Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 6 Issue 1 (2014), Pages 9-22

OSCILLATION CRITERIA FOR EVEN-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF MIXED TYPE

(COMMUNICATED BY IOANNIS P. STAVROULAKIS)

ETHIRAJU THANDAPANI, SANKARAPPAN PADMAVATHI AND SANDRA PINELAS

Abstract. This paper deals with oscillation criteria for even order nonlinear neutral mixed type differential equations of the form

$$\Big(a(t)(x(t)+bx(t-\tau_1)+cx(t+\tau_2))^{(n-1)}\Big)'+p(t)x^{\alpha}(t-\sigma_1)+q(t)x^{\beta}(t+\sigma_2)=0,$$

where $t \geq t_0$ and $n \geq 2$ is an even integer, $\alpha \geq 1$ and $\beta \geq 1$, are ratios of odd positive integers. The results are obtained both for the case $\int^{\infty} a^{-1}(t)dt = \infty$, and in case $\int^{\infty} a^{-1}(t)dt < \infty$. Some examples are given to illustrate our main results.

1. Introduction

In this paper, we study the oscillatory behavior of the following even order nonlinear neutral mixed type differential equation of the form

$$\Big(a(t)(x(t)+bx(t-\tau_1)+cx(t+\tau_2))^{(n-1)}\Big)'+p(t)x^{\alpha}(t-\sigma_1)+q(t)x^{\beta}(t+\sigma_2)=0,\ t \geq t_0, \tag{1.1}$$

where $n \geq 2$ is an even integer. We set $z(t) = x(t) + bx(t-\tau_1) + cx(t+\tau_2)$. Throughout this paper, we assume that

(C$_1$) $a \in C([t_0,\infty),\mathbb{R})$, $a(t) > 0$ and $a'(t) > 0$ for all $t \geq t_0$;
(C$_2$) $p, q \in C([t_0,\infty),\mathbb{R})$, $p(t) > 0$ and $q(t) > 0$ for all $t \geq t_0$;
(C$_3$) b and c are positive constants,$\tau_1, \tau_2, \sigma_1, \sigma_2$ are nonnegative constants and α and β are ratios of odd positive integers.

We shall consider the two cases:

$$\int_{t_0}^{\infty} \frac{1}{a(t)} = \infty, \tag{1.2}$$

2010 *Mathematics Subject Classification.* 34C15.
Key words and phrases. Oscillation, Even order, Mixed type,Nonlinear neutral differential equations.

Submitted June 10, 2013. Published January 5, 2014.

and

$$\int_{t_0}^{\infty} \frac{1}{a(t)} < \infty. \tag{1.3}$$

Differential equations with delayed and advanced arguments (also called mixed differential equations or equations with mixed arguments) occur in many problems of economy, biology and physics (see for example [3, 6, 9, 10, 16]), because differential equations with mixed arguments are much more suitable than delay differential equations for an adequate treatment of dynamic phenomena. The concept of delay is related to a memory of system, the past events are importance for the current behavior, and the concept of advance is related to a potential future events which can be known at the current time which could be useful for decision making. The study of various problems for differential equations with mixed arguments can be seen in [5, 8, 15, 17, 19, 24]. It is well known that the solutions of these types of equations cannot be obtained in closed form. In the absence of closed form solutions a rewarding alternative is to resort to the qualitative study of the solutions of these types of differential equations. But it is not quite clear how to formulate an initial value problem for such equations and existence and uniqueness of solutions becomes a complicated issue. To study the oscillation of solutions of differential equations, we need to assume that there exists a solution of such equation on the half line.

In [20] the authors established some oscillation results for the nth order $(n > 1)$ differential equations of mixed type

$$y^{(n)}(t) - \sum_{i=1}^{k} p_i^n y(t - n\tau_i) - \sum_{j=1}^{l} q_j^n y(t + n\sigma_j) = 0 \tag{1.4}$$

and

$$y^{(n)}(t) + \sum_{i=1}^{k} p_i^n y(t - n\tau_i) + \sum_{j=1}^{l} q_j^n y(t + n\sigma_j) = 0 \tag{1.5}$$

where $p_i, \tau_i, i = 1, 2, ..., k$ and $q_j, \sigma_j, j = 1, 2, ..., l$ are positive constants..

In [25] the author established some oscillation results for the solutions of the neutral equations of mixed type

$$\frac{d}{dt}(x(t) + cx(t - r)) + \sum_{i=1}^{k} p_i x(t - \tau_i) + \sum_{j=1}^{l} q_j x(t + \sigma_j) = 0 \tag{1.6}$$

and

$$\frac{d}{dt}(x(t) + cx(t - r)) - \sum_{i=1}^{k} p_i x(t + \tau_i) - \sum_{j=1}^{l} q_j x(t - \sigma_j = 0 \tag{1.7}$$

where $c \in \mathbb{R}, r \in (0, \infty), p_i, q_j \in (0, \infty)$ and $\tau_i, \sigma_j \in [0, \infty)$ for $i = 1, 2, ..., k, j = 1, 2, ..., l$.

Grace[11] obtained some oscillation theorems for the odd order neutral differential equation

$$(x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2))^{(n)} = q_1 x(t - \sigma_1) + q_2 x(t + \sigma_2), \; t \geq t_0 \tag{1.8}$$

where $n \geq 1$ is odd. In [13] the authors established some oscillation criteria for the following mixed neutral equation

$$(x(t) + p_1 x(t - \tau_1) + p_2 x(t + \tau_2))'' = q_1 x(t - \sigma_1) + q_2 x(t + \sigma_2),\ t \geq t_0 \tag{1.9}$$

with q_1 and q_2 are nonnegative real valued functions.

Zhang et al.[30] studied the even order nonlinear neutral functional equations

$$(x(t) + p(t)x(\tau(t)))^{(n)} + q(t)f(x(\sigma(t))) = 0,\ t \geq t_0, \tag{1.10}$$

where n is even, $0 \leq p(t) < 1$. The authors established a comparison theorem for (1.10) and obtained results which improved and generalized some known results.

In 2011, Zhang et al. [29] studied the oscillatory behavior of the following higher order half quasilinear delay differential equation

$$(r(t)(x^{(n-1)}(t))^{\alpha})' + q(t)x^{\beta}(\tau(t)) = 0,\ t \geq t_0, \tag{1.11}$$

under the condition $\int_{t_0}^{\infty} \frac{1}{r^{\frac{1}{\alpha}}} dt < \infty$. The authors obtained some sufficient conditions, which guarantee that every solution of (1.11) is oscillatory or tends to zero.

In 2012, Y.B.Sun, Z.L.Han, S.R.Sun, Ch.Zhang [26] studied the oscillation criteria for even order nonlinear neutral differential equations

$$(r(t)(x(t) + p(t)x(\tau(t)))^{(n-1)})' + q(t)f(x(\sigma(t))) = 0,\ t \geq t_0, \tag{1.12}$$

where $\int_{t_0}^{\infty} r^{-1}(t)dt = \infty, \int_{t_0}^{\infty} r^{-1}(t)dt < \infty, \tau(t) \leq t, \sigma(t) \leq t, 0 \leq p(t) \leq p_0 < \infty$. The authors obtained some oscillation theorems, which guarantee that every solution of equation (1.12) is oscillatory. For the particular case when $n = 2$, equation (1.1) reduces to the following equation

$$(r(t)(x(t) + p(t)x(\tau(t)))')' + q(t)f(x(\sigma(t))) = 0,\ t \geq t_0. \tag{1.13}$$

Han et al.[14] established the oscillation criteria for the solutions of (1.13), where $\int_{t_0}^{\infty} r^{-1}(t)dt = \infty, \tau(t) \leq t, \sigma(t) \leq t, 0 \leq p(t) \leq p_0 < \infty$.

In [28] the authors obtained several sufficient conditions for the oscillation of solutions of second order neutral differential equation of the form

$$(a(t)([x(t)+b(t)x(t-\sigma_1)+c(t)x(t+\sigma_2)]^{\alpha})')'+q(t)x^{\beta}(t-\tau_1)+p(t)x^{\gamma}(t+\tau_2) = 0,\ t \geq t_0 \tag{1.14}$$

where $\int_{t_0}^{\infty} a^{-1}(t)dt = \infty, 0 \leq b(t) \leq b, 0 \leq c(t) \leq c$ and p and q are nonnegative continuous real valued functions.

Motivated by the above observations, in this paper we establish some sufficient conditions for the oscillation of all solutions of equation (1.1) when the condition (1.2) or (1.3) is satisfied.

In Section 2, we establish some preliminary lemmas and in Section 3, we present sufficient conditions for the oscillation all solutions of equation (1.1). Examples are provided to illustrate the main results.

2. Some preliminary lemmas

In this section, we present some useful lemmas, which will be used in the proofs of our main results.

Lemma 2.1. *[[23]] Let $u \in C^n([t_0,\infty),\mathbb{R}^+)$. If $u^{(n)}(t)$ is eventually of one sign for all large t, then there exists a $t_x > t_1$, for some $t_1 > t_0$, and an integer l, $0 \le l \le n$, with $n+l$ even for $u^{(n)}(t) \ge 0$ or $n+l$ odd for $u^{(n)}(t) \le 0$ such that $l > 0$ implies that $u^{(k)}(t) > 0$ for $t > t_x,\ k = 0,1,...,l-1$, and $l \le n-1$, implies that $(-1)^{l+k}u^{(k)}(t) > 0$ for $t > t_x,\ k = l, l+1, ..., n-1$.*

Lemma 2.2. *[[1]] Let u be as in Lemma 2.1. Assume that $u^{(n)}(t)$ is not identically zero on any interval $[t_0,\infty)$, and there exists a $t_1 \ge t_0$ such that $u^{(n-1)}(t)u^{(n)}(t) \le 0$ for all $t \ge t_1$. If $\lim_{t\to\infty} u(t) \ne 0$, then for every $\lambda,\ 0 < \lambda < 1$, there exists $T \ge t_1$, such that for all $t \ge T$,*

$$u(t) \ge \frac{\lambda}{(n-1)!}t^{n-1}u^{(n-1)}(t).$$

Lemma 2.3. *Assume that condition (1.2) holds. Furthermore, assume that x is an eventually positive solution of equation (1.1). Then there exists $t_1 \ge t_0$, such that*

$$z(t) > 0, z'(t) > 0, z^{(n-1)}(t) > 0 \text{ and } z^{(n)}(t) \le 0,\ \text{for all } t \ge t_1.$$

The proof is similar to that of Meng and Xu [[22], Lemma 2.3] and so omitted.

Lemma 2.4. *[[21]] Assume that $\alpha \in (0,\infty)$ and $c \ge 0$ and $d \ge 0$. Then*

$$c^\alpha + d^\alpha \ge (c+d)^\alpha \text{ if } 0 < \alpha < 1,$$

and

$$c^\alpha + d^\alpha \ge \frac{1}{2^{\alpha-1}}(c+d)^\alpha \text{ if } \alpha \ge 1.$$

Lemma 2.5. *[[27]] Assume that for large t*

$$q(s) \ne 0 \text{ for all } s \in [t,t^*],$$

where t^* satisfies $\sigma(t^*) = t$. Then

$$x'(t) + q(t)[x(\sigma(t))]^\alpha = 0,\ t \ge t_0,$$

has an eventually positive solution if and only if the corresponding inequality

$$x'(t) + q(t)[x(\sigma(t))]^\alpha \le 0,\ t \ge t_0,$$

has an eventually positive solution.

In [7, 12, 18, 30], the authors investigated the oscillatory behavior of the following equation

$$x'(t) + q(t)[x(\sigma(t))]^\alpha = 0,\ t \ge t_0, \tag{2.1}$$

where $q \in C([t_0,\infty),\mathbb{R}^+)$, $\sigma \in C([t_0,\infty),\mathbb{R})$, $\sigma(t) < t$, $\lim_{t\to\infty}\sigma(t) = \infty$ and $\alpha \in (0,\infty)$ is a ratio of odd positive integers.

Let $\alpha \in (0,1)$. Then it is shown that every solution of the sublinear equation (2.1) oscillates if and only if

$$\int_{t_0}^{\infty} q(s)ds = \infty. \tag{2.2}$$

Let $\alpha = 1$. Then equation (2.1) reduces to the linear delay differential equation

$$x'(t) + q(t)x(\sigma(t)) = 0, \ t \geq t_0, \tag{2.3}$$

and it is shown that every solution of equation (2.3) oscillates if

$$\liminf_{t\to\infty} \int_{\sigma(t)}^{t} q(s)ds > \frac{1}{e}. \tag{2.4}$$

Let $\alpha \in (1, \infty)$ and $\sigma(t) = t - \sigma$. Then equation (2.3) reduces to

$$x'(t) + q(t)x^{\alpha}(t - \sigma) = 0, \ t \geq t_0, \tag{2.5}$$

for which the following results was obtained: If there exists $\lambda \in (\sigma^{-1} \ln \alpha, \infty)$ such that

$$\liminf_{t\to\infty} q(t)exp(e^{-\lambda t}) > 0, \tag{2.6}$$

then every solution of equation (2.5) oscillates.

3. Oscillation Results

In this section, we state and prove our main results. Define for all $t \geq t_0$,

$$R(t) = P(t) + Q(t),$$

where

$$P(t) = \min \{p(t), p(t - \tau_1), p(t + \tau_2)\},$$

and

$$Q(t) = \min \{q(t), q(t - \tau_1), q(t + \tau_2)\}. \tag{3.1}$$

Theorem 3.1. *Assume that condition (1.2) holds and $1 \leq \alpha \leq \beta$. If*

$$\int_{t_0}^{\infty} R(t)dt = \infty, \tag{3.2}$$

then every solution of equation (1.1) is oscillatory.

Proof. Suppose, on the contrary, x is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a constant $t_1 \geq t_0$, such that $x(t) > 0$, for all $t \geq t_1$. From the definition of z, we have $z(t) > 0$ for all $t \geq t_1$. From the equation (1.1), we obtain

$$(a(t)z^{(n-1)}(t))' = -(p(t)x^{\alpha}(t - \sigma_1) + q(t)x^{\beta}(t + \sigma_2)) < 0, \ t \geq t_1.$$

Therefore, by Lemma 2.3 $a(t)z^{(n-1)}(t)$ is a positive decreasing function. Furthermore, we have

$$(a(t)z^{(n-1)}(t))' + p(t)x^{\alpha}(t - \sigma_1) + q(t)x^{\beta}(t + \sigma_2) = 0, \tag{3.3}$$

$$b^{\alpha}(a(t-\tau_1)z^{(n-1)}(t-\tau_1))' + b^{\alpha}p(t-\tau_1)x^{\alpha}(t-\tau_1-\sigma_1) + b^{\alpha}q(t-\tau_1)x^{\beta}(t-\tau_1+\sigma_2) = 0, \tag{3.4}$$

and

$$c^{\alpha}(a(t+\tau_2)z^{(n-1)}(t+\tau_2))' + c^{\alpha}p(t+\tau_2)x^{\alpha}(t+\tau_2-\sigma_1) + c^{\alpha}q(t+\tau_2)x^{\beta}(t+\tau_2+\sigma_2) = 0. \tag{3.5}$$

Combining (3.3), (3.4), (3.5) and using Lemma 2.4 and (3.1) we obtain for $t \geq t_1$,

$$(a(t)z^{(n-1)}(t))' + b^\alpha(a(t-\tau_1)z^{(n-1)}(t-\tau_1))' + c^\alpha(a(t+\tau_2)z^{(n-1)}(t+\tau_2))'$$
$$+P(t)\frac{1}{4^{\alpha-1}}z^\alpha(t-\sigma_1) + Q(t)\frac{1}{4^{\alpha-1}}z^\alpha(t+\sigma_2) \leq 0, \ t \geq t_1. \tag{3.6}$$

But $z(t) > 0$ and increasing, we have

$$(a(t)z^{(n-1)}(t))' + b^\alpha(a(t-\tau_1)z^{(n-1)}(t-\tau_1))' + c^\alpha(a(t+\tau_2)z^{(n-1)}(t+\tau_2))'$$
$$+\frac{1}{4^{\alpha-1}}R(t)z^\alpha(t-\sigma_1) \leq 0, \ t \geq t_1. \tag{3.7}$$

Integrating (3.7) from t_1 to t, we have

$$\int_{t_1}^{t} (a(s)z^{(n-1)}(s))' ds + \int_{t_1}^{t} b^\alpha(a(s-\tau_1)z^{(n-1)}(s-\tau_1))' ds$$
$$+\int_{t_1}^{t} c^\alpha(a(s+\tau_2)z^{(n-1)}(s+\tau_2))' ds + \int_{t_1}^{t} \frac{1}{4^{\alpha-1}}R(s)z^\alpha(s-\sigma_1) ds \leq 0, \ t \geq t_1,$$

again we get

$$\frac{1}{4^{\alpha-1}}\int_{t_1}^{t} R(s)z^\alpha(s-\sigma_1) ds \leq -\int_{t_1}^{t} (a(s)z^{(n-1)}(s))' ds$$
$$-b^\alpha\int_{t_1}^{t} (a(s-\tau_1)z^{(n-1)}(s-\tau_1))' ds - c^\alpha\int_{t_1}^{t} (a(s+\tau_2)z^{(n-1)}(s+\tau_2))' ds$$
$$\leq a(t_1)z^{(n-1)}(t_1) - a(t)z^{(n-1)}(t)$$
$$+b^\alpha\left(a(t_1-\tau_1)z^{(n-1)}(t_1-\tau_1) - a(t-\tau_1)z^{(n-1)}(t-\tau_1)\right)$$
$$+c^\alpha\left(a(t_1+\tau_2)z^{(n-1)}(t_1+\tau_2) - a(t+\tau_2)z^{(n-1)}(t+\tau_2)\right). \tag{3.8}$$

Since $z'(t) > 0$ for $t \geq t_1$, we can find a constant $M > 0$ such that $z(t-\sigma_1) \geq M$,for all $t \geq t_1$. Then from (3.8) and the fact that $a(t)z^{(n-1)}(t)$ is positive, we obtain

$$\int_{t_1}^{\infty} R(s) ds < \infty,$$

which is in contradiction with (3.2). This completes the proof. □

Theorem 3.2. *Assume that condition (1.2) holds. Further assume that* $\alpha = 1$*and* $\sigma_1 > \tau_1$. *If either*

$$\liminf_{t\to\infty} \int_{t-(\sigma_1-\tau_1)}^{t} \frac{R(s)(s-\sigma_1)^{n-1}}{a(s-\sigma_1)} ds > \frac{(1+b+c)(n-1)!}{\lambda_0 e}, \tag{3.9}$$

or

$$\limsup_{t\to\infty} \int_{t-(\sigma_1-\tau_1)}^{t} \frac{R(s)(s-\sigma_1)^{n-1}}{a(s-\sigma_1)} ds > \frac{(1+b+c)(n-1)!}{\lambda_0}, \tag{3.10}$$

for some $\lambda_0 \in (0,1)$,*then every solution of equation (1.1) is oscillatory.*

Proof. Suppose, on the contrary, x is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a constant $t_1 \geq t_0$, such that $x(t) > 0$, for all $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we have (3.7). By Lemma 2.2 and (3.7), for every $\lambda, 0 < \lambda < 1$, we obtain

$$\begin{aligned}&(a(t)z^{(n-1)}(t))' + b^{\alpha}(a(t-\tau_1)z^{(n-1)}(t-\tau_1))' + c^{\alpha}(a(t+\tau_2)z^{(n-1)}(t+\tau_2))'\\&+\frac{R(t)}{4^{\alpha-1}}\left(\frac{\lambda}{(n-1)!}(t-\sigma_1)^{n-1}z^{(n-1)}(t-\sigma_1)\right)^{\alpha} \leq 0,\ t \geq t_1.\end{aligned}$$

Let $y(t) = a(t)z^{(n-1)}(t) > 0$. Then for all t large enough, we have

$$\begin{aligned}&(y(t) + b^{\alpha}y(t-\tau_1) + c^{\alpha}y(t+\tau_2))'\\&+\frac{R(t)}{4^{\alpha-1}a^{\alpha}(t-\sigma_1)}\left(\frac{\lambda}{(n-1)!}(t-\sigma_1)^{n-1}\right)^{\alpha} y^{\alpha}(t-\sigma_1) \leq 0,\ t \geq t_1. \quad (3.11)\end{aligned}$$

Next, set

$$w(t) = y(t) + b^{\alpha}y(t-\tau_1) + c^{\alpha}y(t+\tau_2).$$

Since y is decreasing, it follows that

$$w(t) \leq (1 + b^{\alpha} + c^{\alpha})y(t-\tau_1),\ t \geq t_1. \quad (3.12)$$

Combining (3.11) and (3.12), we get

$$w'(t) + \frac{R(t)}{4^{\alpha-1}(1+b^{\alpha}+c^{\alpha})^{\alpha}a^{\alpha}(t-\sigma_1)}\left(\frac{\lambda}{(n-1)!}(t-\sigma_1)^{n-1}\right)^{\alpha} w^{\alpha}(t-\sigma_1+\tau_1) \leq 0. \quad (3.13)$$

Hence for $\alpha = 1$, we have

$$w'(t) + \frac{R(t)}{(1+b+c)a(t-\sigma_1)}\left(\frac{\lambda}{(n-1)!}(t-\sigma_1)^{n-1}\right) w(t-\sigma_1+\tau_1) \leq 0. \quad (3.14)$$

Therefore, w is a positive solution of (3.14). Now, we consider the following two cases, depending on whether (3.9) or (3.10) holds.
Case (i):It is easy to see that if (3.9) holds, then we can choose a constant $0 < \lambda_0 < 1$, such that

$$\liminf_{t\to\infty} \int_{t-(\sigma_1-\tau_1)}^{t} \frac{R(s)(s-\sigma_1)^{n-1}\lambda}{a(s-\sigma_1)(n-1)!(1+b+c)}ds > \frac{1}{e}. \quad (3.15)$$

But according to the Lemma 2.5, (3.15) guarantees that (3.14) has no positive solution, which is a contradiction.
Case (ii): Using the definition of w and (3.7), we obtain

$$\begin{aligned}&w'(t) = y'(t) + b^{\alpha}y'(t-\tau_1) + c^{\alpha}y'(t+\tau_2)\\&\leq (a(t)z^{(n-1)}(t))' + b^{\alpha}(a(t-\tau_1)z^{(n-1)}(t-\tau_1))' + c^{\alpha}(a(t+\tau_2)z^{(n-1)}(t+\tau_2))'\\&\leq -\frac{1}{4^{\alpha-1}}R(t)z^{\alpha}(t-\sigma_1) \leq 0,\ t \geq t_1. \quad (3.16)\end{aligned}$$

Noting that $\alpha = 1$ and $\sigma_1 \geq \tau_1$, there exists $t_2 \geq t_1$, such that

$$w(t-\sigma_1+\tau_1) \geq w(t), t \geq t_2. \quad (3.17)$$

Integrating (3.14) from $t-\sigma_1+\tau_1$ to t, we have

$$w(t)-w(t-\sigma_1+\tau_1)+\frac{\lambda}{(1+b+c)(n-1)!}\int_{t-\sigma_1+\tau_1}^{t}\frac{(s-\sigma_1)^{n-1}R(s)}{a(s-\sigma_1)}w(s-\sigma_1+\tau_1)ds \le 0,$$

where$t \ge t_2$.Thus

$$w(t)-w(t-\sigma_1+\tau_1)+\frac{\lambda}{(1+b+c)(n-1)!}w(t-\sigma_1+\tau_1)\int_{t-\sigma_1+\tau_1}^{t}\frac{(s-\sigma_1)^{n-1}R(s)}{a(s-\sigma_1)}ds \le 0,$$

where $t \ge t_2$. From the above inequality, we obtain

$$\frac{w(t)}{w(t-\sigma_1+\tau_1)}-1+\frac{\lambda}{(1+b+c)(n-1)!}\int_{t-\sigma_1+\tau_1}^{t}\frac{(s-\sigma_1)^{n-1}R(s)}{a(s-\sigma_1)}ds \le 0,\ \ t \ge t_2.$$

Hence from (3.17), we have

$$\frac{\lambda}{(1+b+c)(n-1)!}\int_{t-\sigma_1+\tau_1}^{t}\frac{(s-\sigma_1)^{n-1}R(s)}{a(s-\sigma_1)}ds \le 1,\ \ t \ge t_2. \tag{3.18}$$

Taking the sup limit as $t \to \infty$ in (3.18), we get

$$\limsup_{t\to\infty}\int_{t-(\sigma_1-\tau_1)}^{t}\frac{R(s)(s-\sigma_1)^{n-1}}{a(s-\sigma_1)}ds \le \frac{(1+b+c)(n-1)!}{\lambda}. \tag{3.19}$$

If (3.10) holds, we can choose a constant $0<\lambda_0<1$, such that

$$\limsup_{t\to\infty}\int_{t-(\sigma_1-\tau_1)}^{t}\frac{R(s)(s-\sigma_1)^{n-1}}{a(s-\sigma_1)}ds > \frac{(1+b+c)(n-1)!}{\lambda},$$

which is in contradiction with (3.18). This completes the proof. □

Theorem 3.3. *Assume that condition (1.2) holds and* $1 \le \beta \le \alpha$. *If*

$$\int_{t_0}^{\infty}R(t)dt=\infty, \tag{3.20}$$

then every solution of equation (1.1) is oscillatory.

Proof. The proof is similar to that of Theorem 3.1 and hence the details are omitted. □

Theorem 3.4. *Assume that condition (1.2) holds. Further assume that* $\beta = 1$*and* $\sigma_1 > \tau_1$. *If either*

$$\liminf_{t\to\infty}\int_{t-(\sigma_1-\tau_1)}^{t}\frac{R(s)(s-\sigma_1)^{n-1}}{a(s-\sigma_1)}ds > \frac{(1+b+c)(n-1)!}{\lambda_0 e}, \tag{3.21}$$

or

$$\limsup_{t\to\infty}\int_{t-(\sigma_1-\tau_1)}^{t}\frac{R(s)(s-\sigma_1)^{n-1}}{a(s-\sigma_1)}ds > \frac{(1+b+c)(n-1)!}{\lambda_0}, \tag{3.22}$$

for some $\lambda_0 \in (0,1)$*,then every solution of equation (1.1) is oscillatory.*

Proof. The proof is similar to that of Theorem 3.2 and hence the details are omitted. □

Corollary 3.5. *Assume that condition (1.2) holds,* $\sigma_1 - \tau_1 > 0$ *and* $\alpha \in (1,\infty)$*. If there exists* $\mu \in ((\sigma_1 - \tau_1)^{-1} \ln \alpha, \infty)$ *such that*

$$\liminf_{t\to\infty} R(t) \left(\frac{(t-\sigma_1)^{n-1}}{a(t-\sigma_1)(n-1)!} \right)^{\alpha} exp(e^{-\mu t}) > 0, \tag{3.23}$$

then every solution of equation (1.1) is oscillatory.

Proof. According to Lemma 2.5, the condition (3.23) guarantees that (3.13) with $\alpha > 1$ has no positive solution. Hence by Theorem 3.2, every solution of equation (1.1) is oscillatory. This completes the proof. □

Corollary 3.6. *Assume that condition (1.2) holds,* $\sigma_1 - \tau_1 > 0$ *and* $\beta \in (1,\infty)$*. If there exists* $\nu \in ((\sigma_1 - \tau_1)^{-1} \ln \beta, \infty)$ *such that*

$$\liminf_{t\to\infty} R(t) \left(\frac{(t-\sigma_1)^{n-1}}{a(t-\sigma_1)(n-1)!} \right)^{\beta} exp(e^{-\nu t}) > 0, \tag{3.24}$$

then every solution of equation (1.1) is oscillatory.

Proof. According to Lemma 2.5, the condition (3.24) guarantees that (3.13) with $\beta > 1$ has no positive solution. Hence by Theorem 3.2, every solution of equation (1.1) is oscillatory. This completes the proof. □

Theorem 3.7. *Assume that condition (1.3) holds and* $\sigma_1 - \tau_1 > 0$*. Suppose, further that the first order differential equation*

$$w'(t) + \frac{R(t)}{4^{\alpha-1}(1+b^\alpha+c^\alpha)^\alpha a^\alpha(t-\sigma_1)} \left(\frac{\lambda_0}{(n-1)!}(t-\sigma_1)^{n-1} \right)^{\alpha} w^\alpha(t-\sigma_1+\tau_1) = 0 \tag{3.25}$$

is oscillatory for some constant $\lambda_0 \in (0,1)$*. If*

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[k_1 R(s) \left(\frac{1}{(n-2)!}(s-\sigma_1)^{n-2} \right)^{\alpha} \delta(s) - \frac{(1+b^\alpha+c^\alpha)}{4a(s+\tau_2)\delta(s)} \right] ds = \infty, \tag{3.26}$$

for all constants $k_1 > 0$*, then equation (1.1) is almost oscillatory.*

Proof. Suppose, on the contrary, x is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there exists a constant $t_1 \geq t_0$, such that $x(t) > 0$, for all $t \geq t_1$. It follows from the equation (1.1) and Kiguradze's Lemma 2.1 that there exist three possible cases:

(*i*) $z(t) > 0, z'(t) > 0, z^{(n-1)}(t) > 0, z^{(n)}(t) \leq 0;$
(*ii*) $z(t) > 0, z'(t) > 0, z^{(n-2)}(t) > 0, z^{(n-1)}(t) < 0;$
(*iii*) $z(t) > 0, z'(t) < 0, z^{(n-2)}(t) > 0, z^{(n-1)}(t) < 0;$

for $t \geq t_2 \geq t_1$, t_1 is sufficiently large. Assume that case(i) holds. From the proof of Theorem 3.2 we get

$$w'(t) + \frac{R(t)}{4^{\alpha-1}(1+b^\alpha+c^\alpha)^\alpha a^\alpha(t-\sigma_1)} \left(\frac{\lambda_0}{(n-1)!}(t-\sigma_1)^{n-1} \right)^{\alpha} w^\alpha(t-\sigma_1+\tau_1) \leq 0.$$

By [[12],Corollary 3.2.2], w is a positive solution of

$$w'(t)+\frac{R(t)}{4^{\alpha-1}(1+b^{\alpha}+c^{\alpha})^{\alpha}a^{\alpha}(t-\sigma_1)}\left(\frac{\lambda_0}{(n-1)!}(t-\sigma_1)^{n-1}\right)^{\alpha}w^{\alpha}(t-\sigma_1+\tau_1)=0$$

for every $\lambda\in(0,1)$, which contradicts the fact that (3.25) is oscillatory.

Assume that case (ii) holds. Define the function u by

$$u(t)=\frac{a(t+\tau_2)z^{(n-1)}(t+\tau_2)}{z^{(n-2)}(t)},\ t\geq t_2. \tag{3.27}$$

Clearly, $u(t)<0$ for $t\geq t_2$. Noting that $a(t+\tau_2)z^{(n-1)}(t+\tau_2)$ is decreasing, we obtain

$$a(s+\tau_2)z^{(n-1)}(s+\tau_2)\leq a(t+\tau_2)z^{(n-1)}(t+\tau_2),\ s\geq t\geq t_2. \tag{3.28}$$

Dividing (3.28) by $a(s+\tau_2)$ and integrating it from t to $l(l\geq t)$, we have

$$z^{(n-2)}(l+\tau_2)\leq z^{(n-2)}(t+\tau_2)+a(t+\tau_2)z^{(n-1)}(t+\tau_2)\int_t^l\frac{du}{a(u)}.$$

Letting $l\to\infty$, we get

$$0\leq z^{(n-2)}(t)+a(t+\tau_2)z^{(n-1)}(t+\tau_2)\delta(t),$$

that is

$$-1\leq\frac{a(t+\tau_2)z^{(n-1)}(t+\tau_2)\delta(t)}{z^{(n-2)}(t)}.$$

Therefore, from (3.28), we obtain

$$-1\leq u(t)\delta(t)\leq 0,\ t\geq t_2. \tag{3.29}$$

Next, we define the function w as

$$w(t)=\frac{a(t)z^{(n-1)}(t)}{z^{(n-2)}(t)},\ t\geq t_2. \tag{3.30}$$

Clearly, $w(t)<0$ for $t\geq t_2$. Noting that $a(t)z^{(n-1)}(t)$ is decreasing, we have

$$a(t+\tau_2)z^{(n-1)}(t+\tau_2)\leq a(t)z^{(n-1)}(t),$$

then $u(t)\leq w(t)$. Thus, by (3.30), we get

$$-1\leq w(t)\delta(t)\leq 0,\ t\geq t_2. \tag{3.31}$$

Next, define the function v as

$$v(t)=\frac{a(t-\tau_1)z^{(n-1)}(t-\tau_1)}{z^{(n-2)}(t)},\ t\geq t_2. \tag{3.32}$$

Clearly, $v(t)<0$ for $t\geq t_2$. Noting that $a(t-\tau_1)z^{(n-1)}(t-\tau_1)$ is decreasing, we have

$$a(t)z^{(n-1)}(t)\leq a(t-\tau_1)z^{(n-1)}(t-\tau_1),$$

then $u(t)\leq w(t)\leq v(t)$. Thus, by (3.32), we get

$$-1\leq v(t)\delta(t)\leq 0,\ t\geq t_2. \tag{3.33}$$

Differentiating (3.27), we obtain

$$u'(t)\leq\frac{(a(t+\tau_2)z^{(n-1)}(t+\tau_2))'}{z^{(n-2)}(t)}-\frac{u^2(t)}{a(t+\tau_2)}. \tag{3.34}$$

Differentiating (3.30) and from (3.28), we obtain

$$w'(t) \leq \frac{(a(t)z^{(n-1)}(t))'}{z^{(n-2)}(t)} - \frac{w^2(t)}{a(t+\tau_2)}. \tag{3.35}$$

Differentiating (3.32) and from (3.28), we obtain

$$v'(t) \leq \frac{(a(t-\tau_1)z^{(n-1)}(t-\tau_1))'}{z^{(n-2)}(t)} - \frac{v^2(t)}{a(t+\tau_2)}. \tag{3.36}$$

Combining (3.34),(3.35) and (3.36), we get

$$w'(t) + b^\alpha v'(t) + c^\alpha u'(t) \leq \frac{1}{z^{(n-2)}(t)}(a(t)z^{(n-1)}(t))' + b^\alpha(a(t-\tau_1)z^{(n-1)}(t-\tau_1))'$$
$$+c^\alpha(a(t+\tau_2)z^{(n-1)}(t+\tau_2))' - \frac{1}{a(t+\tau_2)}\left(w^2(t) + b^\alpha v^2(t) + c^\alpha u^2(t)\right). \tag{3.37}$$

Therefore, by (3.7) and (3.37), we obtain

$$w'(t) + b^\alpha v'(t) + c^\alpha u'(t) \leq -\frac{z^\alpha(t-\sigma_1)}{z^{(n-2)}(t)4^{\alpha-1}}R(t)$$
$$-\frac{1}{a(t+\tau_2)}\left(w^2(t) + b^\alpha v^2(t) + c^\alpha u^2(t)\right). \tag{3.38}$$

On the other hand, by Lemma 2.2, we get

$$z(t) \geq \frac{\lambda}{(n-2)!}t^{n-2}z^{(n-2)}(t). \tag{3.39}$$

for every $\lambda \in (0,1)$ and for all sufficiently large t. Then there exists a constant $M > 0$ such that

$$w'(t) + b^\alpha v'(t) + c^\alpha u'(t) \leq -\frac{R(t)}{4^{\alpha-1}}\frac{z^\alpha(t-\sigma_1)}{(z^{(n-2)}(t-\sigma_1))^\alpha}(z^{(n-2)}(t-\sigma_1))^{\alpha-1}\frac{z^{(n-2)}(t-\sigma_1)}{z^{(n-2)}(t)}$$
$$-\frac{1}{a(t+\tau_2)}\left(w^2(t) + b^\alpha v^2(t) + c^\alpha u^2(t)\right)$$
$$\leq -\left(\frac{M}{4}\right)^{\alpha-1}R(t)\left(\frac{\lambda}{(n-2)!}(t-\sigma_1)^{n-2}\right)^\alpha - \frac{1}{a(t+\tau_2)}\left(w^2(t) + b^\alpha v^2(t) + c^\alpha u^2(t)\right)$$

Multiplying the above inequality by $\delta(t)$ and integrating from t_2 to t, we obtain

$$\delta(t)w(t) - \delta(t_2)w(t_2) + \int_{t_2}^{t}\frac{w(s)}{a(s+\tau_2)}ds + \int_{t_2}^{t}\frac{w^2(s)\delta(s)}{a(s+\tau_2)}ds$$
$$+b^\alpha\left(\delta(t)v(t) - \delta(t_2)v(t_2) + \int_{t_2}^{t}\frac{v(s)}{a(s+\tau_2)}ds + \int_{t_2}^{t}\frac{v^2(s)\delta(s)}{a(s+\tau_2)}ds\right)$$
$$+c^\alpha\left(\delta(t)u(t) - \delta(t_2)u(t_2) + \int_{t_2}^{t}\frac{u(s)}{a(s+\tau_2)}ds + \int_{t_2}^{t}\frac{u^2(s)\delta(s)}{a(s+\tau_2)}ds\right)$$
$$+\left(\frac{M}{4}\right)^{\alpha-1}\int_{t_2}^{t}R(s)\left(\frac{\lambda}{(n-2)!}(s-\sigma_1)^{n-2}\right)^\alpha\delta(s)ds \leq 0. \tag{3.40}$$

It follows from (3.40), taking into account that $-1 \le w(t)\delta(t) \le 0,\ -1 \le v(t)\delta(t) \le 0$ and $-1 \le u(t)\delta(t) \le 0$,

$$\delta(t)w(t) - \delta(t_2)w(t_2) + b^\alpha(\delta(t)v(t) - \delta(t_2)v(t_2))$$
$$+c^\alpha(\delta(t)u(t) - \delta(t_2)u(t_2)) + \left(\frac{M}{4}\right)^{\alpha-1}\left(\frac{\lambda}{(n-2)!}\right)^\alpha \int_{t_2}^{t} \delta(s)R(s)(s-\sigma_1)^{\alpha(n-2)}ds$$
$$-\frac{1+b^\alpha+c^\alpha}{4}\int_{t_2}^{t}\frac{1}{a(s+\tau_2)\delta(s)} \le 0.$$

Therefore,

$$\delta(t)w(t) + b^\alpha\delta(t)v(t) + c^\alpha\delta(t)u(t)$$
$$+\int_{t_2}^{t}\left[k_1\left(\frac{1}{(n-2)!}(s-\sigma_1)^{n-2}\right)^\alpha R(s)\delta(s) - \frac{1+b^\alpha+c^\alpha}{4}\frac{1}{a(s+\tau_2)\delta(s)}\right]ds$$
$$\le \delta(t_2)w(t_2) + b^\alpha\delta(t_2)v(t_2) + c^\alpha\delta(t_2)u(t_2).$$

From (3.26) and the above inequality, we get a contradiction to (3.29),(3.31) and (3.33).

Assume that case(iii) holds. Similar to the proof of that of [[4],Lemma2], there exists a constant $k > 0$ such that

$$x(t) \ge kz(t). \tag{3.41}$$

The conclusion of the proof is similar to that of case (ii) and we can obtain the contradiction to (3.26), and so is omitted. This completes the proof. □

4. Examples

In this section we present some examples to illustrate the main results.

Example 4.1. *Consider the even order nonlinear mixed type differential equation*

$$(x(t) + x(t-\pi) + 4x(t+2\pi))^{(iv)} + 2x(t-3\pi) + 2x(t+\pi) = 0,\ t \ge 0. \tag{4.1}$$

Here $a(t) = 1, p(t) = q(t) = 2, b = 1, c = 4, \tau_1 = \pi, \tau_2 = 2\pi, \sigma_1 = 3\pi, \sigma_2 = \pi$ *and* $\alpha = \beta = 1$. *It satisfies all the conditions of the Theorem 3.2. Hence, every solution of equation (4.1) oscillates. For, example,* $x(t) = \sin t$ *is an oscillatory solution of equation (4.1).*

Example 4.2. *Consider the even order nonlinear mixed type differential equation*

$$(2(x(t) + x(t-\pi) + x(t+\pi))''')' + x(t-2\pi) + x(t+\pi) = 0,\ t \ge 0. \tag{4.2}$$

Here $a(t) = 2, p(t) = q(t) = 1, b = 1, c = 1, \tau_1 = \pi, \tau_2 = \pi, \sigma_1 = 2\pi, \sigma_2 = 2\pi$ *and* $\alpha = \beta = 1$. *It satisfies all the conditions of the Theorem 3.4. Hence, every solution of equation (4.2) oscillates. For, example,* $x(t) = \sin t$ *is an oscillatory solution of equation (4.2).*

Example 4.3. *Consider the fourth-order differential equation*

$$\left(\frac{1}{e^{2t}}(x(t) + x(t-1) + x(t+1))'''\right)' + \frac{1}{e^{4t}}(e^6 + e^7)x^3(t-2)$$
$$+\frac{1}{e^{4t+10}}x^3(t+3) = 0, \tag{4.3}$$

where $t \geq 0$. *Here* $a(t) = 1/e^{2t}, p(t) = \frac{1}{e^{4t}}, q(t) = \frac{1}{e^{4t+10}}, b = c = 1, \tau_1 = \tau_2 = 1, \sigma_1 = 2, \sigma_2 = 3$ *and* $\alpha = \beta = 3$. *Then one can see that all conditions of Theorem 3.4 are satisfied except the condition (3.2). Therefore all the solutions of equation (4.3) not necessarily oscillatory. In fact* $x(t) = e^t$ *is an oscillatory solution of equation (4.3).*

Example 4.4. *Consider the fourth-order differential equation*

$$(e^t z'''(t))' + \left(\frac{e^{t-1/2} + e^{t-1}}{16}\right) x(t-2) + \frac{e^t}{16} x(t+1) = 0, \ t \geq 2 \tag{4.4}$$

where $z(t) = x(t) + x(t-1) + x(t+1)$. *We can see that all conditions of Theorem 3.2 satisfied except the condition (1.2). Therefore all the solutions of equation (4.4) not necessarily oscillatory. In fact* $x(t) = e^{-t/2}$ *is one such nonoscillatory solution, since it satisfies the equation (4.4).*

Example 4.5. *Consider the fourth-order differential equation*

$$\left(\frac{1}{e^{2t}}(x(t) + \frac{1}{3}x(t-1) + \frac{1}{3}x(t+1))'''\right)' + \frac{1}{e^{2t}}(e^2 + \frac{1}{3}e)x(t-2) + \frac{1}{3e^{2t}}x(t+1) = 0, \ t \geq 0 \tag{4.5}$$

Here $a(t) = 1/e^{2t}, p(t) = \frac{1}{e^{2t}}(e^2 + \frac{1}{3}e), q(t) = \frac{1}{3e^{2t}}, b = c = 1/3, \tau_1 = 1, \tau_2 = 1, \sigma_1 = 2, \sigma_2 = 1$ *and* $\alpha = \beta = 1$. *Then one can see that all conditions of Theorem 3.4 are satisfied except the condition (3.2). Therefore all the solutions of equation (4.5) not necessarily oscillatory. In fact* $x(t) = e^t$ *is one such nonoscillatory solution, since it satisfies the equation (4.5).*

Acknowledgements

The authors thank the referees for his/her suggestions and corrections which improved the content of the paper.

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Ethiraju Thandapani
Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai- 600 005, India.
E-mail address: ethandapani@yahoo.co.in.

Sankarappan Padmavathi
Ramanujan Institute for Advanced Study in Mathematics
University of Madras, Chennai- 600 005, India.
E-mail address: padmavathi.sankarappan@gmail.com.

Sandra Pinelas
Academia Militar, Departamento de Ciências Exactas e Naturais, Av.
Conde Castro Guimarães,2720-113 Amadora, Portugal.
E-mail address: sandra.pinelas@gmail.com.