-1}, Tx_n), d_{j(\alpha)}(Ty_{n-1}, Ty_n)\big\}\Big) \\ & \le \varphi_\alpha\big(\delta_{n-1}^{j(\alpha)}\big). \end{aligned} \tag{3.7}$$

Combining (3.5)-(3.6), we deduce that

$$\delta_n^\alpha \le \varphi_\alpha\big(\delta_{n-1}^{j(\alpha)}\big) \text{ for every } n \in \mathbb{N}, \alpha \in I. \tag{3.8}$$

Since φ_α is a monotone non-decreasing function, it follows from (3.4), (3.8) and the condition 2) that

$$\begin{aligned} \delta_n^\alpha \le \varphi_\alpha\big(\delta_{n-1}^{j(\alpha)}\big) \le \varphi_\alpha\big(\varphi_{j(\alpha)}\big(\delta_{n-2}^{j^2(\alpha)}\big)\big) & \le \varphi_\alpha\big(\varphi_{j(\alpha)}\big(\dots\varphi_{j^{n-1}(\alpha)}\big(\delta_0^{j^n(\alpha)}\big)\dots\big)\big) \\ & \le \overline{\varphi}_\alpha^n\big(\delta_0^{j^n(\alpha)}\big) \le \overline{\varphi}_\alpha^n\big(p(\alpha)\big). \end{aligned}$$

Put $\overline{\varphi}_\alpha^n\big(p(\alpha)\big) = b_n^\alpha$, for each $\alpha \in I$ and $n \in \mathbb{N}$. Then, we have $\delta_n^\alpha \le b_n^\alpha$ for every $\alpha \in I, n \in \mathbb{N}$. Using the triangle inequality, we get

$$\begin{aligned} & \max\big\{d_\alpha(Tx_n, Tx_{n+p}), d_\alpha(Ty_n, Ty_{n+p}), d_\alpha(Tz_n, Tz_{n+p})\big\} \\ & \le \max\Big\{\sum_{i=0}^{p-1} d_\alpha(Tx_{n+i}, Tx_{n+i+1}), \sum_{i=0}^{p-1} d_\alpha(Ty_{n+i}, Ty_{n+i+1}), \sum_{i=0}^{p-1} d_\alpha(Tz_{n+i}, Tz_{n+i+1})\Big\} \\ & \le \sum_{i=0}^{p-1} \max\big\{d_\alpha(Tx_{n+i}, Tx_{n+i+1}), d_\alpha(Ty_{n+i}, Ty_{n+i+1}), d_\alpha(Tz_{n+i}, Tz_{n+i+1})\big\} \\ & = \sum_{i=0}^{p-1} \delta_{n+i}^\alpha \le \sum_{i=0}^{p-1} b_{n+i}^\alpha. \end{aligned} \tag{3.9}$$

Having in mind that $\overline{\varphi}_\alpha(t) < t$ for every $t > 0$, and $p(\alpha) > 0$, we obtain

$$\overline{\varphi}_\alpha^n\big(p(\alpha)\big) = \overline{\varphi}_\alpha\big(\overline{\varphi}_\alpha^{n-1}(p(\alpha))\big) < \overline{\varphi}_\alpha^{n-1}\big(p(\alpha)\big) < \dots < \overline{\varphi}_\alpha\big(p(\alpha)\big) < p(\alpha). \tag{3.10}$$

Since $\frac{\overline{\varphi}_\alpha(t)}{t}$ is a monotone non-decreasing function, it follows from (3.10) that

$$\frac{b_{n+1}^\alpha}{b_n^\alpha} = \frac{\overline{\varphi}_\alpha\big(\overline{\varphi}_\alpha^n(p(\alpha))\big)}{\overline{\varphi}_\alpha^n\big(p(\alpha)\big)} \le \frac{\overline{\varphi}_\alpha\big(p(\alpha)\big)}{p(\alpha)} < 1.$$

This implies that $\sum_{m=0}^\infty b_m^\alpha$ is a convergent series. Hence $\sum_{i=0}^{p-1} b_{n+i}^\alpha \to 0$ as $n \to \infty$ for all p. It follows from (3.9) that $d_\alpha(Tx_n, Tx_{n+p}) \to 0, d_\alpha(Ty_n, Ty_{n+p}) \to 0$ and $d_\alpha(Tz_n, Tz_{n+p}) \to 0$ as $n \to \infty$ for all p. Thus $\{Tx_n\}, \{Ty_n\}$ and $\{Tz_n\}$ are

Cauchy sequences. By the sequential completeness of X, $\{Tx_n\}, \{Ty_n\}$ and $\{Tz_n\}$ are convergent. Since T is an ICS mapping, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} x_n = x, \lim_{n\to\infty} y_n = y, \lim_{n\to\infty} z_n = z. \tag{3.11}$$

Suppose that the assumption (a) holds. By (3.2), (3.11) and the continuity of F, we get

$$\begin{aligned} x &= \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n, z_n) = F(x, y, z), \\ y &= \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n, y_n) = F(y, x, y), \\ z &= \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} F(z_n, y_n, x_n) = F(z, y, x). \end{aligned}$$

This shows that F has a triple fixed point.

Suppose now the assumption (b) holds. Since $\{x_n\}$, $\{z_n\}$ are non-decreasing and $x_n \to x$, $z_n \to z$, and as $\{y_n\}$ is non-increasing and $y_n \to y$, by the assumption (b), we have

$$x_n \le x, \ y_n \ge y \text{ and } z_n \le z,$$

for all n. Using the triangle inequality and the contractive condition (3.1), we have

$$\begin{aligned} &d_\alpha\big(Tx, TF(x, y, z)\big) \le d_\alpha(Tx, Tx_{n+1}) + d_\alpha\big(Tx_{n+1}, TF(x, y, z)\big) \\ &= d_\alpha(Tx, Tx_{n+1}) + d_\alpha\big(TF(x_n, y_n, z_n), TF(x, y, z)\big) \\ &\le d_\alpha(Tx, Tx_{n+1}) + \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx_n, Tx), d_{j(\alpha)}(Ty_n, Ty), d_{j(\alpha)}(Tz_n, Tz)\big\}\Big) \\ &\le d_\alpha(Tx, Tx_{n+1}) + \max\big\{d_{j(\alpha)}(Tx_n, Tx), d_{j(\alpha)}(Ty_n, Ty), d_{j(\alpha)}(Tz_n, Tz)\big\}. \end{aligned}$$

Letting $n \to \infty$ in the above inequality and by the continuity of T, we have $d_\alpha\big(Tx, TF(x, y, z)\big) = 0$ for all α, this implies that $Tx = TF(x, y, z)$. Since T is injective, we get that $x = F(x, y, z)$. Similarly, we have $y = F(y, x, y)$, $z = F(z, y, x)$. The proof is completed. □

Corollary 3.2. *Let $(X, \le)$ be a partially ordered set and $\mathcal{P} = \{d_\alpha(x, y) : \alpha \in I\}$ be a family of pseudometrics on X such that $(X, \mathcal{P})$ is a Hausdorff sequentially complete uniform space. Let $T : X \to X$ is an ICS mapping and $F : X^3 \to X$ be a mapping having the mixed monotone property on X. Suppose that:*

1) For every $\alpha \in I$ there exists $\varphi_\alpha \in \Phi$ such that

$$d_\alpha\big(TF(x, y, z), TF(u, v, w)\big) \le \varphi_\alpha\left(\frac{d_{j(\alpha)}(Tx, Tu) + d_{j(\alpha)}(Ty, Tv) + d_{j(\alpha)}(Tz, Tw)}{3}\right),$$

for all $x \le u$, $y \ge v$ and $z \le w$;

2) For each $\alpha \in I$, there exists $\overline{\varphi}_\alpha \in \Phi$ such that

$$\sup\{\varphi_{j^n(\alpha)}(t) : \ n = 0, 1, \ldots\} \le \overline{\varphi}_\alpha(t) \text{ for all } t > 0,$$

and $\dfrac{\overline{\varphi}_\alpha(t)}{t}$ is non-decreasing on $(0, +\infty)$;

3) There are $x_0, y_0, z_0 \in X$ such that $x_0 \le F(x_0, y_0, z_0)$, $y_0 \ge F(y_0, x_0, y_0)$, $z_0 \le F(z_0, y_0, x_0)$ and $d_{j^n(\alpha)}\big(Tx_0, TF(x_0, y_0, z_0)\big) + d_{j^n(\alpha)}\big(Ty_0, TF(y_0, x_0, y_0)\big) + d_{j^n(\alpha)}\big(Tz_0, TF(z_0, y_0, x_0)\big) < p(\alpha) < \infty$, for every $\alpha \in I, n \in \mathbb{N}$.

Also, assume either a) F is continuous, or

b) X has the property:

i) If a non-decreasing sequence $\{x_n\}$ in X converges to x then $x_n \le x$ for all n;

ii) If a non-increasing sequence $\{y_n\}$ in X converges to y then $y_n \geq y$ for all n.

Then F has a triple fixed point, that is, there exist $x, y, z \in X$ such that

$$F(x,y,z) = x,\ F(y,x,y) = y,\ F(z,y,x) = z.$$

Proof. Since φ_α is non-decreasing for all α, it is easy to see that

$$\varphi_\alpha\Big(\frac{d_{j(\alpha)}(Tx,Tu) + d_{j(\alpha)}(Ty,Tv) + d_{j(\alpha)}(Tz,Tw)}{3}\Big)$$
$$\leq \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx,Tu), d_{j(\alpha)}(Ty,Tv), d_{j(\alpha)}(Tz,Tw)\big\}\Big),$$

and

$$\max\Big\{d_{j^n(\alpha)}\big(Tx_0, TF(x_0,y_0,z_0)\big), d_{j^n(\alpha)}\big(Ty_0, TF(y_0,x_0,y_0)\big), d_{j^n(\alpha)}\big(Tz_0, TF(z_0,y_0,x_0)\big)\Big\}$$
$$\leq d_{j^n(\alpha)}\big(Tx_0, TF(x_0,y_0,z_0)\big) + d_{j^n(\alpha)}\big(Ty_0, TF(y_0,x_0,y_0)\big) + d_{j^n(\alpha)}\big(Tz_0, TF(z_0,y_0,x_0)\big).$$

Thus, we can apply Theorem 3.1. □

One can proved that the triple fixed point is in fact unique, provide that we have to add the properties for of partial order on X^3 and the mapping $j : I \to I$.

Definition 3.3. ([1]) A uniform space $(X, \mathcal{P})$ is said to be *j-bounded* if for every $\alpha \in I$ and $x, y \in X$ there exists $q = q(x,y,\alpha)$ such that

$$d_{j^n(\alpha)}(x,y) \leq q(x,y,\alpha) < \infty, \text{ for all } n \in \mathbb{N}.$$

Now, we shall prove the uniqueness of a triple fixed point. Let $(X, \leq)$ be a partially ordered set. Then, we define a partial order on X^3 in the following way: For $(x,y,z), (u,v,w) \in X^3$,

$$(x,y,z) \leq (u,v,w) \Leftrightarrow x \leq u,\ y \geq v \text{ and } z \leq w.$$

We say that (x,y,z) and (u,v,w) are comparable if

$$(x,y,z) \leq (u,v,w) \quad \text{or} \quad (u,v,w) \leq (x,y,z).$$

Also, we say that (x,y,z) is equal to (u,v,w) if and only if $x = u,\ y = v$ and $z = w$.

Theorem 3.4. *Suppose that the conditions of Theorem 3.1 are fulfilled. If X is j-bounded and for every $(x,y,z), (u,v,w) \in X^3$ there exists $(a,b,c) \in X^3$ which is comparable to them, then F has a unique triple fixed point.*

Proof. By Theorem 3.1, we conclude that the set of triple fixed points of F is nonempty. Assume that $(x,y,z), (u,v,w)$ are triple fixed points of F, that is

$$x = F(x,y,z), y = F(y,x,y), z = F(z,y,x),$$
$$u = F(u,v,w), v = F(v,u,v), w = F(w,v,u).$$

We shall show that (x,y,z) and (u,v,w) are equal. By assumption, there exists $(a,b,c) \in X^3$ such that (a,b,c) is comparable to (x,y,z) and (u,v,w). Set $a_0 = a$,

$b_0 = b, c_0 = c$. By induction, we construct the sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ defined by

$$\begin{aligned} a_{n+1} &= F(a_n, b_n, c_n), \\ b_{n+1} &= F(b_n, a_n, b_n), \\ c_{n+1} &= F(c_n, b_n, a_n), \end{aligned} \tag{3.12}$$

for $n = 0, 1, 2, \ldots$

Suppose that $(x, y, z) \leq (a, b, c) = (a_0, b_0, c_0)$. Since F has the mixed monotone property, we have

$$\begin{aligned} a_1 &= F(a_0, b_0, c_0) \geq F(x, y, z) = x, \\ b_1 &= F(b_0, a_0, b_0) \leq F(y, x, y) = y, \\ c_1 &= F(c_0, b_0, a_0) \geq F(z, y, x) = z. \end{aligned}$$

Hence $(a_1, b_1, c_1) \geq (x, y, z)$. Recursively, we get that

$$(a_n, b_n, c_n) \geq (x, y, z) \text{ for every } n = 0, 1, 2, \ldots \tag{3.13}$$

By (3.12), (3.13) and (3.1), we have

$$\begin{aligned} & d_\alpha(Tx, Ta_{n+1}) \\ &= d_\alpha\big(TF(x, y, z), TF(a_n, b_n, c_n)\big) \\ &\leq \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx, Ta_n), d_{j(\alpha)}(Ty, Tb_n), d_{j(\alpha)}(Tz, Tc_n)\big\}\Big), \end{aligned} \tag{3.14}$$

$$\begin{aligned} & d_\alpha(Ty, Tb_{n+1}) \\ &= d_\alpha\big(TF(y, x, y), TF(b_n, a_n, b_n)\big) \\ &\leq \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Ty, Tb_n), d_{j(\alpha)}(Tx, Ta_n), d_{j(\alpha)}(Ty, Tb_n)\big\}\Big) \\ &\leq \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx, Ta_n), d_{j(\alpha)}(Ty, Tb_n), d_{j(\alpha)}(Tz, Tc_n)\big\}\Big), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} & d_\alpha(Tz, Tc_{n+1}) \\ &= d_\alpha\big(TF(z, y, x), TF(c_n, b_n, a_n)\big) \\ &\leq \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tz, Tc_n), d_{j(\alpha)}(Ty, Tb_n), d_{j(\alpha)}(Tx, Ta_n)\big\}\Big), \end{aligned} \tag{3.16}$$

It follows from (3.14)-(3.16) that

$$\begin{aligned} & \max\big\{d_\alpha(Tx, Ta_{n+1}), d_\alpha(Ty, Tb_{n+1}), d_\alpha(Tz, Tc_{n+1})\big\} \\ &\leq \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx, Ta_n), d_{j(\alpha)}(Ty, Tb_n), d_{j(\alpha)}(Tz, Tc_n)\big\}\Big), \end{aligned} \tag{3.17}$$

for every $n = 0, 1, 2, \ldots$

Since X is j-bounded, it follows from (3.17) and the condition 2) (in Theorem 3.1) that

$$\begin{aligned}
&\max\big\{d_\alpha(Tx,Ta_{n+1}),d_\alpha(Ty,Tb_{n+1}),d_\alpha(Tz,Tc_{n+1})\big\}\\
&\le \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx,Ta_n),d_{j(\alpha)}(Ty,Tb_n),d_{j(\alpha)}(Tz,Tc_n)\big\}\Big)\\
&\le \varphi_\alpha\Big(\varphi_{j(\alpha)}\Big(\max\big\{d_{j^2(\alpha)}(Tx,Ta_{n-1}),d_{j^2(\alpha)}(Ty,Tb_{n-1}),d_{j^2(\alpha)}(Tz,Tc_{n-1})\big\}\Big)\Big)\\
&\le \varphi_\alpha\Big(\varphi_{j(\alpha)}\Big(\dots\varphi_{j^{n-1}(\alpha)}\Big(\max\big\{d_{j^n(\alpha)}(Tx,Ta_1),d_{j^n(\alpha)}(Ty,Tb_1),\\
&\qquad\qquad d_{j^n(\alpha)}(Tz,Tc_1)\big\}\Big)\dots\Big)\Big)\\
&\le \overline{\varphi}_\alpha^n\Big(\max\big\{d_{j^n(\alpha)}(Tx,Ta_1),d_{j^n(\alpha)}(Ty,Tb_1),d_{j^n(\alpha)}(Tz,Tc_1)\big\}\Big)\\
&\le \overline{\varphi}_\alpha^n\Big(\max\big\{q(Tx,Ta_1,\alpha),q(Ty,Tb_1,\alpha),q(Tz,Tc_1,\alpha)\big\}\Big).
\end{aligned} \tag{3.18}$$

Denote $r_n^\alpha=\overline{\varphi}_\alpha^n\Big(\max\big\{q(Tx,Ta_1,\alpha),q(Ty,Tb_1,\alpha),q(Tz,Tc_1,\alpha)\big\}\Big)$. By the same argument as in the proof of Theorem 3.1, we can deduce that $\sum_{n=0}^\infty r_n^\alpha$ is convergent. This implies that $r_n^\alpha\to 0$ as $n\to\infty$. It follows from (3.18) that $\max\big\{d_\alpha(Tx,Ta_{n+1}),d_\alpha(Ty,Tb_{n+1}),d_\alpha(Tz,Tc_{n+1})\big\}\to 0$ as $n\to\infty$ for all α. Since T is ICS, this implies that there are $x,y,z\in X$ such that $x=\lim_{n\to\infty}a_n, y=\lim_{n\to\infty}b_n$ and $z=\lim_{n\to\infty}c_n$.

Similarly, we obtain that $u=\lim_{n\to\infty}a_n, v=\lim_{n\to\infty}b_n$ and $w=\lim_{n\to\infty}c_n$. Hence $x=u,y=v$ and $z=w$. □

Corollary 3.5. *In addition to hypotheses of Theorem 3.4, suppose that $x_0\le y_0$ and $z_0\le y_0$ then F has a unique fixed point, that is, there exists $x\in X$ such that $F(x,x,x)=x$.*

Proof. By Theorem 3.4, we conclude that F has a unique triple fixed point (x,y,z). Now we shall show that $x=y=z$. By the mixed monotone property of F, we have

$$\begin{aligned}
x_1&=F(x_0,y_0,z_0)\le F(y_0,x_0,y_0)=y_1,\\
z_1&=F(z_0,y_0,x_0)\le F(y_0,x_0,y_0)=y_1.
\end{aligned}$$

Recursively, we get that

$$x_n\le y_n \text{ and } z_n\le y_n, \tag{3.19}$$

where $x_n=F(x_{n-1},y_{n-1},z_{n-1}), y_n=F(y_{n-1},x_{n-1},y_{n-1}), z_n=F(z_{n-1},y_{n-1},x_{n-1})$, for all $n\ge 0$.

Since (x,y,z) is a unique fixed point of F, we have $x=\lim_{n\to\infty}x_n$, $y=\lim_{n\to\infty}y_n$ and $z=\lim_{n\to\infty}z_n$. By (3.2), (3.19) and (3.1), we have

$$\begin{aligned}
&d_\alpha(Tx_{n+1},Ty_{n+1})\\
&=d_\alpha\big(TF(x_n,y_n,z_n),TF(y_n,x_n,y_n)\big)\\
&\le \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx_n,Ty_n),d_{j(\alpha)}(Ty_n,Tx_n),d_{j(\alpha)}(Tz_n,Ty_n)\big\}\Big)\\
&=\varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx_n,Ty_n),d_{j(\alpha)}(Tz_n,Ty_n)\big\}\Big),
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &d_\alpha(Tz_{n+1}, Ty_{n+1}) \\ &= d_\alpha\big(TF(z_n, y_n, x_n), TF(y_n, x_n, y_n)\big) \\ &\le \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tz_n, Ty_n), d_{j(\alpha)}(Ty_n, Tx_n), d_{j(\alpha)}(Tx_n, Ty_n)\big\}\Big) \\ &= \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx_n, Ty_n), d_{j(\alpha)}(Tz_n, Ty_n)\big\}\Big). \end{aligned} \tag{3.21}$$

It follows from (3.20) and (3.21) that

$$\begin{aligned} &\max\big\{d_\alpha(Tx_{n+1}, Ty_{n+1}), d_\alpha(Tz_{n+1}, Ty_{n+1})\big\} \\ &\le \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx_n, Ty_n), d_{j(\alpha)}(Tz_n, Ty_n)\big\}\Big). \end{aligned} \tag{3.22}$$

Since X is j-bounded, by (3.22) and the condition 2) (in Theorem 3.1) we have

$$\begin{aligned} &\max\big\{d_\alpha(Tx_{n+1}, Ty_{n+1}), d_\alpha(Tz_{n+1}, Ty_{n+1})\big\} \\ &\le \varphi_\alpha\Big(\max\big\{d_{j(\alpha)}(Tx_n, Ty_n), d_{j(\alpha)}(Tz_n, Ty_n)\big\}\Big) \\ &\le \varphi_\alpha\Big(\varphi_{j(\alpha)}\Big(\max\big\{d_{j^2(\alpha)}(Tx_{n-1}, Ty_{n-1}), d_{j^2(\alpha)}(Tz_{n-1}, Ty_{n-1})\big\}\Big)\Big) \\ &\le \varphi_\alpha\Big(\varphi_{j(\alpha)}\Big(\dots\varphi_{j^{n-1}(\alpha)}\Big(\max\big\{d_{j^n(\alpha)}(Tx_1, Ty_1), d_{j^n(\alpha)}(Tz_1, Ty_1)\big\}\Big)\dots\Big)\Big) \\ &\le \overline{\varphi}_\alpha^n\Big(\max\big\{d_{j^n(\alpha)}(Tx_1, Ty_1), d_{j^n(\alpha)}(Tz_1, Ty_1)\big\}\Big) \\ &\le \overline{\varphi}_\alpha^n\Big(\max\big\{q(Tx_1, Ty_1, \alpha), q(Tz_1, Ty_1, \alpha)\big\}\Big). \end{aligned} \tag{3.23}$$

By the same argument as in the proof of Theorem 3.4, we infer that

$$\sum_{n=0}^{\infty}\overline{\varphi}_\alpha^n\Big(\max\big\{q(Tx_1, Ty_1, \alpha), q(Tz_1, Ty_1, \alpha)\big\}\Big)$$

is convergent. This implies that

$$\lim_{n\to\infty}\overline{\varphi}_\alpha^n\Big(\max\big\{q(Tx_1, Ty_1, \alpha), q(Tz_1, Ty_1, \alpha)\big\}\Big) = 0. \tag{3.24}$$

From (3.23), (3.24) and the continuity of T, we have

$$d_\alpha(Tx, Ty) = \lim_{n\to\infty} d_\alpha(Tx_{n+1}, Ty_{n+1}) = 0$$

and

$$d_\alpha(Tz, Ty) = \lim_{n\to\infty} d_\alpha(Tz_{n+1}, Ty_{n+1}) = 0.$$

Thus, $d_\alpha(Tx, Ty) = d_\alpha(Tz, Ty) = 0$ for all $\alpha \in I$. This implies that $Tx = Ty$ and $Tz = Ty$. Since T is an ICS mapping we have $x = y$ and $z = y$, that is, $F(x, x, x) = x$. □

Now we state some examples showing that our results are effective. We denote $\mathbb{R}_+ = [0, +\infty)$ and $\mathbb{N}^* = \{1, 2, \dots\}$.

Example 3.6. Let $X = \mathbb{R}^\infty = \big\{x = \{x_n\} : x_n \in \mathbb{R}, n = 1, 2, \dots\big\}$ and the mapping $P_n : X \to \mathbb{R}$ defined by $P_n(x) = P_n\big(\{x_n\}\big) = x_n$ for each $n = 1, 2, \dots$ Let $I = \mathbb{N}^* \times \mathbb{R}_+$ be the index set and the family of pseudometrics on X defined by

$$d_{(n,r)}(x, y) = r\big|P_n(x) - P_n(y)\big|, (n, r) \in I$$

for every $x, y \in X$. Then $\{d_{(n,r)} : (n,r) \in I\}$ generates the uniform structure on X. We consider the partial ordered " $\leq$ " on X which defined by $x \leq y \Leftrightarrow x_n \leq y_n$ for every $n = 1, 2, \ldots$

Let $T : X \to X$, and $F : X^3 \to X$ are maps defined by

$$Tx = \Big\{ \frac{x_1}{2}, \frac{x_2}{2}, \ldots \Big\}$$

and

$$F(x,y,z) = \Big\{1, 1 + \big(1 - \frac{1}{2}\big)\frac{x_2 - 2y_2 + z_2}{2}, 1 + \big(1 - \frac{1}{3}\big)\frac{x_3 - 2y_3 + z_3}{2}, \ldots \Big\}.$$

It is easy to see that T is ICS. Now we claim that F satisfies Theorem 3.1. For this, for every $(n,r) \in I$, we put $\varphi_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1} t$ for every $t \geq 0$, and denote by $j : I \to I$ a map defined by $j(n,r) = \big(n, 2r(1 - \frac{1}{2n})\big)$ for every $(n,r) \in I$. It is easy to see that $\varphi_{j^k(n,r)}(t) = \dfrac{2(n-1)}{2n-1} t = \varphi_{(n,r)}(t)$ for every $k = 0, 1, 2, \ldots$ Now, we fix the functions $\overline{\varphi}_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1} t$, for every $t \geq 0$ and $(n,r) \in I$. Then, we have

$$\sup \big\{ \varphi_{j^k(n,r)}(t) : k = 0, 1, 2, \ldots \big\} \leq \overline{\varphi}_{(n,r)}(t), \quad \text{for all } t \geq 0$$

and $\dfrac{\overline{\varphi}_{(n,r)}(t)}{t} = \dfrac{2(n-1)}{2n-1}$ is monotone non-decreasing. Next, we show that F has the mixed monotone property. Indeed, if $x^1, x^2, y, z \in X$ and $x^1 \leq x^2$ then $x^1_n \leq x^2_n$ for every $n = 1, 2, \ldots$ It follows that $x^1_n - 2y_n + z_n \leq x^2_n - 2y_n + z_n$, for all n. Hence

$$\big(1 - \frac{1}{n}\big)\frac{x^1_n - 2y_n + z_n}{2} \leq \big(1 - \frac{1}{n}\big)\frac{x^2_n - 2y_n + z_n}{2}$$

or

$$P_n\big(F(x^1, y, z)\big) \leq P_n\big(F(x^2, y, z)\big) \text{ for all } n.$$

Thus $F(x^1, y, z) \leq F(x^2, y, z)$.

Similarly, if $x, y, z^1, z^2 \in X$ and $z^1 \leq z^2$ then we have $F(x, y, z^1) \leq F(x, y, z^2)$.

Now, if $x, y^1, y^2, z \in X$ and $y^1 \leq y^2$ then $y^1_n \leq y^2_n$ for every $n = 1, 2, \ldots$ This implies that $x_n - 2y^1_n + z_n \geq x_n - 2y^2_n + z_n$, for all n. It follows that $\big(1 - \frac{1}{n}\big)\dfrac{x_n - 2y^1_n + z_n}{2} \geq \big(1 - \frac{1}{n}\big)\dfrac{x_n - 2y^2_n + z_n}{2}$ for all n. Hence $F(x, y^1, z) \geq F(x, y^2, z)$.

This proves that F has the mixed monotone property.

Now, we show that F satisfies the contractive condition (3.1) with φ_α and j above mentioned. Indeed, if $x \leq u,\ y \geq v,\ z \leq w$ then

$$\begin{aligned}
&d_{(n,r)}\big(TF(x,y,z), TF(u,v,w)\big) \\
&= r\big|P_n\big(TF(x,y,z)\big) - P_n\big(TF(u,v,w)\big)\big| \\
&= r\Big|\frac{1}{2}\big(1 - \frac{1}{n}\big)\frac{x_n - 2y_n + z_n}{2} - \frac{1}{2}\big(1 - \frac{1}{n}\big)\frac{u_n - 2v_n + w_n}{2}\Big| \\
&= r\frac{n-1}{4n}\big(u_n - x_n + 2(y_n - v_n) + w_n - z_n\big),
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned} d_{j(n,r)}(Tx,Tu) &= d_{\left(n,2r(1-\frac{1}{2n})\right)}(Tx,Tu) \\ &= 2r\big(1-\frac{1}{2n}\big)\Big|\frac{1}{2}x_n-\frac{1}{2}u_n\Big| = r\frac{2n-1}{2n}(u_n-x_n), \end{aligned} \tag{3.26}$$

$$\begin{aligned} d_{j(n,r)}(Ty,Tv) &= d_{\left(n,2r(1-\frac{1}{2n})\right)}(Ty,Tv) \\ &= 2r\big(1-\frac{1}{2n}\big)\Big|\frac{1}{2}y_n-\frac{1}{2}v_n\Big| = r\frac{2n-1}{2n}(y_n-v_n), \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} d_{j(n,r)}(Tz,Tw) &= d_{\left(n,2r(1-\frac{1}{2n})\right)}(Tz,Tw) \\ &= 2r\big(1-\frac{1}{2n}\big)\Big|\frac{1}{2}z_n-\frac{1}{2}w_n\Big| = r\frac{2n-1}{2n}(w_n-z_n). \end{aligned} \tag{3.28}$$

Since (3.26)-(3.28), we have

$$\begin{aligned} &\varphi_{(n,r)}\Big(\max\big\{d_{j(n,r)}(Tx,Tu),d_{j(n,r)}(Ty,Tv),d_{j(n,r)}(Tz,Tw)\big\}\Big) \\ &= \varphi_{(n,r)}\Big(r\frac{2n-1}{2n}\max\{u_n-x_n,y_n-v_n,w_n-z_n\}\Big) \\ &= \frac{2(n-1)}{2n-1}r\frac{2n-1}{2n}\max\{u_n-x_n,y_n-v_n,w_n-z_n\} \\ &= \frac{n-1}{n}\max\{u_n-x_n,y_n-v_n,w_n-z_n\}. \end{aligned} \tag{3.29}$$

It follows from (3.25) and (3.29) that

$$\begin{aligned} &d_{(n,r)}\big(TF(x,y,z),TF(u,v,w)\big) \\ &\le \varphi_{(n,r)}\Big(\max\big\{d_{j(n,r)}(Tx,Tu),d_{j(n,r)}(Ty,Tv),d_{j(n,r)}(Tz,Tw)\big\}\Big). \end{aligned}$$

Now, if we fix $x^0=y^0=z^0=(1,1,\dots)$ then

$$x^0=y^0=z^0=F(x^0,y^0,z^0)=F(y^0,x^0,y^0)=F(z^0,y^0,x^0)$$

and

$$\begin{aligned} \max\Big\{&d_{j^k(n,r)}\big(Tx^0,TF(x^0,y^0,z^0)\big),d_{j^k(n,r)}\big(Ty^0,TF(y^0,x^0,y^0)\big), \\ &d_{j^k(n,r)}\big(Tz^0,TF(z^0,y^0,x^0)\big)\Big\}=0<\infty. \end{aligned}$$

Finally, it is easy to see that F is continuous. Hence, the conditions of Theorem 3.1 are fulfilled for F and F has at least of the triple fixed point.

Example 3.7. Let $X=\mathbb{R}^\infty=\big\{x=\{x_n\}:x_n\in\mathbb{R},n=1,2,\dots\big\}$ and the mapping $P_n:X\to\mathbb{R}$ defined by $P_n(x)=P_n(\{x_n\})=x_n$ for each $n=1,2,\dots$ Let $I=\mathbb{N}^*\times\mathbb{R}_+$ be the index set and the family of pseudometrics on X defined by

$$d_{(n,r)}(x,y)=r\big|P_n(x)-P_n(y)\big|,(n,r)\in I$$

for every $x,y\in X$. Then $\{d_{(n,r)}:(n,r)\in I\}$ generates the uniform structure on X. We consider the partial ordered " $\le$ " on X which defined by $x\le y\Leftrightarrow x_n\le y_n$ for every $n=1,2,\dots$

Consider the map $T=id_X$ and $F:X^3\to X$ defined by

$$F(x,y,z)=\Big\{1,1+\big(1-\frac{1}{2}\big)\frac{x_2-2y_2+z_2}{4},1+\big(1-\frac{1}{3}\big)\frac{x_3-2y_3+z_3}{4},\dots\Big\}.$$

Let $\varphi_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1}t$ with $t \geq 0$ and $j : I \to I$ be defined $j(n,r) = \big(n, r(1-\frac{1}{2n})\big)$ for every $(n,r) \in I$. It is clearly that T is ICS. By the same computation as in Example 3.6, we can show that F satisfies Theorem 3.1 with $\overline{\varphi}_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1}t$.

Now, we check that X is j-bounded. Indeed, for each $(x,y) \in X$ we have

$$\begin{aligned} d_{j^k(n,r)}(x,y) &= d_{\big(n,r(1-\frac{1}{2n})^k\big)}(x,y) \\ &= r\big(1-\frac{1}{2n}\big)^k \big|P_n(x) - P_n(y)\big| \\ &\leq r\big|P_n(x) - P_n(y)\big| = q\big(x,y,(n,r)\big). \end{aligned}$$

This proves that X is j-bounded. It is easy to see that if $(x,y,z),(u,v,w) \in X^3$ then there exists $(a,b,c) \in X^3$ is comparable to them. Thus F satisfies Theorem 3.4. Hence, F has a unique triple fixed point, that is $x = y = z = \{1,1,\dots\}$.

Example 3.8. Let $X = \big\{x = \{x_n\} : x_n \in [1,8], n = 1,2,\dots\big\}$ and the mapping $P_n : X \to \mathbb{R}$ defined by $P_n(x) = P_n\big(\{x_n\}\big) = x_n$ for each $n = 1,2,\dots$ Let $I = \mathbb{N}^* \times \mathbb{R}_+$ be the index set and the family of pseudometrics on X defined by

$$d_{(n,r)}(x,y) = r\big|P_n(x) - P_n(y)\big|, (n,r) \in I$$

for every $x,y \in X$. Then $\{d_{(n,r)} : (n,r) \in I\}$ generates the uniform structure on X. We consider the partial ordered " $\leq$ " on X which defined by $x \leq y \Leftrightarrow x_n \leq y_n$ for every $n = 1,2,\dots$

Let $T : X \to X$, and $F : X^3 \to X$ are maps defined by

$$Tx = \Big\{\ln x_1 + 1, \ln x_2 + 1, \dots\Big\}$$

and

$$F(x,y,z) = \left\{2, 2\left(\frac{\sqrt{x_2 z_2}}{y_2}\right)^{\frac{1}{3}\left(1-\frac{1}{2}\right)}, 2\left(\frac{\sqrt{x_3 z_3}}{y_3}\right)^{\frac{1}{3}\left(1-\frac{1}{3}\right)}, \dots\right\}.$$

It is easy to see that T is ICS. Now we claim that F satisfies Theorem 3.4. For this, for every $(n,r) \in I$, we put $\varphi_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1}t$, for every $t \geq 0$, and denote by $j : I \to I$ a map defined by $j(n,r) = \big(n, r(1-\frac{1}{2n})\big)$ for every $(n,r) \in I$. It is easy to see that

$$\varphi_{j^k(n,r)}(t) = \frac{2(n-1)}{2n-1}t = \varphi_{(n,r)}(t), \text{ for every } t \geq 0, \text{and } k = 0,1,2,\dots$$

Now, we fix the functions $\overline{\varphi}_{(n,r)}(t) = \dfrac{2(n-1)}{2n-1}t$, for every $t \geq 0$ and $(n,r) \in I$. Then, we have

$$\sup\big\{\varphi_{j^k(n,r)}(t) : k = 0,1,2,\dots\big\} \leq \overline{\varphi}_{(n,r)}(t), \text{ for every } t \geq 0$$

and $\dfrac{\overline{\varphi}_{(n,r)}(t)}{t} = \dfrac{2(n-1)}{2n-1}$ is monotone non-decreasing.

Next, we show that F has the mixed monotone property. Indeed, if $x^1, x^2, y, z \in X$ and $x^1 \leq x^2$ then $x_n^1 \leq x_n^2$ for every $n = 1,2,\dots$ It follows that $\frac{\sqrt{x_n^1 z_n}}{y_n} \leq \frac{\sqrt{x_n^2 z_n}}{y_n}$, for every n. Hence

$$2\left(\frac{\sqrt{x_n^1 z_n}}{y_n}\right)^{\frac{1}{3}\left(1-\frac{1}{n}\right)} \leq 2\left(\frac{\sqrt{x_n^2 z_n}}{y_n}\right)^{\frac{1}{3}\left(1-\frac{1}{n}\right)}$$

or

$$P_n\big(F(x^1, y, z)\big) \leq P_n\big(F(x^2, y, z)\big) \text{ for all } n.$$

Thus $F(x^1, y, z) \leq F(x^2, y, z)$.

Similarly, if $x, y, z^1, z^2 \in X$ and $z^1 \leq z^2$ then we have $F(x, y, z^1) \leq F(x, y, z^2)$.

Now, if $x, y^1, y^2, z \in X$ and $y^1 \leq y^2$ then $y_n^1 \leq y_n^2$ for every $n = 1, 2, \ldots$ This implies that $\frac{\sqrt{x_n z_n}}{y_n^1} \geq \frac{\sqrt{x_n z_n}}{y_n^2}$, for all n. It follows that $2\left(\frac{\sqrt{x_n z_n}}{y_n^1}\right)^{\frac{1}{3}\left(1-\frac{1}{n}\right)} \geq 2\left(\frac{\sqrt{x_n z_n}}{y_n^2}\right)^{\frac{1}{3}\left(1-\frac{1}{n}\right)}$ for all n. Thus, $F(x, y^1, z) \geq F(x, y^2, z)$.

This proves that F has the mixed monotone property.

Now, we show that F satisfies the contractive condition (3.1) with φ_α and j above mentioned. Indeed, if $x \leq u,\ y \geq v,\ z \leq w$ then

$$\begin{aligned}
&d_{(n,r)}\big(TF(x,y,z), TF(u,v,w)\big) \\
&= r\big|P_n\big(TF(x,y,z)\big) - P_n\big(TF(u,v,w)\big)\big| \\
&= r\left|\ln 2\left(\frac{\sqrt{x_n z_n}}{y_n}\right)^{\frac{1}{3}\left(1-\frac{1}{n}\right)} + 1 - \ln 2\left(\frac{\sqrt{u_n w_n}}{v_n}\right)^{\frac{1}{3}\left(1-\frac{1}{n}\right)} - 1\right| \\
&= r\frac{1}{3}\big(1-\frac{1}{n}\big)\Big(\frac{1}{2}(\ln u_n - \ln x_n) + (\ln y_n - \ln v_n) + \frac{1}{2}(\ln w_n - \ln z_n)\Big) \\
&\leq r\frac{n-1}{n}\frac{(\ln u_n - \ln x_n) + (\ln y_n - \ln v_n) + (\ln w_n - \ln z_n)}{3},
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
d_{j(n,r)}(Tx, Tu) &= d_{\left(n, r(1-\frac{1}{2n})\right)}(Tx, Tu) \\
&= r\big(1-\frac{1}{2n}\big)|\ln x_n - \ln u_n| = r\frac{2n-1}{2n}(\ln u_n - \ln x_n),
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
d_{j(n,r)}(Ty, Tv) &= d_{\left(n, r(1-\frac{1}{2n})\right)}(Ty, Tv) \\
&= r\big(1-\frac{1}{2n}\big)|\ln y_n - \ln v_n| = r\frac{2n-1}{2n}(\ln y_n - \ln v_n),
\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}
d_{j(n,r)}(Tz, Tw) &= d_{\left(n, r(1-\frac{1}{2n})\right)}(Tz, Tw) \\
&= r\big(1-\frac{1}{2n}\big)|\ln z_n - \ln w_n| = r\frac{2n-1}{2n}(\ln w_n - \ln z_n).
\end{aligned} \tag{3.33}$$

Since (3.31)-(3.33), we have

$$\begin{aligned}
&\varphi_{(n,r)}\Big(\max\big\{d_{j(n,r)}(Tx,Tu),d_{j(n,r)}(Ty,Tv),d_{j(n,r)}(Tz,Tw)\big\}\Big)\\
&=\varphi_{(n,r)}\Big(r\frac{2n-1}{2n}\max\{\ln u_n-\ln x_n,\ln y_n-\ln v_n,\ln w_n-\ln z_n\}\Big)\\
&=\frac{2(n-1)}{2n-1}r\frac{2n-1}{2n}\max\{\ln u_n-\ln x_n,\ln y_n-\ln v_n,\ln w_n-\ln z_n\}\\
&=r\frac{n-1}{n}\max\{\ln u_n-\ln x_n,\ln y_n-\ln v_n,\ln w_n-\ln z_n\}.
\end{aligned} \tag{3.34}$$

It follows from (3.30) and (3.34) that

$$\begin{aligned}
&d_{(n,r)}\big(TF(x,y,z),TF(u,v,w)\big)\\
&\le\varphi_{(n,r)}\Big(\max\big\{d_{j(n,r)}(Tx,Tu),d_{j(n,r)}(Ty,Tv),d_{j(n,r)}(Tz,Tw)\big\}\Big).
\end{aligned}$$

Now, if we fix $x^0=y^0=z^0=(2,2,\dots)$ then

$$x^0=y^0=z^0=F(x^0,y^0,z^0)=F(y^0,x^0,y^0)=F(z^0,y^0,x^0)$$

and

$$\max\Big\{d_{j^k(n,r)}\big(Tx^0,TF(x^0,y^0,z^0)\big),\ d_{j^k(n,r)}\big(Ty^0,TF(y^0,x^0,y^0)\big),\ d_{j^k(n,r)}\big(Tz^0,TF(z^0,y^0,x^0)\big)\Big\}=0<\infty.$$

Finally, it is easy to see that F is continuous. By the same argument as in the Example 3.7, we obtain that X is j-bounded. It implies that the conditions of Corollary 3.5 are fulfilled for F. Thus, F has a unique fixed point, that is $x=\{2,2,\dots\}$.

Remark 3.9. 1) It is not difficult to see that in Example 3.6 X is not j-bounded. Indeed, we have

$$j(n,r)=\big(n,2r(1-\frac{1}{2n})\big),$$

$$j^2(n,r)=j\big(n,2r(1-\frac{1}{2n})\big)=\big(n,2^2r(1-\frac{1}{2n})^2\big),$$

and by induction we have

$$j^k(n,r)=\big(n,2^kr(1-\frac{1}{2n})^k\big),$$

for every $k=1,2,\dots$ Thus, for each $x,y\in X$, we have

$$\begin{aligned}
d_{j^k(n,r)}(x,y)&=d_{\left(n,2^kr(1-\frac{1}{2n})^k\right)}(x,y)\\
&=2^kr\big(1-\frac{1}{2n}\big)^k\big|P_n(x)-P_n(y)\big|\\
&=r\big(\frac{2n-1}{n}\big)^k\big|P_n(x)-P_n(y)\big|.
\end{aligned}$$

Since $\lim_{k\to\infty}(\frac{2n-1}{n})^k=\infty$ for each $n>1$, we can conclude that there no $q\big(x,y,(n,r)\big)<+\infty$ such that $d_{j^k(n,r)}(x,y)<q\big(x,y,(n,r)\big)$ for every $k=0,1,2,\dots$ This proves that X is not j-bounded.

In fact, F have more than one triple fixed point. For this, we consider

$$x=\{1,x_2,1,1,\dots\},\ y=\{1,y_2,1,1,\dots\},\ z=\{1,z_2,1,1,\dots\},$$

with $x_2+y_2=2$ and $z_2=x_2$. It is easy to check that (x,y,z) are triple fixed points of F.

2) It follows from 1) that we can not omit the j-bounded property of X in the Theorem 3.4.

Finally, we give a example which shows that if T is not an ICS mapping, then the conclusion of Theorem 3.1 fails.

Example 3.10. Let $X=\mathbb{R}^\infty=\big\{x=\{x_n\}:x_n\in\mathbb{R}, n=1,2,\dots\big\}$ and the mapping $P_n:X\to\mathbb{R}$ defined by $P_n(x)=P_n\big(\{x_n\}\big)=x_n$ for each $n=1,2,\dots$ Let $I=\mathbb{N}^*$ be the index set and the family of pseudometrics on X defined by

$$d_n(x,y)=\big|P_n(x)-P_n(y)\big|,\ n\in I$$

for every $x,y\in X$. Then $\{d_n:n\in I\}$ generates the uniform structure on X. We consider the partial ordered " $\le$ " on X which defined by $x\le y\Leftrightarrow x_n\le y_n$ for every $n=1,2,\dots$

Let $T:X\to X$, and $F:X^3\to X$ are maps defined by

$$Tx=\{1,1,\dots\}$$

and

$$F(x,y,z)=\big\{2x_1-y_1+1, 2x_2-y_2+1,\dots\big\}.$$

Now we claim that F satisfies Theorem 3.1. For this, for every $n\in\mathbb{N}^*$ we put $\varphi_n(t)=\dfrac{3}{4}t$, for every $t\ge 0$, and denote by $j:I\to I$ a map defined by $j(n)=n$ for every $n\in I$. It is easy to see that $\varphi_{j^k(n)}(t)=\varphi_n(t)=\dfrac{3}{4}t$, for every $k=0,1,2,\dots$ Now, we fix the functions $\overline{\varphi}_n(t)=\dfrac{3}{4}t$, for every $t\ge 0$ and $n\in I$. Then, we have

$$\sup\big\{\varphi_{j^k(n)}(t):k=0,1,2,\dots\big\}=\varphi_n(t)\le\overline{\varphi}_n(t),\ \text{for all } t\ge 0$$

and $\dfrac{\overline{\varphi}_n(t)}{t}=\dfrac{3}{4}$ is monotone non-decreasing.

Firstly, we show that F has the mixed monotone property. Indeed, if $x^1,x^2,y,z\in X$ and $x^1\le x^2$ then $x^1_n\le x^2_n$ for every $n=1,2,\dots$ It follows that $x^1_n-2y_n+1\le x^2_n-2y_n+1$, or

$$P_n\big(F(x^1,y,z)\big)\le P_n\big(F(x^2,y,z)\big)\text{ for all } n.$$

Thus $F(x^1,y,z)\le F(x^2,y,z)$.

Similarly, if $x,y,z^1,z^2\in X$ and $z^1\le z^2$ then we have $F(x,y,z^1)\le F(x,y,z^2)$.

Now, if $x,y^1,y^2,z\in X$ and $y^1\le y^2$ then $y^1_n\le y^2_n$ for every $n=1,2,\dots$ This implies that $x_n-2y^1_n+1\ge x_n-2y^2_n+1$, for all n. Hence $F(x,y^1,z)\ge F(x,y^2,z)$.

This proves that F has the mixed monotone property.

Now, we show that F satisfies the contractive condition (3.1) with φ_α and j above mentioned. Indeed, if $x\le u, y\ge v, z\le w$ then

$$\begin{aligned} d_n\big(TF(x,y,z),TF(u,v,w)\big)&=\big|P_n\big(TF(x,y,z)\big)-P_n\big(TF(u,v,w)\big)\big|\\ &=\big|1-1\big|=0, \end{aligned}\tag{3.35}$$

and

$$d_{j(n)}(Tx,Tu)=d_n(Tx,Tu)=\big|P_n(Tx)-P_n(Tu)\big|=\big|1-1\big|=0,$$

$$d_{j(n)}(Ty,Tv) = d_{j(n)}(Tz,Tw) = 0,$$

$$\varphi_n\Big(\max\big\{d_{j(n)}(Tx,Tu), d_{j(n)}(Ty,Tv), d_{j(n)}(Tz,Tw)\big\}\Big) = 0. \tag{3.36}$$

It follows from (3.35) and (3.36) that

$$\begin{aligned} &d_n\big(TF(x,y,z), TF(u,v,w)\big) \\ &\leq \varphi_n\Big(\max\big\{d_{j(n)}(Tx,Tu), d_{j(n)}(Ty,Tv), d_{j(n)}(Tz,Tw)\big\}\Big). \end{aligned}$$

Now, if we fix $x^0 = z^0 = (1,1,\ldots)$, $y^0 = (0,0,\ldots)$ then

$$x^0 \leq F(x^0,y^0,z^0),\ y^0 \geq F(y^0,x^0,y^0),\ z^0 \leq F(z^0,y^0,x^0)$$

and

$$\begin{aligned} \max\Big\{ &d_{j^k(n)}\big(Tx^0, TF(x^0,y^0,z^0)\big),\ d_{j^k(n)}\big(Ty^0, TF(y^0,x^0,y^0)\big), \\ &d_{j^k(n)}\big(Tz^0, TF(z^0,y^0,x^0)\big)\Big\} = 0 < \infty. \end{aligned}$$

It is easy to see that F is continuous. Hence, the conditions of Theorem 3.1 are fulfilled for F. Since $Tx = \{1,1,\ldots\}$ for all $x \in X$, it is easy to see that T is not ICS mapping. However, F has no triple fixed point.

4. Applications to nonlinear integral equations

In this section, we wish to investigate the existence of a unique solution to a class of nonlinear integral equations, as an application of the tripled fixed point theorems proved in the previous section.

Let us consider the following integral equations

$$\begin{aligned} x(t) = k(t) + \int_0^{\Delta(t)} &\big[K_1(t,s) + K_2(t,s) + K_3(t,s)\big] \\ &\times \Big(f\big(s,x(s)\big) + g\big(s,x(s)\big) + h\big(s,x(s)\big)\Big) ds, \end{aligned} \tag{4.1}$$

where $K_1, K_2, K_3 \in C\big(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}\big)$, $f, g, h \in C\big(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}\big)$, and an unknown function $x(t) \in C\big(\mathbb{R}_+, \mathbb{R}\big)$. The deviation $\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, in general case, unbounded. Note that, since deviation $\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ is unbounded, we can not apply the known tripled fixed point theorems in metric space (see [5], [9]) for the above integral equations.

Adopting in [17], we assume that the functions K_1, K_2, K_3, f, g, h fulfill the following conditions

Assumption 4.1. A) $K_1(t,s) \geq 0$, $K_2(t,s) \leq 0$ and $K_3(t,s) \geq 0$ for all $t, s \geq 0$.

B) For each compact subset $K \subset \mathbb{R}$, there exist the positive numbers λ, μ, η and $\varphi_K \in \Phi$ such that for all $x, y \in \mathbb{R}$, $x \geq y$ and for all $t \in K$,

$$0 \leq f(t,x) - f(t,y) \leq \lambda \varphi_K\big(x-y\big),$$

$$-\mu \varphi_K\big(x-y\big) \leq g(t,x) - g(t,y) \leq 0,$$

$$0 \leq h(t,x) - h(t,y) \leq \eta \varphi_K\big(x-y\big)$$

and

$$\max(\lambda,\mu,\eta) \sup_{t\in K} \int_0^{\Delta(t)} \big(K_1(t,s) - K_2(t,s) + K_3(t,s)\big) ds \leq \frac{1}{6}.$$

C) For each compact subset $K \subset \mathbb{R}$, there exists a compact set $\overline{K} \subset \mathbb{R}$ such that for all $n \in \mathbb{N}$,

$$\Delta^n(K) \subset \overline{K},$$

where $\Delta^0(t) = t,\ \Delta^n(t) = \Delta\big(\Delta^{n-1}(t)\big)$, for all $t \geq 0$ and $n = 1, 2, \ldots$

D) For each compact subset $K \subset \mathbb{R}$, there exists $\overline{\varphi}_K \in \Phi$ such that $\dfrac{\overline{\varphi}_K(t)}{t}$ is non-decreasing and

$$\varphi_{\Delta^n(K)}(t) \leq \overline{\varphi}_K(t)$$

for all $n \in \mathbb{N}$ and for all $t \geq 0$.

Definition 4.2. An element $(\alpha, \beta, \gamma) \in C\big(\mathbb{R}_+, \mathbb{R}\big) \times C\big(\mathbb{R}_+, \mathbb{R}\big) \times C\big(\mathbb{R}_+, \mathbb{R}\big)$ is a *tripled lower* and *upper solution* of the integral equation (4.1) if for any $t \in \mathbb{R}_+$ we have $\alpha(t) \leq \beta(t)$, $\gamma(t) \leq \beta(t)$ and

$$\begin{aligned}\alpha(t) \leq k(t) &+ \int_0^{\Delta(t)} K_1(t,s)\Big(f\big(s,\alpha(s)\big) + g\big(s,\beta(s)\big) + h\big(s,\gamma(s)\big)\Big)ds\\ &+ \int_0^{\Delta(t)} K_2(t,s)\Big(f\big(s,\beta(s)\big) + g\big(s,\alpha(s)\big) + h\big(s,\beta(s)\big)\Big)ds\\ &+ \int_0^{\Delta(t)} K_3(t,s)\Big(f\big(s,\gamma(s)\big) + g\big(s,\beta(s)\big) + h\big(s,\alpha(s)\big)\Big)ds,\end{aligned}$$

$$\begin{aligned}\beta(t) \geq k(t) &+ \int_0^{\Delta(t)} K_1(t,s)\Big(f\big(s,\beta(s)\big) + g\big(s,\alpha(s)\big) + h\big(s,\beta(s)\big)\Big)ds\\ &+ \int_0^{\Delta(t)} K_2(t,s)\Big(f\big(s,\alpha(s)\big) + g\big(s,\beta(s)\big) + h\big(s,\alpha(s)\big)\Big)ds\\ &+ \int_0^{\Delta(t)} K_3(t,s)\Big(f\big(s,\beta(s)\big) + g\big(s,\alpha(s)\big) + h\big(s,\beta(s)\big)\Big)ds\end{aligned}$$

and

$$\begin{aligned}\gamma(t) \leq k(t) &+ \int_0^{\Delta(t)} K_1(t,s)\Big(f\big(s,\gamma(s)\big) + g\big(s,\beta(s)\big) + h\big(s,\alpha(s)\big)\Big)ds\\ &+ \int_0^{\Delta(t)} K_2(t,s)\Big(f\big(s,\beta(s)\big) + g\big(s,\gamma(s)\big) + h\big(s,\beta(s)\big)\Big)ds\\ &+ \int_0^{\Delta(t)} K_3(t,s)\Big(f\big(s,\alpha(s)\big) + g\big(s,\beta(s)\big) + h\big(s,\gamma(s)\big)\Big)ds.\end{aligned}$$

Theorem 4.3. *Consider the integral equation (4.1) with* $K_1, K_2, K_3 \in C\big(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}\big)$ *and* $f, g, h \in C\big(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}\big)$ *and* $k \in C\big(\mathbb{R}_+, \mathbb{R}\big)$ *and suppose that Assumption 4.1 is fulfilled. Then the existence of a tripled lower and upper solution for (4.1) provides the existence of a unique solution of (4.1) in* $C\big(\mathbb{R}_+, \mathbb{R}\big)$.

Proof. Let $X = C(\mathbb{R}_+, \mathbb{R})$. Then, X is a partially ordered set if we defined the following order relation in X:

$$x,\ y \in X, \quad x \leq y \Leftrightarrow x(t) \leq y(t),\ \text{for every } t \in \mathbb{R}_+.$$

For each a compact subset $K \subset \mathbb{R}$, we define

$$p_K(f) = \sup\big\{|f(t)| : t \in K\big\},\ \text{for all } f \in C(\mathbb{R}_+, \mathbb{R}).$$

It is known that the family of seminorms $\{p_K\}$, where K runs over all compact subsets of $\mathbb{R}$, defines a locally convex Hausdorff topology of the space. Hence, X is a Hausdorff sequentially uniform space whose uniformity is generated by the family of pseudometrics

$$d_K(f,g) = p_K(f-g) = \sup\big\{|f(t)-g(t)| : t \in K\big\}.$$

Let us next define the map $j : I \to I$, where the index set I consists of all compact subsets of $\mathbb{R}_+$, by the following way: For an arbitrary compact set $K \subset \mathbb{R}_+$, we put $j(K) := \big[0, \max_{t\in K} \Delta(t)\big]$, and $j^n(K) = j\big(j^{n-1}(K)\big)$, for every $n \in \mathbb{N}$. Then, since $\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous the sets $j(K), j^2(K), j^3(K), \dots$ is also compact.

Consider the map $T = id_X$. It is easy to see that T is ICS. Define $F : X^3 \to X$ by

$$\begin{aligned} F(x,y,z)(t) = &\int_0^{\Delta(t)} K_1(t,s)\Big(f\big(s,x(s)\big) + g\big(s,y(s)\big) + h\big(s,z(s)\big)\Big)ds \\ &+ \int_0^{\Delta(t)} K_2(t,s)\Big(f\big(s,y(s)\big) + g\big(s,x(s)\big) + h\big(s,y(s)\big)\Big)ds \\ &+ \int_0^{\Delta(t)} K_3(t,s)\Big(f\big(s,z(s)\big) + g\big(s,y(s)\big) + h\big(s,x(s)\big)\Big)ds + k(t) \end{aligned}$$

for all $t \in \mathbb{R}_+$.

Now, we show that F has the mixed monotone property. Indeed, for $x_1, x_2 \in C(\mathbb{R}_+, \mathbb{R})$ and $x_1 \le x_2$, that is $x_1(t) \le x_2(t)$ for every $t \in \mathbb{R}_+$, by Assumption 4.1 we have

$$\begin{aligned} &F(x_1,y,z)(t) - F(x_2,y,z)(t) \\ =&\int_0^{\Delta(t)} K_1(t,s)\Big(f\big(s,x_1(s)\big) + g\big(s,y(s)\big) + h\big(s,z(s)\big)\Big)ds \\ &+ \int_0^{\Delta(t)} K_2(t,s)\Big(f\big(s,y(s)\big) + g\big(s,x_1(s)\big) + h\big(s,y(s)\big)\Big)ds \\ &+ \int_0^{\Delta(t)} K_3(t,s)\Big(f\big(s,z(s)\big) + g\big(s,y(s)\big) + h\big(s,x_1(s)\big)\Big)ds + k(t) \\ &- \int_0^{\Delta(t)} K_1(t,s)\Big(f\big(s,x_2(s)\big) + g\big(s,y(s)\big) + h\big(s,z(s)\big)\Big)ds \\ &- \int_0^{\Delta(t)} K_2(t,s)\Big(f\big(s,y(s)\big) + g\big(s,x_2(s)\big) + h\big(s,y(s)\big)\Big)ds \\ &- \int_0^{\Delta(t)} K_3(t,s)\Big(f\big(s,z(s)\big) + g\big(s,y(s)\big) + h\big(s,x_2(s)\big)\Big)ds - k(t) \\ =&\int_0^{\Delta(t)} K_1(t,s)\Big(f\big(s,x_1(s)\big) - f\big(s,x_2(s)\big)\Big)ds \\ &+ \int_0^{\Delta(t)} K_2(t,s)\Big(g\big(s,x_1(s)\big) - g\big(s,x_2(s)\big)\Big)ds \\ &+ \int_0^{\Delta(t)} K_3(t,s)\Big(h\big(s,x_1(s)\big) - h\big(s,x_2(s)\big)\Big)ds \le 0 \end{aligned}$$

for every $t \in \mathbb{R}_+$. This yields $F(x_1, y, z)(t) \le F(x_2, y, z)(t)$ for every $t \in \mathbb{R}_+$, that is $F(x_1, y, z) \le F(x_2, y, z)$.

By the same computation, we arrive at $F(x, y_1, z) \le F(x, y_2, z)$ if $y_1 \ge y_2$ and $F(x, y, z_1) \le F(x, y, z_2)$ if $z_1 \le z_2$. Hence, F has the mixed monotone property.

Next, we show that F satisfies the contractive condition 3.1) of Theorem 3.1. Indeed, for each compact subset K of $\mathbb{R}$ and for $x \ge u$, $y \le v$ and $z \ge w$, that is $x(t) \ge u(t)$, $y(t) \le v(t)$ and $z(t) \ge w(t)$ for every $t \in \mathbb{R}_+$, we have

$$\begin{aligned}
&d_K\big(F(x,y,z), F(u,v,w)\big)\\
&= \sup_{t\in K}\Big|F(x,y,z)(t) - F(u,v,w)(t)\Big|\\
&= \sup_{t\in K}\Big|\int_0^{\Delta(t)} K_1(t,s)\Big(f\big(s,x(s)\big) + g\big(s,y(s)\big) + h\big(s,z(s)\big)\Big)ds\\
&\quad + \int_0^{\Delta(t)} K_2(t,s)\Big(f\big(s,y(s)\big) + g\big(s,x(s)\big) + h\big(s,y(s)\big)\Big)ds\\
&\quad + \int_0^{\Delta(t)} K_3(t,s)\Big(f\big(s,z(s)\big) + g\big(s,y(s)\big) + h\big(s,x(s)\big)\Big)ds + k(t)\\
&\quad - \int_0^{\Delta(t)} K_1(t,s)\Big(f\big(s,u(s)\big) + g\big(s,v(s)\big) + h\big(s,w(s)\big)\Big)ds\\
&\quad - \int_0^{\Delta(t)} K_2(t,s)\Big(f\big(s,v(s)\big) + g\big(s,u(s)\big) + h\big(s,v(s)\big)\Big)ds\\
&\quad - \int_0^{\Delta(t)} K_3(t,s)\Big(f\big(s,w(s)\big) + g\big(s,v(s)\big) + h\big(s,u(s)\big)\Big)ds - k(t)\Big|\\
&= \sup_{t\in K}\Big|\int_0^{\Delta(t)} K_1(t,s)\Big[\Big(f\big(s,x(s)\big) - f\big(s,u(s)\big)\Big) + \Big(g\big(s,y(s)\big) - g\big(s,v(s)\big)\Big)\\
&\qquad\qquad + \Big(h\big(s,z(s)\big) - h\big(s,w(s)\big)\Big)\Big]ds\\
&\quad + \int_0^{\Delta(t)} K_2(t,s)\Big[\Big(f\big(s,y(s)\big) - f\big(s,v(s)\big)\Big) + \Big(g\big(s,x(s)\big) - g\big(s,u(s)\big)\Big)\\
&\qquad\qquad + \Big(h\big(s,y(s)\big) - h\big(s,v(s)\big)\Big)\Big]ds\\
&\quad + \int_0^{\Delta(t)} K_3(t,s)\Big[\Big(f\big(s,z(s)\big) - f\big(s,w(s)\big)\Big) + \Big(g\big(s,y(s)\big) - g\big(s,v(s)\big)\Big)\\
&\qquad\qquad + \Big(h\big(s,x(s)\big) - h\big(s,u(s)\big)\Big)\Big]ds\Big|\\
&= \sup_{t\in K}\Big|\int_0^{\Delta(t)} K_1(t,s)\Big[\Big(f\big(s,x(s)\big) - f\big(s,u(s)\big)\Big) - \Big(g\big(s,v(s)\big) - g\big(s,y(s)\big)\Big)\\
&\qquad\qquad + \Big(h\big(s,z(s)\big) - h\big(s,w(s)\big)\Big)\Big]ds\\
&\quad - \int_0^{\Delta(t)} K_2(t,s)\Big[\Big(f\big(s,v(s)\big) - f\big(s,y(s)\big)\Big) - \Big(g\big(s,x(s)\big) - g\big(s,u(s)\big)\Big)\\
&\qquad\qquad + \Big(h\big(s,v(s)\big) - h\big(s,y(s)\big)\Big)\Big]ds
\end{aligned}$$

$$+\int_0^{\Delta(t)} K_3(t,s)\Big[\Big(f\big(s,z(s)\big)-f\big(s,w(s)\big)\Big)-\Big(g\big(s,v(s)\big)-g\big(s,y(s)\big)\Big)$$
$$+\Big(h\big(s,x(s)\big)-h\big(s,u(s)\big)\Big)\Big]ds\Big|$$
$$\leq \sup_{t\in K}\Big|\int_0^{\Delta(t)} K_1(t,s)\Big[\lambda\varphi_K\big(x(s)-u(s)\big)+\mu\varphi_K\big(v(s)-y(s)\big)+\eta\varphi_K\big(z(s)-w(s)\big)\Big]ds$$
$$-\int_0^{\Delta(t)} K_2(t,s)\Big[\lambda\varphi_K\big(v(s)-y(s)\big)+\mu\varphi_K\big(x(s)-u(s)\big)+\eta\varphi_K\big(v(s)-y(s)\big)\Big]ds$$
$$+\int_0^{\Delta(t)} K_3(t,s)\Big[\lambda\varphi_K\big(z(s)-w(s)\big)+\mu\varphi_K\big(v(s)-y(s)\big)+\eta\varphi_K\big(x(s)-u(s)\big)\Big]ds\Big|$$
$$\leq 2\max\{\lambda,\mu,\eta\}\sup_{t\in K}\int_0^{\Delta(t)}\big[K_1(t,s)-K_2(t,s)+K_3(t,s)\big]\Big[\varphi_K\big(x(s)-u(s)\big)$$
$$+\varphi_K\big(v(s)-y(s)\big)+\varphi_K\big(z(s)-w(s)\big)\Big]ds$$
$$\leq 2\max\{\lambda,\mu,\eta\}\Big[\varphi_K\Big(\sup_{s\in[0,\max_{t\in K}\Delta(t)]}\big|x(s)-u(s)\big|\Big)$$
$$+\varphi_K\Big(\sup_{s\in[0,\max_{t\in K}\Delta(t)]}\big|v(s)-y(s)\big|\Big)+\varphi_K\Big(\sup_{s\in[0,\max_{t\in K}\Delta(t)]}\big|z(s)-w(s)\big|\Big)\Big]$$
$$\times\sup_{t\in K}\int_0^{\Delta(t)}\big[K_1(t,s)-K_2(t,s)+K_3(t,s)\big]ds$$
$$\leq 2\max\{\lambda,\mu,\eta\}\Big[\varphi_K\Big(\sup_{s\in j(K)}\big|x(s)-u(s)\big|\Big)+\varphi_K\Big(\sup_{s\in j(K)}\big|v(s)-y(s)\big|\Big)$$
$$+\varphi_K\Big(\sup_{s\in j(K)}\big|z(s)-w(s)\big|\Big)\Big]\sup_{t\in K}\int_0^{\Delta(t)}\big[K_1(t,s)-K_2(t,s)+K_3(t,s)\big]ds$$
$$\leq 2.\frac{1}{6}\Big[\varphi_K\Big(d_{j(K)}(x,u)\Big)+\varphi_K\Big(d_{j(K)}(y,v)\Big)+\varphi_K\Big(d_{j(K)}(z,w)\Big)\Big]$$
$$\leq\frac{1}{3}\Big[3\varphi_K\Big(\max\big\{d_{j(K)}(x,u),\ d_{j(K)}(y,v),\ d_{j(K)}(y,v)\big\}\Big)\Big]$$
$$=\varphi_K\Big(\max\big\{d_{j(K)}(x,u),\ d_{j(K)}(y,v),\ d_{j(K)}(y,v)\big\}\Big).$$

Condition 2) in Theorem 3.1 is satisfied by D) of Assumption 4.1.

Now, let us (α,β,γ) be a tripled lower and upper solution of the integral equation of (4.1). Then, we have

$$\alpha(t)\leq F(\alpha,\beta,\gamma)(t),\ \beta(t)\geq F(\beta,\alpha,\beta)(t)\quad\text{and}\quad\gamma(t)\leq F(\gamma,\beta,\alpha)(t)$$

for all $t\in\mathbb{R}_+$, that is $\alpha\leq F(\alpha,\beta,\gamma)$, $\beta\geq F(\beta,\alpha,\beta)$ and $\gamma\leq F(\gamma,\beta,\alpha)$. Moreover, for each compact subset $K\subset\mathbb{R}$, by the continuity and assumption, we have

$$\max\Big\{d_{j^n(K)}\big(\alpha,F(\alpha,\beta,\gamma)\big),d_{j^n(K)}\big(\beta,F(\beta,\alpha,\beta)\big),d_{j^n(K)}\big(\gamma,F(\gamma,\beta,\alpha)\big)\Big\}$$
$$\leq\ \max\Big\{d_{[0,\max_{s\in\overline{K}}\Delta(s)]}\big(\alpha,F(\alpha,\beta,\gamma)\big),d_{[0,\max_{s\in\overline{K}}\Delta(s)]}\big(\beta,F(\beta,\alpha,\beta)\big),$$
$$d_{[0,\max_{s\in\overline{K}}\Delta(s)]}\big(\gamma,F(\gamma,\beta,\alpha)\big)\Big\}<\infty.$$

Hence, condition 3) in Theorem 3.1 is satisfied.

Now, suppose that $\{u_n\}$ is a monotone non-decreasing sequence in X that converges to $u \in X$. Then for every $t \in \mathbb{R}_+$, the sequence of real numbers $u_1(t) \le u_2(t) \le \cdots \le u_n(t) \le \cdots$ converges to $u(t)$. Therefore, for every $t \in \mathbb{R}_+$, $n \in \mathbb{N}$, $u_n(t) \le u(t)$. Hence $u_n \le u$, for all $n \in \mathbb{N}$.

Similarly, we can verify that limit $v(t)$ of a monotone non-increasing sequence $v_n(t)$ in X is a lower bound for all elements in the sequence. That is, $v \le v_n$ for all n. Hence, the condition b) in Theorem 3.1 holds.

Using again assumption (C), we have

$$\begin{aligned} d_{j^n(K)}(x,y) &= \sup_{t\in j^n(K)} \big|x(t)-y(t)\big| \\ &\le \sup_{t\in[0,\max_{s\in\overline{K}}\Delta(s)]} \big|x(t)-y(t)\big| = d_{[0,\max_{s\in\overline{K}}\Delta(s)]}(x,y) < +\infty \end{aligned}$$

for all $n \in \mathbb{N}$. This implies that X is j-bounded.

Now, we define on X^3 the following partial order relation:
For $(x,y,z),\ (u,v,w) \in X^3$,

$$(x,y,z) \le (u,v,w) \Leftrightarrow x(t) \le u(t),\ y(t) \ge v(t) \text{ and } z(t) \le w(t)$$

for every $t \in \mathbb{R}_+$. Observe that for every $x,\ y,\ z \in X$, by the uniform topology of X, we easily see that $\max\{x(t),\ y(t),\ z(t)\}$, $\min\{x(t),\ y(t),\ z(t)\}$ for each $t \in \mathbb{R}_+$ are in X and are the upper and lower bounds of $x,\ y,\ z$, respectively in X. This follows that for every $(x,y,z),\ (u,v,w) \in X^3$, there exists a $\big(\max\{x,u\},\ \min\{y,v\},\ \max\{z,w\}\big) \in X^3$ which is comparable to (x,y,z) and (u,v,w).

Therefore, by applying Theorem 3.4, we can conclude that F has a unique tripled fixed point (x,y,z). Finally, since $\alpha \le \beta$, $\gamma \le \beta$ by Corollary 3.5, we have $x = y = z$, that is $x(t) = y(t) = z(t)$ for every $t \in \mathbb{R}_+$. Hence $F(x,x,x) = x$ and x is the unique solution of the equation (4.1). □

Acknowledgements

The author would like to thank the referee for very deep and useful comments, and his/her suggestions that contributed to the improvement of the manuscript.

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