GENERALIZED q-HERMITE POLYNOMIALS AND THE q-DUNKL HEAT EQUATION

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ABSTRACT. Two classes of generalized discrete q-Hermite polynomials are constructed. Several properties of these polynomials, and an explicit relations connecting them with little q-Laguerre and q-Laguerre polynomials are obtained. A relationship with the q-Dunkl heat polynomials and the q-Dunkl associated functions are established.

1. INTRODUCTION

The classical sequence of Hermite polynomials form one of the best known systems of orthogonal polynomials in literature. Their applications cover many domains in applied and pure mathematics. For instance, the Hermite polynomials play central role in the study of the polynomial solutions of the classical heat equation, which is a partial differential equation involving classical derivatives. It is therefore natural that generalizations of the heat equation for generalized operators lead to generalizations of the Hermite polynomials. For instance, Fitouhi introduced and studied in [6] the generalized Hermite polynomials associated with a Sturm-Liouville operator by studying the corresponding heat equation. In [16], Rösler studied the generalized Hermite polynomials and the heat equation for the Dunkl operator in several variables. In [15], Rosenblum associated Chihara's generalized Hermite polynomials with the Dunkl operator in one variable and used them to study the Bose-like oscillator.

During the mid 1970's, G. E. Andrews started a period of very fruitful collaboration with R. Askey (see [1, 2]). Thanks to these two mathematicians, basic hypergeometric series is an active field of research today. Since Askey's primary area of interest is orthogonal polynomials, q-series suddenly provided him and his co-workers, who include W. A. Al-Salam, M. E. H. Ismail, T. H. Koornwinder, W. G. Morris, D. Stanton, and J. A. Wilson, with a very rich environment for deriving q-extensions of the classical orthogonal polynomials of Jacobi, Gegenbauer, Legendre, Laguerre

and Hermite. It is in this context that this paper is built around the construction of a new generalization of the Hermite polynomials, using new q-operators.

In this paper, we introduce some generalized q-derivative operators with parameter α, which, with the q-Dunkl intertwining operator (see [3]) allow us to introduce and study two families of q-discrete orthogonal polynomials that generalize the two classes of discrete q-Hermite polynomials given in [11].
Next, we consider a generalized q-Dunkl heat equation and we show that it is related to the generalized discrete q-Hermite polynomials in the same way as the classical ones in [15] and in [16].

This paper is organized as follows: in Section 2, we recall some notations and useful results. In Section 3, we introduce generalized q-shifted factorials and we use them to construct some generalized q-exponential functions. In Section 4, we introduce a new class of q-derivative operators. By these operators and the q-Dunkl intertwining operator two classes of discrete q-Hermite polynomials are introduced and analyzed in Section 5. In Section 6, we introduce a q-Dunkl heat equation, and we construct two basic sets of solutions of the q-Dunkl heat equation: the set of generalized q-heat polynomials and the set of generalized q-associated functions. In particular, we show that these classes of solutions are closely related to the generalized discrete q-Hermite I polynomials and the generalized discrete q-Hermite II polynomials, respectively.

2. NOTATIONS AND PRELIMINARIES

2.1. **Basic symbols.** We refer to the general reference [9] for the definitions, notations and properties of the q-shifted factorials and the basic hypergeometric series.
Throughout this paper, we fix $q \in (0,1)$ an we write $\mathbb{R}_q = \{\pm q^n, n \in \mathbb{Z}\}$. For a complex number a, the q-shifted factorials are defined by:

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k), n = 1, 2, ...; \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k).$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{2.1}$$

If we change q by q^{-1}, we obtain

$$(a;q^{-1})_n = (a^{-1};q)_n(-a)^n q^{-\frac{n(n-1)}{2}}, \quad a \neq 0. \tag{2.2}$$

We also denote

$$[x]_q = \frac{1 - q^x}{1 - q}, \qquad x \in \mathbb{C} \quad \text{and} \quad n!_q = \frac{(q;q)_n}{(1-q)^n}, \qquad n \in \mathbb{N}.$$

The basic hypergeometric series is defined by

$$_r\phi_s(a_1, ..., a_r; b_1, ..., b_s; q, z) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k...(a_r;q)_k \left((-1)^k q^{\frac{k(k-1)}{2}}\right)^{1+s-r}}{(b_1;q)_k...(b_s;q)_k(q;q)_k} z^k.$$

The two Euler's q-analogues of the exponential function are given by (see [9])

$$E_q(z) = \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}}{(q;q)_k} z^k = (-z;q)_\infty, \tag{2.3}$$

$$e_q(z) = \sum_{k=0}^{\infty} \frac{1}{(q;q)_k} z^k = \frac{1}{(z;q)_\infty}, \quad |z| < 1. \tag{2.4}$$

Note that the function $E_q(z)$ is entire on $\mathbb{C}$. But for the convergence of the second series, we need $|z| < 1$; however, because of its product representation, e_q is continuable to a meromorphic function on $\mathbb{C}$ with simple poles at $z = q^{-n}$, $\quad n$ non-negative integer.

We denote by

$$exp_q(z) := e_q((1-q)z) = \sum_{n=0}^{\infty} \frac{z^n}{n!_q}, \tag{2.5}$$

$$Exp_q(z) := E_q((1-q)z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}} z^n}{n!_q}. \tag{2.6}$$

We have $\lim_{q \to 1^-} exp_q(z) = \lim_{q \to 1^-} Exp_q(z) = e^z$, where e^z is the classical exponential function.

The Rubin's q-exponential function is defined by (see [13])

$$e(z;q^2) = \sum_{n=0}^{\infty} b_n(z;q^2), \tag{2.7}$$

where

$$b_n(z;q^2) = \frac{q^{[\frac{n}{2}]([\frac{n}{2}]+1)}}{n!_q} z^n \tag{2.8}$$

and $[x]$ is the integer part of $x \in \mathbb{R}$.

$e(z;q^2)$ is entire on $\mathbb{C}$ and we have $\lim_{q \to 1^-} e(z;q^2) = e^z$.

The q-Gamma function is given by (see [5, 9])

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^z;q)_\infty} (1-q)^{1-z}, \; z \neq 0, -1, -2, \ldots \tag{2.9}$$

and tends to $\Gamma(z)$ when q tends to 1^-.

We shall need the Jackson q-integrals defined by (see [9, 10])

$$\int_0^a f(x)d_qx = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \quad \int_a^b f(x)d_qx = \int_0^b f(x)d_qx - \int_0^a f(x)d_qx,$$

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n, \tag{2.10}$$

$$\int_{-\infty}^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q)\sum_{n=-\infty}^{\infty} q^n f(-q^n).$$

We denote by $L_{\alpha,q}^1$ the space of complex-valued functions f on $\mathbb{R}_q$ such that

$$\int_{-\infty}^\infty |f(x)||x|^{2\alpha+1} d_qx < \infty.$$

The q-Gamma function has the q-integral representation (see [5, 8])

$$\Gamma_q(z) = \int_0^{(1-q)^{-1}} x^{z-1} Exp_q(-qx) d_q x, \qquad (2.11)$$

and satisfies the relation

$$\int_0^1 t^{2x+1} \frac{(q^2 t^2; q^2)_\infty}{(q^{2y} t^2; q^2)_\infty} d_q t \quad = \quad \frac{\Gamma_{q^2}(x+1)\Gamma_{q^2}(y)}{(1+q)\Gamma_{q^2}(x+y+1)}, \ x > -1, \ y > 0. \ (2.12)$$

2.2. The q-derivatives.
The Jackson's q-derivative D_q (see [9, 10]) is defined by :

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}. \qquad (2.13)$$

We also need a variant D_q^+, called forward q-derivative, given by

$$D_q^+ f(z) = \frac{f(q^{-1}z) - f(z)}{(1-q)z}. \qquad (2.14)$$

Note that $\lim_{q \to 1^-} D_q f(z) = \lim_{q \to 1^-} D_q^+ f(z) = f'(z)$ whenever f is differentiable at z.
Recently, R. L. Rubin introduced in [13, 14] a q-derivative operator ∂_q as follows

$$\partial_q f = D_q^+ f_e + D_q f_o, \qquad (2.15)$$

where f_e and f_o are respectively the even and the odd parts of f.
We note that if f is differentiable at x then $\partial_q f(x)$ tends as $q \to 1^-$ to $f'(x)$.

3. THE GENERALIZED q-EXPONENTIAL FUNCTIONS

3.1. The generalized q-shifted factorials.
For $\alpha > -1$, we define the generalized q-shifted factorials for non-negative integers n by

$$(2n)!_{q,\alpha} \quad := \quad \frac{(1+q)^{2n}\Gamma_{q^2}(\alpha+n+1)\Gamma_{q^2}(n+1)}{\Gamma_{q^2}(\alpha+1)} \quad = \quad \frac{(q^2; q^2)_n (q^{2\alpha+2}; q^2)_n}{(1-q)^{2n}},$$

$$(2n+1)!_{q,\alpha} \quad := \quad \frac{(1+q)^{2n+1}\Gamma_{q^2}(\alpha+n+2)\Gamma_{q^2}(n+1)}{\Gamma_{q^2}(\alpha+1)} \quad = \quad \frac{(q^2; q^2)_n (q^{2\alpha+2}; q^2)_{n+1}}{(1-q)^{2n+1}}.$$

We denote

$$(q; q)_{2n,\alpha} = (q^2; q^2)_n (q^{2\alpha+2}; q^2)_n \ \text{ and } \ (q; q)_{2n+1,\alpha} = (q^2; q^2)_n (q^{2\alpha+2}; q^2)_{n+1}.$$

Remarks.

(1) If $\alpha = -\frac{1}{2}$, then we get

$$(q; q)_{n,-\frac{1}{2}} = (q; q)_n \ \text{ and } \ n!_{q,-\frac{1}{2}} = n!_q. \qquad (3.1)$$

(2) We have the recursion relations

$$(n+1)!_{q,\alpha} = [n+1+\theta_n(2\alpha+1)]_q \, n!_{q,\alpha}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (3.2)

$$(q; q)_{n+1,\alpha} = (1-q)[n+1+\theta_n(2\alpha+1)]_q (q; q)_{n,\alpha},$$

where

$$\theta_n = \begin{cases} 1 & \text{if} \quad n \text{ is even} \\ \\ 0 & \text{if} \quad n \text{ is odd.} \end{cases} \tag{3.3}$$

(3) It is easy to prove the following limits

$$\begin{aligned} \lim_{q \to 1^-} (2n)!_{q,\alpha} &= \frac{2^{2n} n! \Gamma(\alpha + n + 1)}{\Gamma(\alpha + 1)} = \gamma_{\alpha + \frac{1}{2}}(2n), \\ \lim_{q \to 1^-} (2n+1)!_{q,\alpha} &= \frac{2^{2n+1} n! \Gamma(\alpha + n + 2)}{\Gamma(\alpha + 1)} = \gamma_{\alpha + \frac{1}{2}}(2n+1), \end{aligned} \tag{3.4}$$

where γ_μ is the Rosenblum's generalized factorial (see [15]).

3.2. The generalized q-exponential functions.

By means of the generalized q-shifted factorials, we construct three generalized q-exponential functions as follows:

Definition 3.1. *For $z \in \mathbb{C}$, the generalized q-exponential functions are defined by*

$$E_{q,\alpha}(z) := \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}} z^k}{(q;q)_{k,\alpha}}. \tag{3.5}$$

$$e_{q,\alpha}(z) := \sum_{k=0}^{\infty} \frac{z^k}{(q;q)_{k,\alpha}}, \quad |z| < 1. \tag{3.6}$$

$$\psi_\lambda^{\alpha,q}(z) = \sum_{n=0}^{\infty} b_{n,\alpha}(i\lambda z; q^2), \quad \lambda \in \mathbb{C}, \tag{3.7}$$

where

$$b_{n,\alpha}(z; q^2) = \frac{q^{[\frac{n}{2}]([\frac{n}{2}]+1)} z^n}{n!_{q,\alpha}}. \tag{3.8}$$

Note that $\psi_\lambda^{\alpha,q}(z)$ is exactly the q-Dunkl kernel introduced in [3].

For $\alpha = -\dfrac{1}{2}$, it follows from (3.1) that $E_{q,-\frac{1}{2}}(z) = E_q(z)$, $e_{q,-\frac{1}{2}}(z) = e_q(z)$, $b_{n,-\frac{1}{2}}(x; q^2) = b_n(x; q^2)$ and we have $\psi_\lambda^{-\frac{1}{2},q}(x) = e(i\lambda x; q^2)$ the Rubin's q-exponential function (2.7). By (3.4), the q-Dunkl kernel $\psi_\lambda^{\alpha,q}(x)$ tends to $e_{\alpha+\frac{1}{2}}(i\lambda x)$ as $q \to 1^-$, where e_μ is the Rosenblum's generalized exponential function (see [15, 16]).

The generalized q-exponential function $\psi_\lambda^{\alpha,q}(x)$ gives rise to a q-integral transform, called the q-Dunkl transform on the real line, which was introduced and studied in [3]:

$$\mathcal{F}_D^{\alpha,q}(f)(\lambda) = K_\alpha \int_{-\infty}^{+\infty} f(x) \psi_{-\lambda}^{\alpha,q}(x) |x|^{2\alpha+1} d_q x, \quad f \in L_{\alpha,q}^1,$$

where

$$K_\alpha = \frac{(1-q)^\alpha \left(q^{2\alpha+2}; q^2\right)_\infty}{2 \left(q^2; q^2\right)_\infty}.$$

4. THE GENERALIZED q-DERIVATIVES

In this section we introduce a new class of q-derivatives operators which play an important role in the construction of a generalized q-Hermite polynomials.

4.1. The generalized q-derivative operators.

Definition 4.1. *The generalized backward and forward q-derivative operators $D_{q,\alpha}$ and $D_{q,\alpha}^+$ are defined as*

$$D_{q,\alpha}f(z) = \frac{f(z) - q^{2\alpha+1}f(qz)}{(1-q)z}, \tag{4.1}$$

$$D_{q,\alpha}^+ f(z) = \frac{f(q^{-1}z) - q^{2\alpha+1}f(z)}{(1-q)z}. \tag{4.2}$$

The operators given by

$$\boldsymbol{\Delta}_{\alpha,q}f = D_q f_e + D_{q,\alpha}f_o, \tag{4.3}$$
$$\boldsymbol{\Delta}_{\alpha,q}^+ f = D_q^+ f_e + D_{q,\alpha}^+ f_o \tag{4.4}$$

are called the generalized q-derivatives operators.

Remark that for $\alpha = -\dfrac{1}{2}$, we have:

$D_{q,-\frac{1}{2}} = D_q$, $D_{q,-\frac{1}{2}}^+ = D_q^+$, $\boldsymbol{\Delta}_{q,-\frac{1}{2}} = D_q$ and $\boldsymbol{\Delta}_{q,-\frac{1}{2}}^+ = D_q^+$.

The following elementary result is useful in the sequel.

Lemma 4.1.

$$\boldsymbol{\Delta}_{\alpha,q}^+ x^n = q^{-n}[n + \theta_{n+1}(2\alpha + 1)]_q x^{n-1}, \ n = 1, 2, 3, ... \tag{4.5}$$

$$\boldsymbol{\Delta}_{\alpha,q,x}^+ e_{q,\alpha}(qxt) = \frac{t}{1-q}e_{q,\alpha}(tx), \tag{4.6}$$

$$\boldsymbol{\Delta}_{\alpha,q,x}^+ \left[E_{q^2}(-q^2 x^2)e_{q,\alpha}(qxt) \right] = \frac{t-x}{1-q}E_{q^2}(-q^2 x^2)e_{q,\alpha}(tx), \tag{4.7}$$

where the operator $\boldsymbol{\Delta}_{\alpha,q,x}^+$ acts with respect to x and θ is given by (3.3).

Proof. By elementary calculus, we get (4.5).
On the one hand, using the definition of the operator $\boldsymbol{\Delta}_{\alpha,q}^+$ and (4.5), we have

$$\boldsymbol{\Delta}_{\alpha,q,x}^+ e_{q,\alpha}(qxt) = \sum_{n=0}^{\infty} \frac{(qt)^n \boldsymbol{\Delta}_{\alpha,q}^+ x^n}{(q;q)_{n,\alpha}} = \sum_{n=1}^{\infty} \frac{t^n[n + \theta_{n+1}(2\alpha + 1)]_q x^{n-1}}{(q;q)_{n,\alpha}}.$$

On the other hand, using the second recursion relation in (3.2) and changing the index in the second sum in the previous equation, we obtain

$$\boldsymbol{\Delta}_{\alpha,q,x}^+ e_{q,\alpha}(qxt) = \frac{t}{1-q} \sum_{n=0}^{\infty} \frac{t^n x^n}{(q;q)_{n,\alpha}} = \frac{t}{1-q}e_{q,\alpha}(xt).$$

Finally, using the infinite product representation (2.3) of $E_{q^2}(-q^2 x^2)$, we get

$$\boldsymbol{\Delta}_{\alpha,q,x}^+ \left[E_{q^2}(-q^2 x^2)e_{q,\alpha}(qxt) \right] = E_{q^2}(-q^2 x^2)\left[\boldsymbol{\Delta}_{\alpha,q}^+ e_{q,\alpha}(qxt) - \frac{x}{1-q}e_{q,\alpha}(xt) \right],$$

and (4.7) follows from (4.6). ∎

4.2. The q-Dunkl operator.

We can rewrite the q-Dunkl operator introduced in [3] by means of the generalized q-derivative operators introduced in Definition 4.1 as

$$\Lambda_{\alpha,q}f = \mathbf{\Delta}_{\alpha,q}^{+}f_e + \mathbf{\Delta}_{\alpha,q}f_o. \tag{4.8}$$

Indeed.

We have

$$\partial_q f = D_q^+ f_e + D_q f_o,$$

and

$$\Lambda_{\alpha,q}(f)(x) = \partial_q\left[H_{\alpha,q}(f)\right](x) + [2\alpha + 1]_q \frac{f(x) - f(-x)}{2x},$$

where

$$H_{\alpha,q} : f = f_e + f_o \longmapsto f_e + q^{2\alpha+1}f_o.$$

We can write, then,

$$
\begin{aligned}
\Lambda_{\alpha,q}f(x) &= \partial_q f_e(x) + q^{2\alpha+1}\partial_q f_o(x) + \frac{1 - q^{2\alpha+1}}{(1-q)x}f_o(x) \\
&= D_q^+ f_e(x) + q^{2\alpha+1}D_q f_o(x) + \frac{1 - q^{2\alpha+1}}{(1-q)x}f_o(x) \\
&= D_q^+ f_e(x) + D_{q,\alpha}f_o(x) \\
&= \mathbf{\Delta}_{\alpha,q}^{+}f_e(x) + \mathbf{\Delta}_{\alpha,q}f_o(x).
\end{aligned}
$$

It is noteworthy that in the case $\alpha = -\dfrac{1}{2}$, $\Lambda_{\alpha,q}$ reduces to the Rubin's q-derivative operator ∂_q defined in [13] and that for a differentiable function f, the q-Dunkl operator $\Lambda_{\alpha,q}f$ tends to the classical Dunkl operator $\Lambda_\alpha f$ as q tends to 1.

By induction, we prove the following results :

Proposition 4.1.

  (1) *A repeated application of the q-Dunkl operator to the monomial $b_{n,\alpha}(x; q^2)$ gives*

$$\Lambda_{\alpha,q}^{k}b_{n,\alpha}(x; q^2) = b_{n-k,\alpha}(x; q^2), \ k = 0, 1, ...n. \tag{4.9}$$

  (2) *If f is an even function, then*

$$\Lambda_{\alpha,q}^{2n}f(x) = q^{-n(n+1)}\mathbf{\Delta}_{\alpha,q}^{2n}f(q^{-n}x), \ n = 0, 1, 2, ..., \tag{4.10}$$

$$\Lambda_{\alpha,q}^{2n+1}f(x) = q^{-(n+1)^2}\mathbf{\Delta}_{\alpha,q}^{2n+1}f(q^{-n-1}x); \ n = 0, 1, 2, ... \tag{4.11}$$

4.3. The q-Dunkl intertwining operator.

For our further development, we need to extend the notion of the q-Dunkl intertwining operator introduced in [3] to the space $\mathcal{C}(\mathbb{R})$ of continuous functions on $\mathbb{R}$.

Definition 4.2. *The q-Dunkl intertwining operator $V_{\alpha,q}$ is defined on $\mathcal{C}(\mathbb{R})$ by*

$$V_{\alpha,q}f(x) = \frac{C(\alpha;q)}{2}\int_{-1}^{1}W_\alpha(t; q^2)(1+t)f(xt)d_qt, \ x \in \mathbb{R}, \tag{4.12}$$

where

$$C(\alpha;q) = \frac{(1+q)\Gamma_{q^2}(\alpha+1)}{\Gamma_{q^2}(1/2)\Gamma_{q^2}(\alpha+1/2)} \tag{4.13}$$

and

$$W_\alpha(t; q^2) = \frac{(t^2q^2; q^2)_\infty}{(t^2q^{2\alpha+1}; q^2)_\infty}. \tag{4.14}$$

Since the integrand is continuous, the q-integral (4.12) is well-defined.

In the following proposition we shall show that $V_{\alpha,q}$ is the intertwining operator between the generalized q-derivatives and the usual q-derivatives which generalize the transmutation relation (53) in [3].

Proposition 4.2. *Suppose that the function f and its q-derivatives $D_q f$, $D_q^+ f$ and $\partial_q f$ are in $\mathcal{C}(\mathbb{R})$, then*

$$\mathbf{\Delta}_{\alpha,q} V_{\alpha,q}(f) = V_{\alpha,q}(D_q f); \tag{4.15}$$

$$\mathbf{\Delta}_{\alpha,q}^+ V_{\alpha,q}(f) = V_{\alpha,q}(D_q^+ f); \tag{4.16}$$

$$\Lambda_{\alpha,q} V_{\alpha,q}(f) = V_{\alpha,q}(\partial_q f). \tag{4.17}$$

Proof. By splitting f into its even and odd parts $f = f_e + f_o$, and using the fact that D_q changes the parity of the function and the q-integral of an odd function on $[-1, 1]$ is equal zero, we obtain

$$\begin{aligned}
V_{\alpha,q}(D_q f)(x) &= \frac{C(\alpha;q)}{2} \int_{-1}^{1} W_\alpha(t;q^2) t D_q f_e(xt) d_q t \\
&+ \frac{C(\alpha;q)}{2} \int_{-1}^{1} W_\alpha(t;q^2) D_q f_0(xt) d_q t.
\end{aligned} \tag{4.18}$$

On the other hand, from the definition of $\mathbf{\Delta}_{\alpha,q}$ (4.3) we have

$$\begin{aligned}
\mathbf{\Delta}_{\alpha,q} V_{\alpha,q}(f)(x) &= \frac{C(\alpha;q)}{2} \int_{-1}^{1} W_\alpha(t;q^2) t D_q f_e(xt) d_q t \\
&+ \frac{C(\alpha;q)}{2} \int_{-1}^{1} W_\alpha(t;q^2) t^2 D_{q,\alpha} f_0(xt) d_q t.
\end{aligned} \tag{4.19}$$

Then, using the fact that $(1 - q^{2\alpha+1} t^2) W_\alpha(t;q^2) = (1 - q^2 t^2) W_\alpha(qt;q^2)$, we get

$$\begin{aligned}
V_{\alpha,q}(D_q f)(x) - \mathbf{\Delta}_{\alpha,q} V_{\alpha,q}(f)(x) &= \int_{-1}^{1} W_\alpha(t;q^2) \left[D_q f_0(xt) - t^2 D_{q,\alpha} f_0(xt) \right] d_q t \\
&= \int_{-1}^{1} W_\alpha(t;q^2) \frac{(1 - t^2) f_0(xt)}{(1 - q)xt} d_q t \\
&- \int_{-1}^{1} W_\alpha(t;q^2) \frac{(1 - q^{2\alpha+1} t^2) f_0(qxt)}{(1 - q)xt} d_q t \\
&= \int_{-1}^{1} W_\alpha(t;q^2) \frac{(1 - t^2) f_0(xt)}{(1 - q)xt} d_q t \\
&- \int_{-1}^{1} W_\alpha(qt;q^2) \frac{(1 - q^2 t^2) f_0(qxt)}{(1 - q)xt} d_q t.
\end{aligned}$$

Since the integrand in the last Jackson q-integral vanishes at the points -1 and 1, the change of variable $u = qt$ in this q-integral leads to the relation (4.15).

In a similar way, one can obtain (4.16).

Now by using (4.8), (4.15) and (4.16), we obtain

$$\Lambda_{\alpha,q} V_{\alpha,q}(f) = \mathbf{\Delta}_{\alpha,q}^+ V_{\alpha,q} f_e + \mathbf{\Delta}_{\alpha,q} V_{\alpha,q} f_o = V_{\alpha,q} \left(D_q^+ f_e + D_q f_o \right) = V_{\alpha,q} \left(\partial_q f \right).$$

$\blacksquare$

The following result shows the effect of the q-Dunkl intertwining operator on monomial functions and on the q-exponential functions.

Proposition 4.3. *The following relations hold:*

$$V_{\alpha,q}z^n = \frac{(q;q)_n}{(q;q)_{n,\alpha}}z^n, \quad n = 0,1,2,...; \tag{4.20}$$

$$V_{\alpha,q}b_n(z;q^2) = b_{n,\alpha}(z;q^2), \quad n = 0,1,2,...; \tag{4.21}$$

$$V_{\alpha,q}e_q(z) = e_{q,\alpha}(z), \quad |z| < 1; \tag{4.22}$$

$$V_{\alpha,q}E_q(z) = E_{q,\alpha}(z); \tag{4.23}$$

$$V_{\alpha,q}e(\lambda z;q^2) = \psi_{-i\lambda}^{\alpha,q}(z), \quad \lambda \in \mathbb{C}. \tag{4.24}$$

Proof. If we take $x = n - \dfrac{1}{2}$ and $y = \alpha + \dfrac{1}{2}$ in (2.12), we get by using (4.13)

$$
\begin{aligned}
V_{\alpha,q}z^{2n} &= C(\alpha;q)z^{2n}\int_0^1 \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}t^{2n}d_qt = C(\alpha;q)\frac{\Gamma_{q^2}(n+\frac{1}{2})\Gamma_{q^2}(\alpha+\frac{1}{2})}{(1+q)\Gamma_{q^2}(n+\alpha+1)}z^{2n} \\
&= \frac{\Gamma_{q^2}(\alpha+1)\Gamma_{q^2}(n+\frac{1}{2})}{\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(n+\alpha+1)}z^{2n} = \frac{(q;q^2)_n}{(q^{2\alpha+2};q^2)_n}z^{2n}.
\end{aligned}
$$

Similarly, we prove that

$$V_{\alpha,q}z^{2n+1} = C(\alpha;q)z^{2n+1}\int_0^1 \frac{(t^2q^2;q^2)_\infty}{(t^2q^{2\alpha+1};q^2)_\infty}t^{2(n+1)}d_qt = \frac{(q;q^2)_{n+1}}{(q^{2\alpha+2};q^2)_{n+1}}z^{2n+1}.$$

Then (4.20) follows from the two following facts

$$\frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}} = \frac{(q;q^2)_n}{(q^{2\alpha+2};q^2)_n}, \quad \frac{(q;q)_{2n+1}}{(q;q)_{2n+1,\alpha}} = \frac{(q;q^2)_{n+1}}{(q^{2\alpha+2};q^2)_{n+1}}. \tag{4.25}$$

For $|z| < 1$, we have by using (4.20)

$$V_{\alpha,q}e_q(z) = \sum_{k=0}^\infty \frac{V_{\alpha,q}z^k}{(q;q)_k} = \sum_{k=0}^\infty \frac{z^k}{(q;q)_{k,\alpha}} = e_{q,\alpha}(z).$$

The same techniques produce the relations (4.23) and (4.24).   ∎

5. The generalized discrete q-Hermite polynomials

We begin this section by the following useful lemma.

Lemma 5.1.

(1) *For $s \geq 0$, we have*

$$F_q(s) := \int_0^1 t^s E_{q^2}(-q^2t^2)d_qt = (1-q)\frac{(q^2;q^2)_\infty}{(q^{s+1};q^2)_\infty}. \tag{5.1}$$

(2) *For $\lambda > 0$ and n non-negative integer, we have*

$$G_{q,n}^\alpha(\lambda) := \int_0^\infty e_{q^2}(-\lambda y^2)y^{2n+2\alpha+1}d_qy = c_{q,\alpha}(\lambda)\frac{q^{-n^2-(2\alpha+1)n}}{\lambda^n}\left(q^{2\alpha+2};q^2\right)_n, \tag{5.2}$$

     where

$$c_{q,\alpha}(\lambda) = \frac{(1-q)(-q^{2\alpha+2}\lambda, -q^{-2\alpha}/\lambda, q^2; q^2)_\infty}{(-\lambda, -q^2/\lambda, q^{2\alpha+2}; q^2)_\infty}. \tag{5.3}$$

Proof. (1) Let $s \geq 0$. From (2.11), we have by doing the two changes of variables $u = (1-q^2)t$ and $u = t^2$,

$$\Gamma_{q^2}(s) = \frac{1}{(1-q^2)^s} \int_0^1 t^{s-1} E_{q^2}(-q^2 t) d_{q^2} t = \frac{1+q}{(1-q^2)^s} \int_0^1 u^{2s-1} E_{q^2}(-q^2 u^2) d_q u.$$

The relation (5.1) follows then by replacing s by $\dfrac{s+1}{2}$ in the previous equation.

(2) Let $\lambda > 0$ and n non negative integer. Then, from the definition of the Jackson q-integral and the Ramanujan identity (see [9], p. 125), we have

$$\int_0^\infty e_{q^2}(-\lambda y^2) y^s d_q y \;=\; (1-q) \sum_{k=-\infty}^\infty \frac{q^{(s+1)k}}{(-\lambda q^{2k}; q^2)_\infty}$$

$$=\; \frac{(1-q)\left(-\lambda q^{s+1}, -q^{-s+1}/\lambda, q^2; q^2\right)_\infty}{(-\lambda, q^{s+1}, -q^2/\lambda; q^2)_\infty}.$$

In particular for $s = 2n + 2\alpha + 1$, we have

$$\int_0^\infty e_{q^2}(-\lambda y^2) y^{2n+2\alpha+1} d_q y = \frac{(1-q)\left(-\lambda q^{2\alpha+2n+2}, -q^{-2\alpha-2n}/\lambda, q^2; q^2\right)_\infty}{(-\lambda, q^{2\alpha+2n+2}, -q^2/\lambda; q^2)_\infty}.$$

We conclude (5.2) by using the following equalities:

$$\left(-\lambda q^{2\alpha+2n+2}; q^2\right)_\infty = \frac{\left(-q^{2\alpha+2}\lambda; q^2\right)_\infty}{\left(-q^{2\alpha+2}\lambda; q^2\right)_n}, \quad \left(q^{2\alpha+2n+2}; q^2\right)_\infty = \frac{\left(q^{2\alpha+2}; q^2\right)_\infty}{\left(q^{2\alpha+2}; q^2\right)_n}$$

and

$$\left(-q^{-2\alpha-2n}/\lambda; q^2\right)_\infty = q^{-n(n+1)}(q^{2\alpha}\lambda)^{-n}\left(-q^{2\alpha+2}\lambda; q^2\right)_n\left(-(q^{2\alpha}\lambda)^{-1}; q^2\right)_\infty.$$

∎

5.1. The generalized discrete q-Hermite I polynomials.

The discrete q-Hermite I polynomials $\{h_n(x;q)\}_{n=0}^\infty$ are defined in [11] by

$$\begin{aligned}
h_n(x;q) \;&:=\; x^n \, {}_2\phi_0(q^{-n}, q^{-n+1}; -; q^2, q^{2n-1}x^{-2}) \\
&=\; (q;q)_n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k q^{k(k-1)} x^{n-2k}}{(q^2;q^2)_k (q;q)_{n-2k}}.
\end{aligned} \tag{5.4}$$

Definition 5.1. *The generalized discrete q-Hermite I polynomials $\{h_{n,\alpha}(x;q)\}_{n=0}^\infty$ are defined by:*

$$h_{n,\alpha}(x;q) := (q;q)_n \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k q^{k(k-1)} x^{n-2k}}{(q^2;q^2)_k (q;q)_{n-2k,\alpha}}. \tag{5.5}$$

Remarks.

(1) For $\alpha = -\dfrac{1}{2}$, we get $h_{n,-\frac{1}{2}}(x;q) = h_n(x;q)$.

(2) Observe that

$$\frac{h_{n,\alpha}(\sqrt{1-q^2}x; q)}{(1-q^2)^{\frac{n}{2}}} = \frac{n!_q}{(1+q)^n} \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k q^{k(k-1)}((1+q)x)^{n-2k}}{k!_{q^2}(n-2k)!_{q,\alpha}}.$$

So by (3.4), we obtain

$$\lim_{q\to 1^-}\frac{h_{n,\alpha}(\sqrt{1-q^2}x;q)}{(1-q^2)^{\frac{n}{2}}}=\frac{H_n^{\alpha+\frac{1}{2}}(x)}{2^n},$$

where $H_n^{\alpha+\frac{1}{2}}(x)$ is the Rosenblum's generalized Hermite polynomial (see [15]).

(3) Each polynomial $h_{n,\alpha}(.;q)$ has the same parity of its degree n.

Lemma 5.2. *The generalized discrete q-Hermite I polynomials can be written in terms of basic hypergeometric functions as:*

$$\begin{cases}
h_{2n,\alpha}(x;q) &= \dfrac{(q;q)_{2n}}{(q;q)_{2n,\alpha}}x^{2n}\,{}_2\phi_0(q^{-2n},q^{-2n-2\alpha};-;q^2,q^{4n+2\alpha}x^{-2})\\
&= (-1)^n q^{n(n-1)}(q;q^2)_n\,{}_2\phi_1(q^{-2n},0;q^{2\alpha+2};q^2,q^2x^2),\\
h_{2n+1,\alpha}(x;q) &= \dfrac{(q;q)_{2n+1}}{(q;q)_{2n+1,\alpha}}x^{2n+1}\,{}_2\phi_0(q^{-2n},q^{-2n-2\alpha-2};-;q^2,q^{4n+2\alpha+2}x^{-2})\\
&= (-1)^n q^{n(n-1)}\dfrac{(q;q^2)_{n+1}}{1-q^{2\alpha+2}}x\,{}_2\phi_1(q^{-2n},0;q^{2\alpha+4};q^2,q^2x^2).
\end{cases}$$
(5.6)

Proof. We have

$$h_{2n,\alpha}(x;q) = (q;q)_{2n}\sum_{k=0}^n\frac{(-1)^k q^{k(k-1)}x^{2n-2k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}(q^{2\alpha+2};q^2)_{n-k}}.$$

Using the identity

$$(a;q^2)_{n-k}=\frac{(a;q^2)_n}{(a^{-1}q^{2-2n};q^2)_k}\left(-q^2 a^{-1}\right)^k q^{k(k-1)-2nk},$$
(5.7)

we get

$$(q^2;q^2)_{n-k}=\frac{(q^2;q^2)_n}{(q^{-2n};q^2)_k}(-1)^k q^{k(k-1)-2nk}$$
(5.8)

and

$$(q^{2\alpha+2};q^2)_{n-k}=\frac{(q^{2\alpha+2};q^2)_n}{(q^{-2n-2\alpha};q^2)_k}\left(-q^{-2\alpha}\right)^k q^{k(k-1)-2nk}.$$
(5.9)

It follows, then, that

$$\begin{aligned}
h_{2n,\alpha}(x;q) &= \frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}}x^{2n}\\
&\quad\times\sum_{k=0}^n\frac{(-1)^k q^{-k(k-1)}(q^{-2n};q^2)_k(q^{-2n-2\alpha};q^2)_k\left(q^{4n+2\alpha}x^{-2}\right)^k}{(q^2;q^2)_k}\\
&= \frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}}x^{2n}\,{}_2\phi_0(q^{-2n},q^{-2n-2\alpha};-;q^2,q^{4n+2\alpha}x^{-2}).
\end{aligned}$$

Now, we have

$$h_{2n+1,\alpha}(x;q)=(q;q)_{2n+1}\sum_{k=0}^n\frac{(-1)^k q^{k(k-1)}x^{2n+1-2k}}{(q^2;q^2)_k(q^2;q^2)_{n-k}(q^{2\alpha+2};q^2)_{n+1-k}}.$$

Then, using (5.8) and replacing n by $n+1$ in (5.9), we obtain

$$(q^{2\alpha+2};q^2)_{n+1-k} = \frac{(q^{2\alpha+2};q^2)_{n+1}}{(q^{-2n-2\alpha-2};q^2)_k}\left(-q^{-2\alpha}\right)^k q^{k(k-1)-2nk-2k}$$

and

$$
\begin{aligned}
h_{2n+1,\alpha}(x;q) &= \frac{(q;q)_{2n+1}}{(q;q)_{2n+1,\alpha}}x^{2n+1} \\
&\quad \times \sum_{k=0}^{n}\frac{(-1)^k q^{-k(k-1)}(q^{-2n};q^2)_k(q^{-2n-2\alpha-2};q^2)_k\left(q^{4n+2\alpha+2}x^{-2}\right)^k}{(q^2;q^2)_k} \\
&= \frac{(q;q)_{2n+1}}{(q;q)_{2n+1,\alpha}}x^{2n+1}\ {}_2\phi_0(q^{-2n},q^{-2n-2\alpha-2};-;q^2,q^{4n+2\alpha+2}x^{-2}).
\end{aligned}
$$

On the other hand, taking $b = q^{-2n-2\alpha}$ and $z = q^{2n+2\alpha}x^{-2}$ in the following transformation formula (see [11], p.19)

$$
{}_2\phi_0(q^{-2n},b;-;q^2,q^{2n}z) = (b;q^2)_n z^n\ {}_2\phi_1(q^{-2n},0;b^{-1}q^{2-2n};q^2,\frac{q^2}{bz}),
\qquad (5.10)
$$

we obtain

$$
\begin{aligned}
{}_2\phi_0(q^{-2n},q^{-2n-2\alpha};-;q^2,q^{4n+2\alpha}x^{-2}) &= (q^{-2n-2\alpha};q^2)_n q^{2n^2+2\alpha n}x^{-2n} \\
&\quad \times\ {}_2\phi_1(q^{-2n},0;q^{2\alpha+2};q^2,q^2 x^2).
\end{aligned}
$$

By the identity (see [11], p.9)

$$(a;q^2)_n = (a^{-1}q^{2-2n};q^2)_n(-a)^n q^{n(n-1)},\quad a\neq 0, \qquad (5.11)$$

we get

$$
\begin{aligned}
{}_2\phi_0(q^{-2n},q^{-2n-2\alpha};-;q^2,q^{4n+2\alpha}x^{-2}) &= (q^{2\alpha+2};q^2)_n(-1)^n q^{n(n-1)}x^{-2n} \\
&\quad \times\ {}_2\phi_1(q^{-2n},0;q^{2\alpha+2};q^2,q^2 x^2).
\end{aligned}
$$

So, it follows from (4.25) that

$$h_{2n,\alpha}(x;q) = (-1)^n q^{n(n-1)}(q;q^2)_n\ {}_2\phi_1(q^{-2n},0;q^{2\alpha+2};q^2,q^2 x^2).$$

Now, take $b = q^{-2n-2\alpha-2}$ and $z = q^{2n+2\alpha+2}x^{-2}$ in (5.10), to obtain

$$
\begin{aligned}
{}_2\phi_0(q^{-2n},q^{-2n-2\alpha-2};-;q^2,q^{4n+2\alpha+2}x^{-2}) &= (q^{-2n-2\alpha-2};q^2)_n q^{2n^2+2\alpha n+2n}x^{-2n} \\
&\quad \times\ {}_2\phi_1(q^{-2n},0;q^{2\alpha+4};q^2,q^2 x^2)
\end{aligned}
$$

By identity (5.11), we have

$$
\begin{aligned}
{}_2\phi_0(q^{-2n},q^{-2n-2\alpha-2};-;q^2,q^{4n+2\alpha+2}x^{-2}) &= (q^{2\alpha+4};q^2)_n(-1)^n q^{n(n-1)}x^{-2n} \\
&\quad \times\ {}_2\phi_1(q^{-2n},0;q^{2\alpha+4};q^2,q^2 x^2),
\end{aligned}
$$

and by (4.25), we get

$$h_{2n+1,\alpha}(x;q) = (-1)^n q^{n(n-1)}\frac{(q;q^2)_{n+1}}{1-q^{2\alpha+2}}x\ {}_2\phi_1(q^{-2n},0;q^{2\alpha+4};q^2,q^2 x^2).$$

$$\blacksquare$$

The little q-Laguerre polynomials $\{p_n(x;a|q)\}_{n=0}^{\infty}$ are defined in [11] by

$$p_n(x;a|q) = {}_2\phi_1(q^{-n},0;aq;q,qx). \qquad (5.12)$$

Using (5.39) and (5.12), the generalized discrete q-Hermite I polynomials $h_{n,\alpha}(x;q)$ can be expressed in terms of the little q-Laguerre polynomials $p_n(x;a|q)$ as follows:

$$\begin{cases} h_{2n,\alpha}(x;q) &= (-1)^n q^{n(n-1)}(q;q^2)_n p_n(x^2;q^{2\alpha}|q^2), \\ h_{2n+1,\alpha}(x;q) &= (-1)^n q^{n(n-1)}\dfrac{(q;q^2)_{n+1}}{1-q^{2\alpha+2}}x p_n(x^2;q^{2\alpha+2}|q^2). \end{cases}$$

Proposition 5.1. *The following relations hold:*
(1) q-Integral representation of Mehler type

$$h_{n,\alpha}(x;q) = V_{\alpha,q}h_n(x;q). \tag{5.13}$$

(2) Generating function

$$E_{q^2}(-z^2)e_{q,\alpha}(xz) = \sum_{n=0}^{\infty} h_{n,\alpha}(x;q)\frac{z^n}{(q;q)_n}, \quad (|xz| < 1). \tag{5.14}$$

(3) Inversion formula

$$x^n = (q;q)_{n,\alpha}\sum_{k=0}^{[\frac{n}{2}]} \frac{h_{n-2k,\alpha}(x;q)}{(q^2;q^2)_k(q;q)_{n-2k}}. \tag{5.15}$$

(4) Three terms recursion formula

$$xh_{n,\alpha}(x;q) - q^{n-1+(2\alpha+1)\theta_n}(1-q^n)h_{n-1,\alpha}(x;q) = \frac{1-q^{n+1+(2\alpha+1)\theta_n}}{1-q^{n+1}}h_{n+1,\alpha}(x;q). \tag{5.16}$$

Proof. (1) The relation (5.13) follows by application of (4.20) to each term in (5.4).
(2) The generating function for the discrete q-Hermite I polynomials is given by (see [11])

$$E_{q^2}(-z^2)e_q(xz) = \sum_{n=0}^{\infty} h_n(x;q)\frac{z^n}{(q;q)_n}. \tag{5.17}$$

So, by applying the operator $V_{\alpha,q}$, with respect to x, to both sides of equation (5.17) and using (5.13) and (4.22), we obtain (5.14).
(3) follows by application of the operator $V_{\alpha,q}$ to both sides of the well-known relation (see [12]):

$$x^n = (q;q)_n\sum_{k=0}^{[\frac{n}{2}]} \frac{h_{n-2k}(x;q)}{(q^2;q^2)_k(q;q)_{n-2k}},$$

(4.20) and (5.13).
(4) To prove (5.16), we consider, separately, the even and the odd cases in the expression

$$xh_{n,\alpha}(x;q) - q^{n-1+(2\alpha+1)\theta_n}(1-q^n)h_{n-1,\alpha}(x;q).$$

We have

$$\begin{aligned} xh_{2n,\alpha}(x;q) \quad &- \quad q^{2n+2\alpha}(1-q^{2n})h_{2n-1,\alpha}(x;q) \\ &= (q;q)_{2n}\sum_{k=0}^{n} \frac{(-1)^k q^{k(k-1)}x^{2n-2k+1}}{(q^2;q^2)_k(q;q)_{2n-2k,\alpha}} \\ &\quad -q^{2n+2\alpha}(q;q)_{2n}\sum_{k=0}^{n-1} \frac{(-1)^k q^{k(k-1)}x^{2n-2(k+1)+1}}{(q^2;q^2)_k(q;q)_{2n-2(k+1)+1,\alpha}}. \end{aligned}$$

Change k to $k-1$ in the second sum, then combine with the first to get

$$\frac{(q;q)_{2n}x^{2n+1}}{(q;q)_{2n,\alpha}} + (q;q)_{2n}\sum_{k=1}^{n}\frac{(-1)^kq^{k(k-1)}x^{2n-2k+1}}{(q^2;q^2)_k(q;q)_{2n-2k+1,\alpha}}$$
$$\times\left[(1-q^{2n-2k+2\alpha+2}) + q^{2n+2\alpha}q^{-2k+2}(1-q^{2k})\right]$$

Simplify to obtain

$$\frac{(q;q)_{2n}x^{2n+1}}{(q;q)_{2n,\alpha}} + (1-q^{2n+2\alpha+2})(q;q)_{2n}\sum_{k=1}^{n}\frac{(-1)^kq^{k(k-1)}x^{2n-2k+1}}{(q^2;q^2)_k(q;q)_{2n-2k+1,\alpha}}.$$

The last expression can be written as

$$\frac{1-q^{2n+2\alpha+2}}{1-q^{2n+1}}h_{2n+1,\alpha}(x;q).$$

In the odd case, the proof follows the same steps as the even case.

$\blacksquare$

The following result states the affect of the operators $\boldsymbol{\Delta}_{\alpha,q}$ and $\boldsymbol{\Delta}_{\alpha,q}^{+}$ on the generalized q-Hermite I polynomials.

Proposition 5.2.
(1) The forward shift operator:

$$\boldsymbol{\Delta}_{\alpha,q}h_{n,\alpha}(x;q) = \frac{1-q^n}{1-q}h_{n-1,\alpha}(x;q), \quad n = 1,2,... \tag{5.18}$$

or equivalently

$$h_{n,\alpha}(x;q) - q^{(2\alpha+1)\theta_{n+1}}h_{n,\alpha}(qx;q) = (1-q^n)xh_{n-1,\alpha}(x;q). \tag{5.19}$$

(2) The backward shift operator:

$$\boldsymbol{\Delta}_{\alpha,q}^{+}\left[E_{q^2}(-q^2x^2)h_{n,\alpha}(x;q)\right] = -\frac{q^{-n}[\theta_n(2\alpha+1)+n+1]_q}{1-q^{n+1}}E_{q^2}(-q^2x^2)h_{n+1,\alpha}(x;q), \tag{5.20}$$

or equivalently

$$q^{\theta_{n+1}(2\alpha+1)}h_{n,\alpha}(x;q) - (1-x^2)h_{n,\alpha}(q^{-1}x;q) = \frac{q^{-n}[\theta_n(2\alpha+1)+n+1]_q}{[n+1]_q}xh_{n+1,\alpha}(x;q).$$

Proof. (1) It is well known (see [11]) that

$$D_qh_n(x;q) = \frac{1-q^n}{1-q}h_{n-1}(x;q).$$

So, by application of the q-Dunkl intertwining operator to both sides of this result, we obtain (5.18), by the use of (4.15) and (5.13).
(2) Put

$$g(x,t) = E_{q^2}(-q^2x^2)E_{q^2}(-q^2t^2)e_{q,\alpha}(qxt).$$

From (4.7), we have

$$\begin{aligned}
\boldsymbol{\Delta}_{\alpha,q,x}^{+}g(x,t) &= E_{q^2}(-q^2t^2)\frac{t-x}{1-q}E_{q^2}(-q^2x^2)e_{q,\alpha}(xt)\\
&= -E_{q^2}(-q^2x^2)\frac{x-t}{1-q}E_{q^2}(-q^2t^2)e_{q,\alpha}(xt)\\
&= -\boldsymbol{\Delta}_{\alpha,q,t}^{+}g(x,t).
\end{aligned} \tag{5.21}$$

But, using the generating function (5.17), we obtain

$$\mathbf{\Delta}^+_{\alpha,q,x}g(x,t) = \sum_{n=0}^{\infty} \mathbf{\Delta}^+_{\alpha,q,x}\left[E_{q^2}(-q^2x^2)h_{n,\alpha}(x;q)\right]\frac{(qt)^n}{(q;q)_n} \qquad (5.22)$$

and

$$\mathbf{\Delta}^+_{\alpha,q,x}g(x,t) = -\sum_{n=0}^{\infty} E_{q^2}(-q^2x^2)h_{n,\alpha}(x;q)\frac{q^n}{(q;q)_n}\mathbf{\Delta}^+_{\alpha,q}t^n.$$

It follows from (4.5) and (5.21) that

$$\mathbf{\Delta}^+_{\alpha,q,x}g(x,t) = -\sum_{n=1}^{\infty} \frac{E_{q^2}(-q^2x^2)h_{n,\alpha}(x;q)}{(q;q)_n}[\theta_{n+1}(2\alpha+1)+n]_q t^{n-1}. \qquad (5.23)$$

Therefore, by comparing the coefficients of t^n in the two series (5.22) and (5.23), we obtain (5.20). ∎

The following formula can now be proved by induction.

Proposition 5.3. *The Rodrigues-formula for* $\{h_{n,\alpha}(x;q)\}_{n=0}^{\infty}$ *is given by*

$$E_{q^2}(-q^2x^2)h_{n,\alpha}(x;q) = \frac{(q-1)^n q^{\frac{n(n-1)}{2}}(q;q)_n}{(q;q)_{n,\alpha}}\left[\mathbf{\Delta}^+_{\alpha,q}\right]^n\left[E_{q^2}(-q^2x^2)\right]. \qquad (5.24)$$

Proof. Since $h_{0,\alpha}(x;q) = 1$, the formula is clearly true for $n = 0$. Assume that it is true for an integer n. Then, using (5.20) and (3.2), the application of the operator $\mathbf{\Delta}^+_{\alpha,q}$ to the both sides of (5.24) completes the induction proof. ∎

Proposition 5.4. *The polynomials* $\{h_{n,\alpha}(x;q)\}_{n=0}^{\infty}$ *satisfy the following q-difference equations:*

$$q^{2\alpha+1}h_{2n,\alpha}(qx;q) - (q+q^{2\alpha+1}-q^{1-2n}x^2)h_{2n,\alpha}(x;q) + q(1-x^2)h_{2n,\alpha}(q^{-1}x;q) = 0. \qquad (5.25)$$

$$q^{2\alpha+1}h_{2n+1,\alpha}(qx;q)-(1+q^{2\alpha+2}-q^{-2n}x^2)h_{2n+1,\alpha}(x;q)+q(1-x^2)h_{2n+1,\alpha}(q^{-1}x;q) = 0. \qquad (5.26)$$

Proof. By (5.18), we have

$$\mathbf{\Delta}^2_{\alpha,q}h_{2n,\alpha}(x;q) = \frac{(1-q^{2n})(1-q^{2n-1})}{(1-q)^2}h_{2n-2,\alpha}(x;q), \quad n = 1,2,3,....$$

But, from the three recursion term formulas (5.16), we obtain

$$xh_{2n-1,\alpha}(x;q) - q^{2n-2}(1-q^{2n-1})h_{2n-2,\alpha}(x;q) = h_{2n,\alpha}(x;q) \qquad (5.27)$$

and from (5.19), we get

$$h_{2n,\alpha}(x;q) - h_{2n,\alpha}(qx;q) = (1-q^{2n})xh_{2n-1,\alpha}(x;q). \qquad (5.28)$$

It follows, then, from (5.27) and (5.28) that

$$\mathbf{\Delta}^2_{\alpha,q}h_{2n,\alpha}(x;q) = \frac{1}{(1-q)^2}\left[q^2h_{2n,\alpha}(x;q) - q^{2-2n}h_{2n,\alpha}(qx;q)\right]$$

or equivalently,

$$(1-q^2x^2)h_{2n,\alpha}(x;q)-(1+q^{2\alpha}-q^{2-2n}x^2)h_{2n,\alpha}(qx;q)+q^{2\alpha}h_{2n,\alpha}(q^2x;q) = 0. \qquad (5.29)$$

Replace x by $q^{-1}x$ in (5.29) and multiply the result by q, we obtain (5.25).
Following the same steps, we prove (5.26). ∎

To prove the orthogonality property of $h_{n,\alpha}(x;q)$, we need the following lemma:

Lemma 5.3. *The polynomials* $\{h_{n,\alpha}(x;q)\}_{n=0}^{\infty}$ *satisfy*

$$
\int_{-1}^{1} x^p h_{n,\alpha}(x;q) E_{q^2}(-q^2 x^2)|x|^{2\alpha+1} d_q x
$$
$$
= \begin{cases} \dfrac{2(1-q)(q^2;q^2)_\infty}{(q^{2\alpha+2};q^2)_\infty}(q;q)_n q^{[\frac{n}{2}](2[\frac{n+1}{2}]+2\alpha)} & \text{if } p = n, \\ 0 & \text{if } p = 0,1,...,n-1. \end{cases} \tag{5.30}
$$

Proof. Since the parity of the q-polynomials $\{h_{n,\alpha}(x;q)\}$ is the parity of thier degrees and the q-integral in (5.30) of odd function is zero. It is, then, sufficient to consider the cases where p and n are both even or odd.
From the definition (5.4), we can write

$$
\int_{-1}^{1} x^{2p} h_{2n,\alpha}(x;q) E_{q^2}(-q^2 x^2)|x|^{2\alpha+1} d_q x
$$
$$
= 2(q;q)_{2n} \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k-1)} F_q(2k+2p+2\alpha+1)}{(q^2;q^2)_{n-k}(q^2;q^2)_k(q^{2\alpha+2};q^2)_k}, \tag{5.31}
$$

where F_q is the function defined by (5.1), then the above sum becomes

$$
2(1-q)(q^2;q^2)_\infty (q;q)_{2n} \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k-1)}}{(q^2;q^2)_{n-k}(q^2;q^2)_k(q^{2\alpha+2};q^2)_k(q^{2k+2p+2\alpha+2};q^2)_\infty}.
$$

Using (2.1), it is possible to rewrite the sum in (5.31) in the form

$$
\frac{2(1-q)(q^2;q^2)_\infty (q;q)_{2n}}{(q^{2\alpha+2};q^2)_\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k-1)}}{(q^2;q^2)_{n-k}(q^2;q^2)_k}(q^{2k+2\alpha+2};q^2)_p.
$$

By using the transformation formulas (5.8) and the fact that

$$
(q^{2k+2\alpha+2};q^2)_p = \frac{(q^{2\alpha+2};q^2)_p(q^{2p+2\alpha+2};q^2)_k}{(q^{2\alpha+2};q^2)_k}, \tag{5.32}
$$

the sum in (5.31) can be written as

$$
\frac{(-1)^n q^{n(n-1)} 2(1-q)(q^2;q^2)_\infty (q;q)_{2n}(q^{2\alpha+2};q^2)_p}{(q^{2\alpha+2};q^2)_\infty (q^2;q^2)_n}
$$
$$
\times \sum_{k=0}^{n} \frac{(q^{-2n};q^2)_k(q^{2p+2\alpha+2};q^2)_k}{(q^2;q^2)_k(q^{2\alpha+2};q^2)_k} q^{2k}
$$
$$
= \frac{(-1)^n q^{n(n-1)} 2(1-q)(q^2;q^2)_\infty (q;q)_{2n}(q^{2\alpha+2};q^2)_p}{(q^{2\alpha+2};q^2)_\infty (q^2;q^2)_n}
$$
$$
\times {}_2\phi_1(q^{-2n}, q^{2p+2\alpha+2}; q^{2\alpha+2}; q^2, q^2).
$$

By the summation formula (see [11], p.15)

$$
{}_2\phi_1(q^{-2n}, b; c; q^2, q^2) = \frac{(b^{-1}c;q^2)_n}{(c;q^2)_n} b^n,
$$

the sum in (5.31) is equal to

$$\frac{(-1)^n q^{n(n-1)} 2(1-q)(q^2;q^2)_\infty (q;q)_{2n}(q^{2\alpha+2};q^2)_p}{(q^{2\alpha+2};q^2)_\infty (q^2;q^2)_n} \times \frac{(q^{-2p};q^2)_n}{(q^{2\alpha+2};q^2)_n} q^{(2p+2\alpha+2)n}.$$

So, if $p < n$, the q-integral in (5.31) vanishes.
If $p = n$,

$$(q^{-2n};q^2)_n = (q^2;q^2)_n(-1)^n q^{-2n-n(n-1)}, \tag{5.33}$$

and the q-integral in (5.31) is equal to

$$\frac{2(1-q)(q^2;q^2)_\infty (q;q)_{2n} q^{2n^2+2\alpha n}}{(q^{2\alpha+2};q^2)_\infty}.$$

In the case where both p and n are odd integers the proof follows the same steps as in the first case.

$$\int_{-1}^{1} x^{2p+1} h_{2n+1,\alpha}(x;q) E_{q^2}(-q^2 x^2)|x|^{2\alpha+1} d_q x$$

$$= 2(q;q)_{2n+1} \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k-1)} F_q(2k+2p+2\alpha+3)}{(q^2;q^2)_{n-k}(q^2;q^2)_k(q^{2\alpha+2};q^2)_{k+1}}$$

$$= 2(1-q)(q^2;q^2)_\infty (q;q)_{2n+1}$$
$$\times \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k-1)}}{(q^2;q^2)_{n-k}(q^2;q^2)_k(q^{2\alpha+2};q^2)_{k+1}(q^{2k+2p+2\alpha+4};q^2)_\infty}$$

$$= \frac{(-1)^n q^{n(n-1)} 2(1-q)(q^2;q^2)_\infty (q;q)_{2n+1}(q^{2\alpha+4};q^2)_p}{(q^{2\alpha+2};q^2)_\infty (q^2;q^2)_n}$$
$$\times {}_2\phi_1(q^{-2n}, q^{2p+2\alpha+4}; q^{2\alpha+4}; q^2, q^2)$$

$$= \frac{(-1)^n q^{n(n-1)} 2(1-q)(q^2;q^2)_\infty (q;q)_{2n+1}(q^{2\alpha+4};q^2)_p}{(q^{2\alpha+2};q^2)_\infty (q^2;q^2)_n}$$
$$\times \frac{(q^{-2p};q^2)_n}{(q^{2\alpha+4};q^2)_n} q^{(2p+2\alpha+4)n}.$$

So, if $p < n$, the above q-integral vanishes.
If $p = n$, by (5.33) the q-integral is equal to

$$\frac{2(1-q)(q^2;q^2)_\infty (q;q)_{2n+1} q^{2n^2+2\alpha n+2n}}{(q^{2\alpha+2};q^2)_\infty}.$$

$\blacksquare$

Theorem 5.1. *For $n, m = 0, 1, 2, ...$, we have the orthogonality relation*

$$\int_{-1}^{1} h_{m,\alpha}(x;q) h_{n,\alpha}(x;q) E_{q^2}(-q^2 x^2)|x|^{2\alpha+1} d_q x$$
$$= \frac{2(1-q)(q^2;q^2)_\infty (q;q)_n^2 q^{[\frac{n}{2}](2[\frac{n+1}{2}]+2\alpha)}}{(q^{2\alpha+2};q^2)_\infty (q;q)_{n,\alpha}} \delta_{n,m}. \tag{5.34}$$

Proof. Without loss of generality, we assume that $m \leq n$. By (5.5), the q-integral in (5.34) is equal to

$$(q;q)_m \sum_{p=0}^{[\frac{m}{2}]} \frac{(-1)^p q^{p(p-1)}}{(q^2;q^2)_p (q;q)_{m-2p,\alpha}} \int_{-1}^{1} x^{m-2p} h_{n,\alpha}(x;q) E_{q^2}(-q^2 x^2)|x|^{2\alpha+1} d_q x.$$

Then, Lemma (5.3) completes the proof of the theorem. ∎

5.2. The generalized discrete q-Hermite II polynomials.

Recall that the discrete q-Hermite II polynomials $\{\tilde{h}_n(x;q)\}_{n=0}^{\infty}$ are defined by (see [11])

$$\begin{aligned}
\tilde{h}_n(x;q) &= x^n \,_2\phi_0(q^{-n}, q^{-n+1}; -; q^2, -q^2 x^{-2}) \\
&= (q;q)_n \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{-2nk} q^{k(2k+1)} x^{n-2k}}{(q^2;q^2)_k (q;q)_{n-2k}}.
\end{aligned} \tag{5.35}$$

The generating function for the discrete q-Hermite II polynomials is (see [11])

$$e_{q^2}(-z^2) E_q(xz) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} \tilde{h}_n(x;q) z^n. \tag{5.36}$$

The monomial function can be expressed in terms of the discrete q-Hermite II polynomials as (see [11])

$$x^n = (q;q)_n \sum_{k=0}^{[\frac{n}{2}]} \frac{q^{-2nk+3k^2} h_{n-2k}(x;q)}{(q^2;q^2)_k (q;q)_{n-2k}}. \tag{5.37}$$

In the following definition, we introduce a new generalization of the discrete q-Hermite II polynomials.

Definition 5.2. *The generalized discrete q-Hermite II polynomials $\{\tilde{h}_{n,\alpha}(x;q)\}_{n=0}^{\infty}$ are defined by*

$$\tilde{h}_{n,\alpha}(x;q) := (q;q)_n \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{-2nk} q^{k(2k+1)} x^{n-2k}}{(q^2;q^2)_k (q;q)_{n-2k,\alpha}}. \tag{5.38}$$

Remarks.

(1) It is easy to see that for $\alpha = -\frac{1}{2}$, $\tilde{h}_{n,-\frac{1}{2}}(x;q) = \tilde{h}_n(x;q)$.

(2) Note that

$$\frac{\tilde{h}_{n,\alpha}(\sqrt{1-q^2}x;q)}{(1-q^2)^{\frac{n}{2}}} = \frac{n!_q}{(1+q)^n} \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{-2nk} q^{k(2k+1)}((1+q)x)^{n-2k}}{k!_{q^2}(n-2k)!_{q,\alpha}},$$

and by (3.4), we get

$$\lim_{q \to 1^-} \frac{\tilde{h}_{n,\alpha}(\sqrt{1-q^2}x;q)}{(1-q^2)^{\frac{n}{2}}} = \frac{H_n^{\alpha+\frac{1}{2}}(x)}{2^n},$$

where H_n^{μ} is the Rosenblum's generalized Hermite polynomial (see [15]).

Lemma 5.4. *The generalized discrete q-Hermite II polynomials can be written in terms of basic hypergeometric functions as:*

$$
\begin{cases}
\tilde{h}_{2n,\alpha}(x;q) & = \dfrac{(q;q)_{2n}}{(q;q)_{2n,\alpha}} x^{2n} \; {}_2\phi_1(q^{-2n}, q^{-2n-2\alpha}; 0; q^2, -q^{2\alpha+3}x^{-2}) \\
& = (-1)^n q^{-n(2n-1)}(q;q^2)_n \; {}_1\phi_1\left(q^{-2n}; q^{2\alpha+2}; q^2, -q^{2n+1}x^2\right); \\
\tilde{h}_{2n+1,\alpha}(x;q) & = \dfrac{(q;q)_{2n+1}}{(q;q)_{2n+1,\alpha}} x^{2n+1} \; {}_2\phi_1(q^{-2n}, q^{-2n-2\alpha-2}; 0; q^2, -q^{2\alpha+3}x^{-2}) \\
& = (-1)^n q^{-n(2n+1)} x \dfrac{(q;q^2)_{n+1}}{1-q^{2\alpha+2}} \; {}_1\phi_1\left(q^{-2n}; q^{2\alpha+4}; q^2, -q^{2n+3}x^2\right).
\end{cases}
$$
$$(5.39)$$

Proof. We have

$$
\tilde{h}_{2n,\alpha}(x;q) = (q;q)_{2n} \sum_{k=0}^{n} \frac{(-1)^k q^{-4nk} q^{k(2k+1)} x^{2n-2k}}{(q^2;q^2)_k (q^2;q^2)_{n-k} (q^{2\alpha+2};q^2)_{n-k}}.
$$

Using the identities (5.8) and (5.9), we obtain

$$
\begin{aligned}
\tilde{h}_{2n,\alpha}(x;q) & = \frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}} x^{2n} \sum_{k=0}^{n} \frac{(q^{-2n};q^2)_k (q^{-2n-2\alpha};q^2)_k \left(-q^{2\alpha+3}x^{-2}\right)^k}{(q^2;q^2)_k} \\
& = \frac{(q;q)_{2n}}{(q;q)_{2n,\alpha}} x^{2n} \; {}_2\phi_1(q^{-2n}, q^{-2n-2\alpha}; 0; q^2, -q^{2\alpha+3}x^{-2})
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{h}_{2n+1,\alpha}(x;q) & = \frac{(q;q)_{2n+1}}{(q;q)_{2n+1,\alpha}} x^{2n+1} \sum_{k=0}^{n} \frac{(q^{-2n};q^2)_k (q^{-2n-2\alpha-2};q^2)_k \left(-q^{2\alpha+3}x^{-2}\right)^k}{(q^2;q^2)_k} \\
& = \frac{(q;q)_{2n+1}}{(q;q)_{2n+1,\alpha}} x^{2n+1} \; {}_2\phi_1(q^{-2n}, q^{-2n-2\alpha-2}; 0; q^2, -q^{2\alpha+3}x^{-2}).
\end{aligned}
$$

But, we have (see [11], p.16)

$$
{}_2\phi_1\left(q^{-2n}, a^{-1}q^{2-2n}; 0; q^2, \frac{aq^{2n+2}}{z}\right) = (a;q^2)_n q^{2n} z^{-n} \; {}_1\phi_1\left(q^{-2n}; a; q^2, z\right). \quad (5.40)
$$

So, taking $a = q^{2\alpha+2}$ and $z = -q^{2n+1}x^2$ in (5.40), we get

$$
\begin{aligned}
{}_2\phi_1\left(q^{-2n}, q^{-2n-2\alpha}; 0; q^2, -q^{2\alpha+3}x^{-2}\right) = (-1)^n & q^{-2n^2+n} x^{-2n} (q^{2\alpha+2};q^2)_n \\
& \times \; {}_1\phi_1\left(q^{-2n}; q^{2\alpha+2}; q^2, -q^{2n+1}x^2\right)
\end{aligned}
$$
$$(5.41)$$

and taking $a = q^{2\alpha+4}$ and $z = -q^{2n+3}x^2$ in (5.40), we obtain

$$
\begin{aligned}
{}_2\phi_1\left(q^{-2n}, q^{-2n-2\alpha-2}; 0; q^2, -q^{2\alpha+3}x^{-2}\right) = (-1)^n & q^{-2n^2-n} x^{-2n} (q^{2\alpha+4};q^2)_n \\
& \times \; {}_1\phi_1\left(q^{-2n}; q^{2\alpha+4}; q^2, -q^{2n+3}x^2\right),
\end{aligned}
$$
$$(5.42)$$

which achieves the proof. $\blacksquare$

We recall that the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ are defined by (see [11])

$$
L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \; {}_1\phi_1(q^{-n}; q^{\alpha+1}; q, -q^{n+\alpha+1}x).
$$

Using (5.39) the generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$ can also be expressed in terms of the q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ as follows:

$$
\begin{cases}
\tilde{h}_{2n,\alpha}(x;q) & = & (-1)^n q^{-n(2n-1)} \dfrac{(q;q)_{2n}}{(q^{2\alpha+2};q^2)_n} L_n^{(\alpha)}(q^{-2\alpha-1}x^2;q^2), \\[2mm]
\tilde{h}_{2n+1,\alpha}(x;q) & = & (-1)^n q^{-n(2n+1)} \dfrac{(q;q)_{2n+1}}{(q^{2\alpha+2};q^2)_{n+1}} xL_n^{(\alpha+1)}(q^{-2\alpha-1}x^2;q^2).
\end{cases}
$$

Proposition 5.5. *The following properties hold:*

(1) *The generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$ are related to the generalized discrete q-Hermite I polynomials $h_{n,\alpha}(x;q)$ by*

$$
\tilde{h}_{n,\alpha}(x;q) = (i)^{-n} q^{-(\alpha+\frac{1}{2})\theta_{n}+1} h_{n,\alpha}(ixq^{-\alpha-\frac{1}{2}};q^{-1}). \tag{5.43}
$$

(2) *q-Integral representation of Mehler type*

$$
\tilde{h}_{n,\alpha}(x;q) = V_{\alpha,q}\tilde{h}_n(x;q). \tag{5.44}
$$

(3) *Generating function*

$$
e_{q^2}(-z^2)E_{q,\alpha}(xz) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} \tilde{h}_{n,\alpha}(x;q)z^n. \tag{5.45}
$$

(4) *Inversion formula:*

$$
x^n = (q;q)_{n,\alpha} \sum_{k=0}^{[\frac{n}{2}]} \frac{q^{-2nk+3k^2}\tilde{h}_{n-2k,\alpha}(x;q)}{(q^2;q^2)_k(q;q)_{n-2k}}. \tag{5.46}
$$

(5) *Three terms recursion formula*

$$
x\tilde{h}_{n,\alpha}(x;q) - q^{1-2n}(1-q^n)\tilde{h}_{n-1,\alpha}(x;q) = \frac{1-q^{n+1+(2\alpha+1)\theta_n}}{1-q^{n+1}}\tilde{h}_{n+1,\alpha}(x;q). \tag{5.47}
$$

Proof.   (1) (5.43) follows from the relation (2.2).

(2) Applications of the operator $V_{\alpha,q}$ to each term in (5.35) and the result (4.20) give (5.44).

(3) By application of the operator $V_{\alpha,q}$ to both sides of equation (5.36) and using (5.13) and (4.23), we obtain (5.45).

(4) Using the same argument to both sides of result (5.37), (4.20) and (5.44) gives (5.46).

(5) By considering separately the even and the odd cases in

$$
x\tilde{h}_{n,\alpha}(x;q) - q^{1-2n}(1-q^n)\tilde{h}_{n-1,\alpha}(x;q),
$$

(5.47) follows from (5.38) by elementary calculus.

∎

In the following proposition we derive some of the important properties of generalized discrete q-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$ from those of generalized discrete q-Hermite I polynomials $h_{n,\alpha}(x;q)$ by using the identity (5.43).

Proposition 5.6. *The generalized discrete q-Hermite II polynomials* $\tilde{h}_{n,\alpha}(x;q)$ *satisfy the following properties:*

(1) Affect of the forward shift operator

$$\tilde{h}_{n,\alpha}(x;q) - q^{(2\alpha+1)\theta_{n+1}}\tilde{h}_{n,\alpha}(qx;q) = q^{-n+1}(1-q^n)x\tilde{h}_{n-1,\alpha}(qx;q) \qquad (5.48)$$

or equivalently

$$\mathbf{\Delta}_{\alpha,q}\tilde{h}_{n,\alpha}(x;q) = q^{-n+1}\frac{1-q^n}{1-q}\tilde{h}_{n-1,\alpha}(qx;q), \quad n=1,2,... \qquad (5.49)$$

(2) Affect of the backward shift operator

$$\tilde{h}_{n,\alpha}(x;q) - q^{(2\alpha+1)\theta_{n+1}}(1+q^{-2\alpha-1}x^2)\tilde{h}_{n,\alpha}(qx;q) = -q^n\frac{1-q^{-n-1-(2\alpha+1)\theta_n}}{1-q^{-n-1}}$$
$$\times x\tilde{h}_{n+1,\alpha}(x;q)$$

or equivalently

$$\mathbf{\Delta}_{\alpha,q}\left[\omega_\alpha(x;q)\tilde{h}_{n,\alpha}(x;q)\right] = -q^n\frac{1-q^{-n-1-(2\alpha+1)\theta_n}}{(1-q)(1-q^{-n-1})}\omega_\alpha(x;q)\tilde{h}_{n+1,\alpha}(x;q), \quad (5.50)$$

where

$$\omega_\alpha(x;q) = e_{q^2}(-q^{-2\alpha-1}x^2). \qquad (5.51)$$

(3) q-Difference equations

$$(1+q^{-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(qx;q)-(1+q^{-2\alpha}+q^{n-2\alpha-1}x^2)\tilde{h}_{2n,\alpha}(x;q)+q^{-2\alpha}\tilde{h}_{2n,\alpha}(q^{-1}x;q)=0.$$

and

$$(1+q^{-2\alpha-1}x^2)\tilde{h}_{2n+1,\alpha}(qx;q) - (q+q^{-2\alpha-1}+q^{n-2\alpha-1}x^2)\tilde{h}_{2n+1,\alpha}(x;q)$$
$$+q^{-2\alpha}\tilde{h}_{2n+1,\alpha}(q^{-1}x;q)=0.$$

The following formula can be proved by induction:

Proposition 5.7. *Rodrigues-type formula*

$$\omega_\alpha(x;q)\tilde{h}_{n,\alpha}(x;q) = \frac{(q-1)^n q^{\frac{-n(n-1)}{2}}(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n,\alpha}}\Delta_{\alpha,q}^n\omega_\alpha(x;q), \qquad (5.52)$$

where $\omega_\alpha(x;q)$ *is the function given by (5.51).*

Proof. Since $\tilde{h}_{0,\alpha}(x;q) = 1$, the formula is clearly true for $n = 0$. Assume that (5.52) is true for an integer n. Then, applying the operator $\Delta_{\alpha,q}$ to both sides of equation (5.52) and using (5.50), we get

$$\omega_\alpha(x;q)\tilde{h}_{n+1,\alpha}(x;q) = -q^{-n}\frac{(1-q)(1-q^{-n-1})}{1-q^{-n-1-(2\alpha+1)\theta_n}}\Delta_{\alpha,q}\left[\omega_\alpha(x;q)\tilde{h}_{n,\alpha}(x;q)\right]$$

$$= -q^{-n}\frac{(1-q)(1-q^{-n-1})}{1-q^{-n-1-(2\alpha+1)\theta_n}}\frac{(q-1)^n q^{\frac{-n(n-1)}{2}}(q^{-1};q^{-1})_n}{(q^{-1};q^{-1})_{n,\alpha}}$$
$$\times\Delta_{\alpha,q}^{n+1}\omega_\alpha(x;q).$$

Finally, the use of (3.2), where q is replaced by q^{-1}, shows that (5.52) is also true for $(n+1)$. This achieves the proof. ∎

To establish an orthogonality relation for the q-polynomials $\{\tilde{h}_{n,\alpha}(x;q)\}$, we need the following lemma.

Lemma 5.5. *The polynomials* $\{\tilde{h}_{n,\alpha}(x;q)\}_{n=0}^{\infty}$ *satisfy*

$$\int_{-\infty}^{\infty} x^p \tilde{h}_{n,\alpha}(x;q)\omega_\alpha(x;q)|x|^{2\alpha+1} d_q x$$

$$= \begin{cases} \dfrac{2q^{-n^2}(1-q)(-q,-q,q^2;q^2)_\infty\, (q;q)_n}{(-q^{-2\alpha-1},-q^{2\alpha+3},q^{2\alpha+2};q^2)_\infty} & \text{if } p=n, \\[2mm] 0 & \text{if } p=0,1,...,n-1. \end{cases}$$
$$\tag{5.53}$$

Proof. As in the proof of (5.30), we consider two cases. First, we compute the following q-integral

$$\int_{-\infty}^{\infty} x^{2p} \tilde{h}_{2n,\alpha}(x;q)\omega_\alpha(x;q)|x|^{2\alpha+1} d_q x. \tag{5.54}$$

From the definition (5.38), the q-integral (5.54) is equal to

$$2(q;q)_{2n}(-1)^n q^{-2n^2+n} \sum_{k=0}^{n} \frac{(-1)^k q^{2k^2-k} G^\alpha_{q,k+p}(q^{-2\alpha-1})}{(q^2;q^2)_{n-k}(q^2;q^2)_k(q^{2\alpha+2};q^2)_k},$$

where $G^\alpha_{q,k+p}(q^{-2\alpha-1}) = c_{q,\alpha}(q^{-2\alpha-1})q^{-(k+p)^2}\left(q^{2\alpha+2};q^2\right)_{k+p}$ is given by (5.2). Then the above sum becomes

$$2c_{q,\alpha}(q^{-2\alpha-1})(q;q)_{2n}(-1)^n q^{-2n^2+n}$$

$$\times \sum_{k=0}^{n} \frac{(-1)^k q^{2k^2-k} q^{-(k+p)^2}}{(q^2;q^2)_{n-k}(q^2;q^2)_k(q^{2\alpha+2};q^2)_k} \left(q^{2\alpha+2};q^2\right)_{k+p}.$$

Using (5.8) and the fact that

$$\left(q^{2\alpha+2};q^2\right)_{k+p} = \left(q^{2p+2\alpha+2};q^2\right)_k \left(q^{2\alpha+2};q^2\right)_p, \tag{5.55}$$

the above sum can be written as

$$2c_{q,\alpha}(q^{-2\alpha-1})(-1)^n q^{-2n^2+n-p^2}(q;q^2)_n \left(q^{2\alpha+2};q^2\right)_p$$

$$\times \sum_{k=0}^{n} \frac{(q^{-2n};q^2)_k \left(q^{2p+2\alpha+2};q^2\right)_k}{(q^2;q^2)_k(q^{2\alpha+2};q^2)_k} q^{2(n-p)k}$$

$$= \quad 2c_{q,\alpha}(q^{-2\alpha-1})(-1)^n q^{-2n^2+n-p^2}(q;q^2)_n \left(q^{2\alpha+2};q^2\right)_p$$

$$\times {}_2\phi_1(q^{-2n}, q^{2p+2\alpha+2}; q^{2\alpha+2}; q^2, q^{2n-2p}).$$

By the summation formula (see [11], p.15)

$${}_2\phi_1(q^{-2n}, b; c; q^2, \frac{cq^{2n}}{b}) = \frac{(b^{-1}c;q^2)_n}{(c;q^2)_n},$$

the q-integral in (5.54) becomes

$$2c_{q,\alpha}(q^{-2\alpha-1})(-1)^n q^{-2n^2+n-p^2}(q;q^2)_n \left(q^{2\alpha+2};q^2\right)_p \times \frac{(q^{-2p};q^2)_n}{(q^{2\alpha+2};q^2)_n}.$$

The only case that the above product is not zero is when $p=n$, where we have

$$2c_{q,\alpha}(q^{-2\alpha-1})(q;q^2)_n(-1)^n q^{-3n^2+n} \left(q^{-2n};q^2\right)_n.$$

By using the fomula (5.33), the q-integral in (5.54) equals

$$\frac{2(1-q)(-q,-q,q^2;q^2)_\infty}{(-q^{-2\alpha-1},-q^{2\alpha+3},q^{2\alpha+2};q^2)_\infty}q^{-4n^2}(q;q)_{2n}.$$

In the case where both p and n are odd integers the proof follows the same steps as in the first case

$$\int_{-\infty}^{\infty}x^{2p+1}\tilde{h}_{2n+1,\alpha}(x;q)\omega_\alpha(x;q)|x|^{2\alpha+1}d_qx$$

$$= 2(q;q)_{2n+1}(-1)^n q^{-2n^2-n}\sum_{k=0}^{n}\frac{(-1)^k q^{2k^2+k}G^\alpha_{q,k+p+1}(q^{-2\alpha-1})}{(q^2;q^2)_{n-k}(q^2;q^2)_k(q^{2\alpha+2};q^2)_{k+1}},$$

$$(5.56)$$

where $G^\alpha_{q,k+p+1}(q^{-2\alpha-1}) = c_{q,\alpha}(q^{-2\alpha-1})q^{-(k+p+1)^2}\left(q^{2\alpha+2};q^2\right)_{k+p+1}$ is given by (5.2). Then the above sum becomes

$$2c_{q,\alpha}(q^{-2\alpha-1})(q;q)_{2n+1}(-1)^n q^{-2n^2-n}$$

$$\times\sum_{k=0}^{n}\frac{(-1)^k q^{2k^2+k}q^{-(k+p+1)^2}}{(q^2;q^2)_{n-k}(q^2;q^2)_k(q^{2\alpha+2};q^2)_{k+1}}\left(q^{2\alpha+2};q^2\right)_{k+p+1}.$$

Using (5.8) and (5.55), it can be written as

$$2c_{q,\alpha}(q^{-2\alpha-1})(-1)^n q^{-2n^2-n-(p+1)^2}(q;q^2)_{n+1}\left(q^{2\alpha+2};q^2\right)_{p+1}$$

$$\times\sum_{k=0}^{n}\frac{(q^{-2n};q^2)_k\left(q^{2p+2\alpha+4};q^2\right)_k}{(q^2;q^2)_k(q^{2\alpha+2};q^2)_{k+1}}q^{2(n-p)k}$$

$$= 2c_{q,\alpha}(q^{-2\alpha-1})(-1)^n q^{-2n^2-n-(p+1)^2}(q;q^2)_{n+1}\left(q^{2\alpha+4};q^2\right)_p$$

$$\times\sum_{k=0}^{n}\frac{(q^{-2n};q^2)_k\left(q^{2p+2\alpha+4};q^2\right)_k}{(q^2;q^2)_k(q^{2\alpha+4};q^2)_k}q^{2(n-p)k}$$

$$= 2c_{q,\alpha}(q^{-2\alpha-1})(-1)^n q^{-2n^2-n-(p+1)^2}(q;q^2)_{n+1}\left(q^{2\alpha+4};q^2\right)_p$$

$$\times{}_2\phi_1(q^{-2n},q^{2p+2\alpha+4};q^{2\alpha+4};q^2,q^{2n-2p})$$

$$= 2c_{q,\alpha}(q^{-2\alpha-1})(-1)^n q^{-2n^2-n-(p+1)^2}(q;q^2)_{n+1}\left(q^{2\alpha+4};q^2\right)_p\times\frac{(q^{-2p};q^2)_n}{(q^{2\alpha+4};q^2)_n}.$$

The only case that the above product is not zero is when $p = n$, and in this case the q-integral in (5.56) equals

$$2c_{q,\alpha}(q^{-2\alpha-1})(q;q^2)_{n+1}(-1)^n q^{-3n^2-3n-1}\left(q^{-2n};q^2\right)_n.$$

By using the fomula (5.33), the q-integral in (5.56) equals

$$\frac{2(1-q)(-q,-q,q^2;q^2)_\infty}{(-q^{-2\alpha-1},-q^{2\alpha+3},q^{2\alpha+2};q^2)_\infty}q^{-(2n+1)^2}(q;q)_{2n+1}.$$

$$\blacksquare$$

Theorem 5.2. *The sequence of the q-polynomials $\{\tilde{h}_{n,\alpha}(x;q)\}_{n=0}^{\infty}$, satisfies the orthogonality relation*

$$\int_{-\infty}^{\infty} \tilde{h}_{n,\alpha}(x;q)\tilde{h}_{m,\alpha}(x;q)\omega_\alpha(x;q)|x|^{2\alpha+1}d_qx \;=\; \frac{2q^{-n^2}(1-q)(-q,-q,q^2;q^2)_\infty}{(-q^{-2\alpha-1},-q^{2\alpha+3},q^{2\alpha+2};q^2)_\infty}$$
$$\times \frac{(q;q)_n^2}{(q;q)_{n,\alpha}}\delta_{n,m}.$$

$$(5.57)$$

Proof. Without loss of generality, we assume that $m \leq n$, by (5.38) the q-integral in (5.57) is equal

$$(q;q)_m \sum_{p=0}^{[\frac{m}{2}]} \frac{(-1)^p q^{-2mp}q^{p(2p+1)}}{(q^2;q^2)_p(q;q)_{m-2p,\alpha}} \int_{-\infty}^{\infty} x^{m-2p}\tilde{h}_{n,\alpha}(x;q)\omega_\alpha(x;q)|x|^{2\alpha+1}d_qx.$$

Then the result (5.53) concludes the proof of the theorem. ∎

6. THE q-DUNKL HEAT EQUATION

Let us consider the following q-Dunkl heat equation:

$$D_{q^2,t}u = \Lambda_{\alpha,q,x}^2 u, \qquad (6.1)$$

where $D_{q^2,t}$ is the partial q^2-derivative in time defined by (2.13) and $\Lambda_{\alpha,q,x}$ is the q-Dunkl operator in space defined by (4.8).

Taking into account that $\lim_{q\to1} D_{q^2,t}u(t,x) = \dfrac{\partial u}{\partial t}(t,x)$ and $\lim_{q\to1}\Lambda_{\alpha,q,x}^2u(t,x) = \Lambda_{\alpha,x}^2u(t,x)$, and clearly, the standard Heat equation for Dunkl operator is recovered when $q \to 1$ (see [15, 16]).

If $\alpha = -\dfrac{1}{2}$, then (6.1) reduces to the q-heat equation $D_{q^2,t}u = \partial_{q,x}^2 u$ studied in [7].

6.1. **The q-Dunkl heat polynomials.** We define the q-Dunkl heat polynomial of degre n as

$$v_{n,\alpha}(x,t;q) = n!_q \sum_{k=0}^{[\frac{n}{2}]} \frac{t^k}{k!_{q^2}} b_{n-2k,\alpha}(x;q^2). \qquad (6.2)$$

We easily derive from (4.9) that

$$\Lambda_{\alpha,x}v_{n,\alpha}(x,t;q) = [n]_q v_{n-1,\alpha}(x,t;q), \qquad (6.3)$$

and we have

$$D_{q^2,t}v_{n,\alpha}(x,t;q) = [n]_q[n-1]_q v_{n-2,\alpha}(x,t;q). \qquad (6.4)$$

It Follows from (6.3) and (6.4) that all q-Dunkl heat polynomials $v_{n,\alpha}(x,t;q)$ are solutions of (6.1).

Note that when $\alpha = -\dfrac{1}{2}$, $v_{n,-\frac{1}{2}}(x,t;q) = v_n(x,t;q)$ the q-heat polynomials defined in [7]. According to the limit in (3.4), the q-Dunkl heat polynomials are the q-analogue of the generalized heat polynomials introduced by M. Rosenblum in [15].

Proposition 6.1. *The generating function of the sequence* $\{v_{n,\alpha}(x,t;q)\}_{n=0}^{\infty}$ *is given by*

$$exp_{q^2}\left(tz^2\right)\psi_x^{\alpha,q}(-iz) = \sum_{n=0}^{\infty} v_{n,\alpha}(x,t;q)\frac{z^n}{n!_q}. \tag{6.5}$$

The expansion of a monomial in terms of the $v_{n,\alpha}(x,t;q)$ *is given by*

$$b_{n,\alpha}(x;q^2) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-t)^k q^{k(k-1)} v_{n-2k,\alpha}(x,t;q)}{k!_{q^2}(n-2k)!_q}. \tag{6.6}$$

Proof. (6.5) follows by using (2.5) and (3.7) and expanding the q-exponential functions in (6.5) as power series, and taking the Cauchy product of the results.
Multiply both sides of (6.5) with $Exp_{q^2}(-tz^2)$ and next compare the coefficients of z^n, we obtain (6.6). ∎

It is not very difficult to verify that the q-Dunkl heat polynomials are closely related to the q-polynomials $h_{n,\alpha}(x;q)$ as follows:

Proposition 6.2.

$$v_{2n,\alpha}(x,t;q) = \frac{q^{-n(n-1)}}{(i\beta)^{2n}} h_{2n,\alpha}(i\beta q^n x;q), \tag{6.7}$$

$$v_{2n+1,\alpha}(x,t;q) = \frac{q^{-n^2}}{(i\beta)^{2n+1}} h_{2n+1,\alpha}(i\beta q^n x;q), \tag{6.8}$$

where

$$\beta = \beta(t,q) = \sqrt{\frac{1-q}{t(1+q)}}. \tag{6.9}$$

6.2. **The q-Dunkl heat kernel.** In order to find the q-Dunkl heat kernel of (6.1), we suggest to apply the q-Dunkl transform as described in [7] for q-Rubin Fourier transform to the function $k_\alpha(x,t;q)$ defined by

$$k_\alpha(x,t;q) = C_\alpha(t;q)exp_{q^2}\left(-\frac{q^{-2\alpha}x^2}{t(1+q)^2}\right), \quad x \in \mathbb{R}, \quad t > 0, \tag{6.10}$$

where

$$C_\alpha(t;q) = \frac{\left(-\frac{q^{-2\alpha}(1-q)}{t(1+q)}, -\frac{q^{2\alpha+2}t(1+q)}{(1-q)}, q^{2\alpha+2};q^2\right)_\infty}{2(1-q)\left(-\frac{q^2(1-q)}{t(1+q)}, -\frac{t(1+q)}{(1-q)}, q^2;q^2\right)_\infty}. \tag{6.11}$$

Since

$$\Lambda_{\alpha,x}k_\alpha(x,t;q) = D_q^+ k_\alpha(x,t;q) = -\frac{x}{q^{2\alpha+2}(1+q)t}k_\alpha(q^{-1}x,t;q), \tag{6.12}$$

and

$$\begin{aligned}\Lambda_{\alpha,x}^2 k_\alpha(x,t;q) &= -\frac{q^{-2\alpha-2}}{(1+q)t}\left[\frac{1-q^{2\alpha+2}}{1-q} - \frac{x^2}{t(1+q)}\right]k_\alpha(q^{-1}x,t;q) \\ &= D_{q^2,t}k_\alpha(x,t;q).\end{aligned}$$

$k_\alpha(x,t;q)$ is solution of q-Dunkl heat equation (6.1) and called the q-Dunkl heat kernel. It generalizes the q-source solution of q-heat equation $k(x,t;q)$ studied in [7] by taking $\alpha = -\frac{1}{2}$ in (6.10).

Proposition 6.3. *The q-Dunkl heat kernel $k_\alpha(x,t;q)$ has the following properties:*

$$\int_{-\infty}^{\infty} k_\alpha(x,t;q)b_{2n,\alpha}(x;q^2)|x|^{2\alpha+1}d_qx \quad = \frac{t^n}{n!_{q^2}}, \quad t > 0, \quad n = 0,1,2.... \quad (6.13)$$

$$\mathcal{F}_D^{\alpha,q}k_\alpha(.,t;q)(x) = K_\alpha exp_{q^2}\left(-tx^2\right). \quad (6.14)$$

Proof. Take $\lambda = \dfrac{q^{-2\alpha}(1-q)}{t(1+q)}$ in (5.2), to get

$$\int_0^{\infty} e_{q^2}\left(-\frac{q^{-2\alpha}(1-q)}{t(1+q)}y^2\right)y^{2\alpha+2n+1}d_qy = c_{q,\alpha}(\lambda)\frac{q^{-n(n+1)}(1+q)^nt^n}{(1-q)^n}\left(q^{2\alpha+2};q^2\right)_n,$$

where

$$c_{q,\alpha}(\lambda) = \frac{(1-q)\left(-\frac{q^2(1-q)}{t(1+q)}, -\frac{t(1+q)}{(1-q)}, q^2;q^2\right)_\infty}{\left(-\frac{q^{-2\alpha}(1-q)}{t(1+q)}, -\frac{q^{2\alpha+2}t(1+q)}{(1-q)}, q^{2\alpha+2};q^2\right)_\infty}. \quad (6.15)$$

Now, by definitions of $k_\alpha(x,t;q)$ (6.10) and of $b_{n,\alpha}(x;q^2)$ (3.8) the result (6.13) follows.

By definition of $\psi_{-x}^{\alpha;q}(y)$ in (3.7), and since $k_\alpha(y,t;q)$ is even in y, we have, by using the Fubini theorem,

$$\begin{aligned}
\mathcal{F}_D^{\alpha,q}(k_\alpha(.,t;q))(x) &= K_\alpha \int_{-\infty}^{\infty} k_\alpha(y,t;q)\sum_{n=0}^{\infty} b_{2n,\alpha}(-ixy;q^2)|y|^{2\alpha+1}d_qy \\
&= K_\alpha \sum_{n=0}^{\infty}(-1)^nx^{2n}\int_{-\infty}^{\infty} k_\alpha(y,t;q)b_{2n,\alpha}(y;q^2)|y|^{2\alpha+1}d_qy.
\end{aligned}$$

So, by (6.13), we obtain

$$\mathcal{F}_D^{\alpha,q}(k_\alpha(.,t;q))(x) = K_\alpha \sum_{n=0}^{\infty}(-1)^nx^{2n}\frac{t^n}{n!_{q^2}}.$$

The conclusion follows from (2.5). $\blacksquare$

6.3. **The q-associated functions for q-Dunkl operator.** We extend the notion of the q-associated functions defined in [7] by defining the q-Dunkl associated functions in terms of the q-Dunkl derivatives of the q-Dunkl heat kernel as follows:

$$w_{n,\alpha}(x,t;q) = (-(1+q))^n\Lambda_{\alpha,q}^n k_\alpha(x,t;q), \quad n = 0,1,2,..., \ t > 0.$$

Consequently, the q-Dunkl associated functions are solutions of q-Dunkl heat equation (6.1).

Proposition 6.4. *For $n = 0,1,2,...$*

$$\Lambda_{\alpha,q}^{2n}k_\alpha(x,t;q) = \frac{q^{n(n-1)-n(2\alpha+1)}(q;q)_{2n,\alpha}}{t^n(1-q^2)^n(q;q)_{2n}}\tilde{h}_{2n,\alpha}(q^{-n}\gamma x;q)k_\alpha(q^{-n}x,t;q), \quad (6.16)$$

$$\begin{aligned}
\Lambda_{\alpha,q}^{2n+1}k_\alpha(x,t;q) &= -\frac{q^{n^2-\frac{1}{2}-(n+1)(2\alpha+1)}(q;q)_{2n+1,\alpha}}{t^{n+\frac{1}{2}}(1-q^2)^{n+\frac{1}{2}}(q;q)_{2n+1}} \\
&\times \tilde{h}_{2n+1,\alpha}(q^{-n-1}\gamma x;q)k_\alpha(q^{-n-1}x,t;q),
\end{aligned} \quad (6.17)$$

where

$$\gamma = \sqrt{\frac{q(1-q)}{t(1+q)}}.$$

Proof. Referring to (6.10) and (5.51), we can write

$$k_\alpha(x,t;q) = C_\alpha(t;q)\omega_\alpha(\gamma x;q).$$

Using (4.10) and (4.11), we obtain

$$\begin{aligned}
\Lambda_{\alpha,q}^{2n} k_\alpha(x,t;q) &= C_\alpha(t;q)\Lambda_{\alpha,q}^{2n}\omega_\alpha(\gamma x;q) \\
&= C_\alpha(t;q)q^{-n(n+1)}\gamma^{2n}[\Delta_{\alpha,q}^{2n}\omega_\alpha](\gamma q^{-n}x;q),
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_{\alpha,q}^{2n+1} k_\alpha(x,t;q) &= C_\alpha(t;q)\Lambda_{\alpha,q}^{2n+1}\omega_\alpha(\gamma x;q) \\
&= C_\alpha(t;q)q^{-(n+1)^2}\gamma^{2n+1}[\Delta_{\alpha,q}^{2n+1}\omega_\alpha](\gamma q^{-n-1}x;q).
\end{aligned}$$

From Rodrigues-type formula (5.52) we have

$$\begin{aligned}
\Lambda_{\alpha,q}^{2n} k_\alpha(x,t;q) &= \frac{\gamma^{2n}q^{-n(n+1)}(q-1)^{-2n}q^{n(2n-1)}(q^{-1};q^{-1})_{2n,\alpha}}{(q^{-1};q^{-1})_{2n}} \\
&\quad \times \tilde{h}_{2n,\alpha}(q^{-n}\gamma x;q)k_\alpha(q^{-n}x,t;q),
\end{aligned}$$

and

$$\begin{aligned}
\Lambda_{\alpha,q}^{2n+1} k_\alpha(x,t;q) &= \frac{\gamma^{2n+1}q^{-(n+1)^2}(q-1)^{-2n-1}q^{n(2n+1)}(q^{-1};q^{-1})_{2n+1,\alpha}}{(q^{-1};q^{-1})_{2n+1}} \\
&\quad \times \tilde{h}_{2n+1,\alpha}(q^{-n-1}\gamma x;q)k_\alpha(q^{-n-1}x,t;q).
\end{aligned}$$

Use the fact that

$$\frac{(q^{-1};q^{-1})_{2n,\alpha}}{(q^{-1};q^{-1})_{2n}} = q^{-n(2\alpha+1)}\frac{(q;q)_{2n,\alpha}}{(q;q)_{2n}}, \qquad \frac{(q^{-1};q^{-1})_{2n+1,\alpha}}{(q^{-1};q^{-1})_{2n+1}} = q^{-(n+1)(2\alpha+1)}\frac{(q;q)_{2n,\alpha}}{(q;q)_{2n}},$$

to obtain the desired result. $\blacksquare$

The above calculation shows that the q-Dunkl associated functions are closely related to the q-polynomials $\tilde{h}_{n,\alpha}(x;q)$ as follows:

Proposition 6.5. *For $n = 0, 1, 2, \ldots$*

$$w_{2n,\alpha}(x,t;q) = \frac{q^{n(n-2\alpha-1)}(q;q)_{2n,\alpha}}{(t\gamma)^{2n}(q;q)_{2n}}\tilde{h}_{2n,\alpha}(q^{-n}\gamma x;q)k_\alpha(q^{-n}x,t;q), \qquad (6.18)$$

$$w_{2n+1,\alpha}(x,t;q) = \frac{q^{(n+1)(n-2\alpha-1)}(q;q)_{2n+1,\alpha}}{(t\gamma)^{2n+1}(q;q)_{2n+1}}\tilde{h}_{2n+1,\alpha}(q^{-n-1}\gamma x;q)k_\alpha(q^{-n-1}x,t;q),$$

$$(6.19)$$

where

$$\gamma = \sqrt{\frac{q(1-q)}{t(1+q)}}.$$

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