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BEST APPROXIMATION OF FUNCTIONS OF GENERALIZED ZYGMUND CLASS BY MATRIX-EULER SUMMABILITY MEANS OF FOURIER SERIES

(COMMUNICATED BY HÜSEIN BOR)

SHYAM LAL AND SHIREEN

Abstract. In this paper, new best approximations of the function $f \in Z_r^{(w)}$, $(r \geq 1)$ class by Matrix-Euler means $(\Delta.E_1)$ of its Fourier Series have been determined.

1. Introduction

The degree of approximation of a function f belonging to Lipschitz class by the Cesàro mean and $f \in H_\alpha$ by the Fejér means has been studied by Alexits [4] and Prössdorf [7] respectively. But till now no work seems to have been done to obtain best approximation of functions belonging to generalized Zygmund class, $Z_r^{(w)}$, $(r \geq 1)$ by product summability means of the form $(\Delta.E_1)$. $Z_r^{(w)}$ class is a generalization of Z_α, $Z_{\alpha,r}$, $Z^{(w)}$ classes. The Matrix-Euler $(\Delta.E_1)$ summability means includes $(N, p_n).E_1$, $(N, p_n, q_n).E_1$ and $(C, 1).E_1$ means as particular cases. In attempt to make an advance study in this direction, in this paper, best approximations of the function belonging to generalized Zygmund class $Z_r^{(w)}$, $(r \geq 1)$ have been obtained.

2. Definition and Notations

Let $\sum_{n=0}^{\infty} u_n$ be an infinite series having n^{th} partial sum $s_n = \sum_{\nu=0}^{n} u_\nu$.

Let f be a 2π periodic function, integrable in the Lebesgue sense over $[0, 2\pi]$. Let the Fourier series of f be given by

$$f(x) := \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{1}$$

with n^{th} partial sum $s_n(f; x)$.

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Let $T = (a_{n,k})$ be an infinite triangular matrix satisfying the (Silverman-Töeplitz [8]) conditions of regularity, i.e.,

(i). $\sum\limits_{k=0}^{n} a_{n,k} = 1$ as $n \to \infty$,
(ii). $a_{n,k} = 0$ for $k > n$,
(iii). $\sum\limits_{k=0}^{n} |a_{n,k}| \leq M$, a finite constant.

The sequence-to-sequence transformation

$$t_n^{\Delta} := \sum_{k=0}^{n} a_{n,k} s_k = \sum_{k=0}^{n} a_{n,n-k} s_{n-k}$$

defines the sequence t_n^{Δ} of triangular matrix means of the sequence $\{s_n\}$, generated by the sequence of coefficients $(a_{n,k})$.

If $t_n^{\Delta} \to s$ as $n \to \infty$, then the series $\sum\limits_{n=0}^{\infty} u_n$ is summable to s by triangular matrix Δ- method (Zygmund [1], p.74).

Let $E_n^{(1)} := \dfrac{1}{2^n} \sum\limits_{k=0}^{n} \dbinom{n}{k} s_k$. If $E_n^{(1)} \to s$ as $n \to \infty$, then $\sum\limits_{n=0}^{\infty} u_n$ is said to be summable to s by the Euler's method, E_1 (Hardy [5]).

The triangular matrix Δ-transform of E_1 transform defines the $(\Delta.E_1)$ transform $t_n^{\Delta E}$ of the partial sums s_n of the series $\sum\limits_{n=0}^{\infty} u_n$ by

$$t_n^{\Delta E} := \sum_{k=0}^{n} a_{n,k} E_k^{(1)} = \sum_{k=0}^{n} a_{n,k} \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} s_\nu.$$

If $t_n^{\Delta E} \to s$ as $n \to \infty$, then the series $\sum\limits_{n=0}^{\infty} u_n$ is said to be summable $(\Delta.E_1)$ to s.

$$\begin{aligned} s_n \to s \quad &\Rightarrow \quad E_n^{(1)} = \frac{1}{2^n} \sum_{\nu=0}^{n} \binom{n}{\nu} s_\nu \to s \ as \ n \to \infty, \ E_1 \text{ method is regular}, \\ &\Rightarrow \quad t_n^{\Delta}(E_n^{(1)}) = t_n^{\Delta E} \to s \ as \ n \to \infty, \ \Delta \text{ method is regular}, \\ &\Rightarrow \quad (\Delta.E_1) \text{ method is regular}. \end{aligned}$$

Some important particular cases of triangular Matrix-Euler means $(\Delta.E_1)$ are

(i). $(H, \frac{1}{n+1}).(E_1)$ means, when $a_{n,k} = \frac{1}{(n-k+1)\log(n+1)}$.
(ii). $(N, p_n).E_1$ means, when $a_{n,k} = \frac{p_{n-k}}{P_n}$, where $P_n = \sum\limits_{k=0}^{n} p_k \neq 0$.
(iii). $(N, p_n, q_n).E_1$ means, when $a_{n,k} = \frac{p_{n-k} q_k}{R_n}$, where $R_n = \sum\limits_{k=0}^{n} p_k q_{n-k} \neq 0$.

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic and continuous functions defined on $[0, 2\pi]$ under the supremum norm.

$E_n(f) := \inf\limits_{t_n} \|f - t_n\|$ is the best n-order approximation of a function $f \in C_{2\pi}$ (Bernstein [6]), where $t_n(x) = \dfrac{1}{2} a_0 + \sum\limits_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x)$.

Zygmund modulus of continuity of f is defined by

$$w(f,h) := \sup_{o\leq t\leq h, x\in R} \left| f(x+t)+f(x-t)-2f((x) \right|.$$

For $0<\alpha\leq 1$, the function space

$$Z_{(\alpha)} := \{f\in C_{2\pi} : |f(x+t)+f(x-t)-2f(x)| = O(|t|^{\alpha})\}$$

is a Banach space under the norm $\|.\|_{(\alpha)}$ defined by

$$\|f\|_{(\alpha)} \quad := \quad \sup_{0\leq x\leq 2\pi} |f(x)| + \sup_{\substack{x,t\\ t\neq 0}} \frac{|f(x+t)+f(x-t)-2f(x)|}{|t|^{\alpha}}.$$

Let

$$L^r[0,2\pi] := \left\{ f : [0,2\pi]\to\mathbb{R} : \int_0^{2\pi} |f(x)|^r \, dx < \infty \right\}, r\geq 1,$$

be the space of all 2π-periodic, integrable functions.

We define the norm $\|.\|_r$ by

$$\|f\|_r := \begin{cases} \left\{ \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^r \, dx \right\}^{\frac{1}{r}} \text{ for } 1\leq r<\infty \\ ess \sup_{0<x<2\pi} |f(x)| \text{ for } r=\infty. \end{cases}$$

For $f\in L^r[0,2\pi], r\geq 1$, the integral Zygmund modulus of continuity is defined by

$$w_r(f,h) := \sup_{0<t\leq h} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+t)+f(x-t)-2f(x)|^r \, dx \right\}^{\frac{1}{r}}, \ for f\in L^r[0,2\pi] \ where \ 1\leq r<\infty,$$

$$w(f,h) = w_{\infty}(f,h) := \sup_{o<t\leq h} \max_x \left| f(x+t)+f(x-t)-2f((x) \right|, \ for f\in C_{2\pi} \ where \ r=\infty.$$

It is known (Zygmund [1], p.45) that $w_r(f,h)\to 0$ as $h\to 0$.

Define

$$Z_{(\alpha),r} := \left\{ f\in L^r[0,2\pi] : \left(\int_0^{2\pi} |f(x+t)+f(x-t)-2f(x)|^r \, dx \right)^{\frac{1}{r}} = O\left(|t|^{\alpha}\right) \right\}.$$

The space $Z_{(\alpha),r}$, $r\geq 1$, $0<\alpha\leq 1$ is a Banach space under the norm $\|.\|_{\alpha,r}$:

$$\|f\|_{\alpha,r} := \|f\|_r + \sup_{t\neq 0} \frac{\|f(.+t)+f(.-t)-2f(.)\|_r}{|t|^{\alpha}}.$$

$$\|f\|_{0,r} := \|f\|_r .$$

The class of function $Z^{(w)}$ is defined as

$$Z^{(w)} := \{f\in C_{2\pi} : |f(x+t)+f(x-t)-2f(x)| = O(w(t))\}$$

where w is a Zygmund modulus of continuity, that is, w is a positive, non-decreasing continuous function with the property: $w(0)=0$, $w(t_1+t_2)\leq w(t_1)+w(t_2)$.

Let $w:[0,2\pi]\to\mathbb{R}$ be an arbitrary function with $w(t)>0$ for $0<t\leq 2\pi$ and $\lim_{t\to 0^+} w(t) = w(0) = 0$.

We define

$$Z_r^{(w)} := \left\{ f\in L^r[0,2\pi] : 1\leq r<\infty, \ \sup_{t\neq 0} \frac{\|f(.+t)+f(.-t)-2f(.)\|_r}{w(t)} < \infty \right\}$$

and

$$\|f\|_r^{(w)} := \|f\|_r + \sup_{t\neq 0} \frac{\|f(.+t)+f(.-t)-2f(.)\|_r}{w(t)},\ r \geq 1.$$

Clearly $\|\|_r^{(w)}$ is a norm on $Z_r^{(w)}$.

The completeness of the space $Z_r^{(w)}$ can be discussed considering the completeness of L^r $(r \geq 1)$.

Define

$$\|f\|_r^{(v)} := \|f\|_r + \sup_{t\neq 0} \frac{\|f(.+t)+f(.-t)-2f(.)\|_r}{v(t)},\ r \geq 1.$$

Let $\left(\frac{w(t)}{v(t)}\right)$ be positive, non decreasing. Then

$$\|f\|_r^{(v)} \leq \max\left(1, \frac{w(2\pi)}{v(2\pi)}\right)\|f\|_r^{(w)} < \infty.$$

Thus,

$$Z_r^{(w)} \subset Z_r^{(v)} \subset L^r, \quad r \geq 1$$

Remarks.

(i). If we take $w(t) = t^\alpha$ then $Z^{(w)}$ reduces to Z_α class.

(ii). By taking $w(t) = t^\alpha$ in $Z_r^{(w)}$, it reduces to $Z_{\alpha,r}$.

(iii). If we take $r \to \infty$ then $Z_r^{(w)}$ class reduces to $Z^{(w)}$

We write,

$\phi(x,t) = f(x+t) + f(x-t) - 2f(x), \Delta a_{n,k} = a_{n,k} - a_{n,k+1}, \quad 0 \leq k \leq n-1.$

$$K_n^{\Delta E} = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k} \frac{\sin{(n-k+1)}(\frac{t}{2})\cos^{n-k}(\frac{t}{2})}{\sin{(\frac{t}{2})}}.$$

3. Theorems

In this paper, we prove the following theorems:

Theorem 3.1. *Let the lower triangular matrix $A = (a_{n,k})$ satisfying the following conditions:*

$$a_{n,k} \geq 0(n = 0,1,2,...; k = 0,1,2,..n) \quad , \quad \sum_{k=0}^{n} a_{n,k} = 1, \tag{2}$$

$$\sum_{k=0}^{n-1} |\Delta a_{n,k}| = O\left(\frac{1}{n+1}\right) \quad \text{and} \quad (n+1)\,a_{n,n} = O\,(1)\,. \tag{3}$$

If $f : [0, 2\pi] \to \mathbb{R}$ be a 2π-periodic, Lebesgue integrable and belonging to the generalized Zygmund class $Z_r^{(w)}$, $r \geq 1$; w, v be Zygmund modulus of continuity and $\frac{w(t)}{v(t)}$ be positive, non-decreasing then best approximation of f by triangular matrix-Euler means $t_n^{\Delta E} = \sum\limits_{k=0}^{n} a_{n,k} \frac{1}{2^k} \sum\limits_{\nu=0}^{k} \binom{k}{\nu} s_\nu$ of its Fourier series (1) is given by

$$E_n(f) = \inf_{t_n^{\Delta E}} \|t_n^{\Delta E} - f\|_r^{(v)} = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi} \frac{w(t)}{t^2 v(t)}dt\right). \tag{4}$$

Theorem 3.2. *Let $A = (a_{n,k})$ be the lower triangular matrix satisfying the conditions (2) and (3) and in addition to Theorem 3.1, $\frac{w(t)}{tv(t)}$ is non-increasing. For $f \in Z_r^{(w)}$, $r \geq 1$, its best approximation by triangular matrix-Euler means $t_n^{\Delta E}$ satisfies*

$$E_n(f) = O\left(\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \log(n+1)\pi\right). \tag{5}$$

4. Lemmas

Following Lemmas are required to prove the theorems:

Lemma 4.1. Under our conditions on $(a_{n,k})$,
$K_n^{\Delta E}(t) = O(n+1)$, for $0 < t \leq (n+1)^{-1}$.

Proof. For $0 < t \leq (n+1)^{-1}$, $\sin\frac{t}{2} \geq \frac{t}{\pi}$, $\sin nt \leq nt$, $|\cos t| \leq 1$, we have

$$\begin{aligned}
\left|K_n^{\Delta E}(t)\right| &= \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k} \frac{\sin{(n-k+1)(\frac{t}{2})}\cos^{n-k}(\frac{t}{2})}{\sin{(\frac{t}{2})}}\right| \\
&\leq \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k} \frac{(n-k+1)(\frac{t}{2})\left|\cos^{n-k}(\frac{t}{2})\right|}{(\frac{t}{\pi})} \\
&\leq \frac{1}{4}(n+1)\sum_{k=0}^{n} a_{n,k} \\
&\leq \frac{1}{4}(n+1) \\
&= O(n+1).
\end{aligned}$$

□

Lemma 4.2. $K_n^{\Delta E}(t) = O\left(\frac{1}{(n+1)t^2}\right)$, for $(n+1)^{-1} < t < \pi$.

Proof. For $(n+1)^{-1} < t < \pi,\ \sin\frac{t}{2} \geq \frac{t}{\pi}$, using Abel's lemma, we get

$$\begin{aligned}
\left|K_n^{\Delta E}(t)\right| &= \left|\frac{1}{2\pi}\sum_{k=0}^{n} a_{n,k}\frac{\sin(n-k+1)(\frac{t}{2})\cos^{n-k}(\frac{t}{2})}{\sin(\frac{t}{2})}\right| \\
&\leq \frac{1}{2t}\left|\sum_{k=0}^{n-1}(a_{n,k}-a_{n,k+1})\sum_{\nu=0}^{k}\sin(n-\nu+1)\left(\frac{t}{2}\right)\cos^{n-\nu}\left(\frac{t}{2}\right)\right. \\
&\qquad \left. + a_{n,n}\sum_{k=0}^{n}\sin(n-k+1)\left(\frac{t}{2}\right)\cos^{n-k}\left(\frac{t}{2}\right)\right| \\
&\leq \frac{1}{2t}\left[\sum_{k=0}^{n-1}|\Delta a_{n,k}|\left|\frac{\sin(2n-k+2)(\frac{t}{4})\sin(n+1)(\frac{t}{4})}{\sin(\frac{t}{4})}\right| + a_{n,n}\left|\frac{\sin(n+2)(\frac{t}{4})\sin(n+1)(\frac{t}{4})}{\sin(\frac{t}{4})}\right|\right] \\
&\leq \frac{\pi}{t^2}\left[\sum_{k=o}^{n-1}|\Delta a_{n,k}| + a_{n,n}\right]\max_{0\leq k\leq n}\left|\sin(2n-k+2)\left(\frac{t}{2}\right)sin(n+1)\left(\frac{t}{2}\right)\right| \\
&= \frac{\pi}{t^2}\left[\sum_{k=o}^{n-1}|\Delta a_{n,k}| + a_{n,n}\right] \\
&= \frac{\pi}{t^2}\left[O\left(\frac{1}{n+1}\right) + O\left(\frac{1}{n+1}\right)\right]\ by\ (3) \\
&= O\left(\frac{1}{(n+1)t^2}\right).
\end{aligned}$$

□

5. Proof of the Theorem3.1

Following Titchmarsh [3], $s_k(f;x)$ of Fourier series (1) is given by

$$s_k(f;x) - f(x) = \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{sin\left(k+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}dt, \quad k = 0,1,2\ldots .$$

Then

$$\begin{aligned}
\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(s_k(f;x)-f(x)) &= \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\frac{sin\left(k+\frac{1}{2}\right)t}{sin\left(\frac{t}{2}\right)}dt \\
or \qquad E_n^1(x) - f(x) &= \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{1}{2^n\sin(\frac{t}{2})}\left\{I_m\sum_{k=0}^{n}\binom{n}{k}e^{i(k+\frac{1}{2})t}\right\}dt \\
&= \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{1}{2^n\sin(\frac{t}{2})}\left\{I_m\sum_{k=0}^{n}\binom{n}{k}e^{ikt}e^{i\frac{t}{2}}\right\}dt \\
&= \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{1}{2^n\sin(\frac{t}{2})}\left\{I_m\left(1+e^{it}\right)^n.e^{i\frac{t}{2}}\right\}dt \\
&= \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\frac{\sin\left\{(n+1)\left(\frac{t}{2}\right)\right\}\cos^n\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}dt.
\end{aligned}$$

Now

$$\begin{aligned} t_n^{\Delta E}(x)-f(x) &= \frac{1}{P_n}\sum_{k=0}^{n} a_{n,k}\left\{E_{n-k}^{1}(x)-f(x)\right\} \\ &= \frac{1}{2\pi}\int_0^{\pi}\phi(x,t)\sum_{k=0}^{n} a_{n,k}\frac{\sin\left\{(n-k+1)(\frac{t}{2})\right\}\cos^{n-k}\left(\frac{t}{2}\right)}{\sin\left(\frac{t}{2}\right)}dt. \end{aligned}$$

Let

$$l_n(x)=t_n^{\Delta E}(x)-f(x) = \int_0^{\pi}\phi(x,t)K_n^{\Delta E}(t)dt.$$

Then

$$l_n(x+y)+l_n(x-y)-2l_n(x) = \int_0^{\pi}(\phi(x+y,t)+\phi(x-y,t)-2\phi(x,t))K_n^{\Delta E}(t)dt.$$

By generalized Minkowski's inequality (Chui[2], p.37), we get

$$\begin{aligned} \|l_n(.+y)+l_n(.-y)-2l_n(.)\|_r &\leq \int_0^{\pi}\|\phi(.+y,t)+\phi(.-y,t)-2\phi(.,t)\|_r\left|K_n^{\Delta E}(t)\right|dt \\ &= \int_0^{\frac{1}{n+1}}\left(\|\phi(.+y,t)+\phi(.-y,t)-2\phi(.,t)\|_r\left|K_n^{\Delta E}(t)\right|\right)dt \\ &\quad + \int_{\frac{1}{n+1}}^{\pi}\left(\|\phi(.+y,t)+\phi(.-y,t)-2\phi(.,t)\|_r\left|K_n^{\Delta E}(t)\right|\right)dt \\ &= I_1+I_2, say \qquad (6) \end{aligned}$$

Clearly

$$\begin{aligned} |\phi(x+y,t)+\phi(x-y,t)-2\phi(x,t)| &\leq |f(x+y+t)+f(x+y-t)-2f(x+y)| \\ &\quad +|f(x-y+t)+f(x-y-t)-2f(x-y)| \\ &\quad +2\,|f(x+t)+f(x-t)-2f(x)|\,. \end{aligned}$$

Applying Minkowski's inequality, we have

$$\begin{aligned} \|\phi(.+y,t)+\phi(.-y,t)-2\phi(.,t)\|_r &\leq \|f(.+y+t)+f(.+y-t)-2f(.+y)\|_r \\ &\quad +\|f(.-y+t)+f(.-y-t)-2f(.-y)\|_r \\ &\quad +2\,\|f(.+t)+f(.-t)-2f(.)\|_r \\ &= O\left(w(t)\right). \qquad (7) \end{aligned}$$

Also

$$\begin{aligned}\|\phi(.+y,t)+\phi(.-y,t)-2\phi(.,t)\|_r &\leq \|f(.+t+y)+f(.+t-y)-2f(.+t)\|_r \\ &\quad +\|f(.-t+y)-f(.-t-y)-2f(.-t)\|_r \\ &\quad +2\|f(.+y)+f(.-y)-2f(.)\|_r \\ &= O\left(w(y)\right). \qquad (8)\end{aligned}$$

For v is positive, non decreasing, $t \leq y$, we obtained

$$\begin{aligned}\|\phi(.+y,t)+\phi(.-y,t)-2\phi(.,t)\|_r &= O\left(w(t)\right) \\ &= O\left(v(t)\left(\frac{w(t)}{v(t)}\right)\right) \\ &= O\left(v(y)\left(\frac{w(t)}{v(t)}\right)\right).\end{aligned}$$

Since $\frac{w(t)}{v(t)}$ is positive, non-decreasing, if $t \geq y$, then $\frac{w(t)}{v(t)} \geq \frac{w(y)}{v(y)}$, so that

$$\begin{aligned}\|\phi(.+y,t)+\phi(.-y,t)-2\phi(.,t)\|_r &= O\left(w(y)\right) \\ &= O\left(v(y)\left(\frac{w(t)}{v(t)}\right)\right). \qquad (9)\end{aligned}$$

Using lemma (4.1) and (9) we obtain

$$\begin{aligned}I_1 &= \int_0^{\frac{1}{n+1}} \|\phi(.+y,t)+\phi(.-y,t)-2\phi(.,t)\|_r \left|K_n^{\Delta E}(t)\right| dt \\ &= O\left(\int_0^{\frac{1}{n+1}} v(y)\frac{w(t)}{v(t)}(n+1)dt\right) \\ &= O\left((n+1)v(y)\int_0^{\frac{1}{n+1}} \frac{w(t)}{v(t)}dt\right) \\ &= O\left((n+1)v(y)\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\int_0^{\frac{1}{n+1}} dt\right) \\ &= O\left(v(y)\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right). \qquad (10)\end{aligned}$$

Also,using Lemma (4.2) and (9) we get

$$\begin{aligned}I_2 &= \int_{\frac{1}{n+1}}^{\pi} \|\phi(.+y,t)+\phi(.-y,t)-2\phi(.,t)\|_r \left|K_n^{\Delta E}(t)\right| dt \\ &= O\left(\int_{\frac{1}{n+1}}^{\pi} v(y)\frac{w(t)}{v(t)}\frac{1}{(n+1)t^2}dt\right) \\ &= O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi} v(y)\frac{w(t)}{t^2 v(t)}dt\right). \qquad (11)\end{aligned}$$

By (6), (10) and (11), we have

$$\begin{aligned}\|l_n(.+y)+l_n(.-y)-2l_n(.)\|_r &= O\left(v(y)\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right)\\ &\quad +O\left(\frac{1}{n+1}\int\limits_{\frac{1}{n+1}}^{\pi} v(y)\frac{w(t)}{t^2v(t)}dt\right).\end{aligned}$$

Thus,

$$\begin{aligned}\sup_{y\neq 0}\frac{\|l_n(.+y)+l_n(.-y)-2l_n(.)\|_r}{v(y)} &= O\left(\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right)\\ &\quad +O\left(\frac{1}{n+1}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2v(t)}dt\right). \qquad (12)\end{aligned}$$

Clearly

$$|\phi(x,t)| = |f(x+t)+f(x-t)-2f(x)|$$

Applying Minkowski's inequality, we have

$$\begin{aligned}\|\phi(.,t)\|_r &\leq \|f(x+t)+f(x-t)-2f(x)\|_r\\ &= O\left(w(t)\right). \qquad (13)\end{aligned}$$

Using(13), Lemma (4.1),Lemma (4.2) we obtain

$$\begin{aligned}\|l_n(.)\|_r &= \left\|t_n^{\Delta E}-f\right\|_r \leq \left(\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right)\|\phi(.,t)\|_r\left|K_n^{\Delta E}(t)\right|dt\\ &= \int_0^{\frac{1}{n+1}}\|\phi(.,t)\|_r\left|(K_n^{\Delta E}(t)\right|dt+\int_{\frac{1}{n+1}}^{\pi}\|\phi(.,t)\|_r\left|K_n^{\Delta E}(t)\right|dt\\ &= O\left((n+1)\int_0^{\frac{1}{n+1}}w(t)dt\right)+O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2}dt\right)\\ &= O\left(w\left(\frac{1}{(n+1)}\right)\right)+O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2}dt\right). \qquad (14)\end{aligned}$$

Now, By (12) and (14)

$$\begin{aligned}\|l_n(.)\|_r^{(v)} &= \|l_n(.)\|_r+\sup_{y\neq 0}\frac{\|l_n(.+y)+l_n(.-y)-2l_n(.)\|_r}{v(y)}\\ &= O\left(w\left(\frac{1}{n+1}\right)\right)+O\left(\frac{1}{(n+1)}\int\limits_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2}dt\right)\\ &\quad +O\left(\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right)+O\left(\frac{1}{(n+1)}\int\limits_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{v(t)t^2}dt\right).\end{aligned}$$

Using the fact that $w(t) = \frac{w(t)}{v(t)}.v(t) \;\leq\; v(\pi)\frac{w(t)}{v(t)}$, $0 < t \leq \pi$, we get

$$\|l_n(.)\|_r^{(v)} \;=\; O\left(\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\right) + O\left(\frac{1}{(n+1)}\int\limits_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{v(t)t^2}dt\right). \tag{15}$$

Since w and v are zygmund modulus of continuity such that $\frac{w(t)}{v(t)}$ is positive, non decreasing, therefore

$$\frac{1}{n+1}\int\limits_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{v(t)t^2}dt \geq \frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\left(\frac{1}{n+1}\right)\int\limits_{\frac{1}{n+1}}^{\pi}\frac{dt}{t^2} \geq \frac{w\left(\frac{1}{n+1}\right)}{2v\left(\frac{1}{n+1}\right)}.$$

Then

$$\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)} \;=\; O\left(\frac{1}{(n+1)}\int\limits_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2v(t)}dt\right). \tag{16}$$

By (15) and (16), we have

$$\left\|t_n^{\Delta E} - f\right\|_r^{(v)} \;=\; O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2v(t)}dt\right).$$

$$E_n(f) = \inf_{t_n^{\Delta E}}\|t_n^{\Delta E} - f\|_r^{(v)} = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2v(t)}dt\right).$$

This completes the proof of theorem 3.1.

6. Proof of the Theorem 3.2

Following the proof of the theorem 3.1,

$$E_n(f) == O\left(\frac{1}{(n+1)}\int\limits_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2v(t)}dt\right).$$

Since $\frac{w(t)}{tv(t)}$ is positive, non increasing, therefore by second mean value theorem of integral calculus,

$$\begin{aligned} E_n(f) &= O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2v(t)}dt\right) \\ &= O\left(\frac{1}{(n+1)}\frac{(n+1)w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}\int_{\frac{1}{n+1}}^{\pi}\frac{dt}{t}\right) \\ &= O\left(\frac{w\left(\frac{1}{n+1}\right)}{v\left(\frac{1}{n+1}\right)}log(n+1)\pi\right). \end{aligned}$$

This completes the proof of theorem 3.2.

7. Applications

Following corollaries can be obtained from the Theorem 3.1.

Corollary 7.1. *Let* $f \in Z_{(\alpha),r}$, $r \geq 1$, $0 < \alpha \leq 1$ *then*

$$E_n(f) = \inf_{t_n^{\Delta E}} \left\|t_n^{\Delta E} - f\right\|_{(\beta),r} = \begin{cases} O\left(\frac{1}{(n+1)^{\alpha-\beta}}\right), & 0 \leq \beta < \alpha < 1, \\ O\left(\frac{log(n+1)\pi}{n+1}\right), & \beta = 0, \alpha = 1. \end{cases}$$

Proof. Taking $w(t) = t^\alpha$, $v(t) = t^\beta$ in Theorem 3.1, proof of this corollary can be obtained. □

Corollary 7.2. *The best approximation of a function* $f \in Z_r^{(w)}$ *by* $(H, \frac{1}{n+1}).E_1$ *means*

$$t_n^{HE} = \frac{1}{\log(n+1)}\sum_{k=0}^{n}\frac{1}{n-k+1}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu$$

of the Fourier series (1) is given by

$$E_n(f) = \inf_{t_n^{HE}} \left\|t_n^{HE} - f\right\|_r^{(v)} = O\left(\frac{1}{(n+1)}\int_{\frac{1}{n+1}}^{\pi}\frac{w(t)}{t^2v(t)}dt\right).$$

Corollary 7.3. *If we take* $a_{n,k} = \frac{p_{n-k}}{P_n}$, *where* $P_n = \sum_{k=0}^{n} p_k \neq 0$ *in Theorem 3.1, then best approximation of a function* $f \in Z_r^{(w)}$ *by* $(N, p_n).E_1$ *means*

$$t_n^{NE} = \frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}s_\nu$$

of the fourier series (1) is given by

$$E_n(f) = \inf_{t_n^{NE}} \left\| t_n^{NE} - f \right\|_r^{(v)} = O\left(\frac{1}{(n+1)} \int_{\frac{1}{n+1}}^{\pi} \frac{w(t)}{t^2 v(t)} dt \right).$$

Corollary 7.4. *If we take* $a_{n,k} = \frac{p_{n-k} q_k}{R_n}$, *where* $R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0$ *in Theorem 3.1, then best approximation of a function* $f \in Z_r^{(w)}$ *by* $(N,p,q).E_1$ *means*

$$t_n^{NE} = \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \frac{1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} s_\nu$$

of the fourier series (1) is given by

$$E_n(f) = \inf_{t_n^{NE}} \left\| t_n^{NE} - f \right\|_r^{(v)} = O\left(\frac{1}{(n+1)} \int_{\frac{1}{n+1}}^{\pi} \frac{w(t)}{t^2 v(t)} dt \right).$$

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Shyam Lal
Department of Mathematics, Faculty of Science, Banaras Hindu University
Varanasi-221005, India

E-mail address: shyam_lal@rediffmail.com

Shireen
Department of Mathematics, Faculty of Science, Banaras Hindu University
Varanasi-221005, India.

E-mail address: shrn.maths.bhu@gmail.com